Available online at www.sciencedirect.com

ScienceDirect

IFAC PapersOnLine 59-8 (2025) 60–65

Lyapunov criterion for output-to-state stability of distributed parameter systems ⋆

Qiaoling Chen * Andrii Mironchenko ** Fabian Wirth *

* *Faculty of Computer Science and Mathematics, University of Passau, 94032 Passau, Germany (e-mail: qiaoling.chen@uni-passau.de, fabian.lastname@uni-passau.de).*
** *Department of Mathematics, University of Bayreuth, 95447, Bayreuth, Germany (e-mail: andrii.mironchenko@uni-bayreuth.de)*

Abstract: We investigate output-to-state stability (OSS) for evolution equations in Banach spaces. We establish the direct OSS Lyapunov theorem and illustrate it with an example. Furthermore, we extend the notion of uniform global asymptotic stability modulo output (UGASMO) to infinite-dimensional systems and show that OSS implies UGASMO, and UGASMO guarantees the vanishing output vanishing state property.

Copyright © 2025 The Authors. This is an open access article under the CC BY-NC-ND license (https://creativecommons.org/licenses/by-nc-nd/4.0/)

Keywords: Output-to-state stability, Lyapunov methods, Stability analysis, Distributed parameter systems, Nonlinear systems, Detectability

1. INTRODUCTION

The notion of detectability, introduced in (Wonham, 1968) for linear systems, is paramount for observer design and dynamic feedback control. A linear system is called detectable if two trajectories producing the same output converge to each other. The generalization of the detectability concept to nonlinear systems in a way that allows for powerful characterization of this concept and for efficient tools for its verification is a challenging problem, and several approaches have been proposed for this, see (Sontag, 2008, Section 8) and Allan et al. (2021) for an overview.

One proposed concept is zero-detectability, a condition where trajectories corresponding to zero output are attracted to the zero equilibrium. This notion naturally extends to situations where inputs and outputs are small or bounded, implying that the states should also remain small or bounded, respectively. For linear systems, zero-detectability is equivalent to detectability (Sontag, 1998).

In parallel with that, the input-to-state stability (ISS) framework has revolutionized the robust stability analysis of nonlinear control systems and their interconnections (Sontag, 1989; Jiang et al., 1994; Sontag and Wang, 1996b; Dashkovskiy et al., 2010), see the monograph (Mironchenko, 2023) for a comprehensive treatment of the ISS theory of ordinary differential equations (ODEs). ISS characterizes how the state of a system responds to inputs, ensuring that bounded inputs lead to bounded states and that the state converges to zero if the input decays to zero.

The success of the ISS theory for analysis of nonlinear ODE control systems and their interconnections has led to the breakthrough in nonlinear detectability theory, based on the concepts of output-to-state stability (OSS) and input-output-to-state stability (IOSS). For systems with outputs, the notion of OSS, introduced as the dual of ISS (Sontag and Wang, 1996a), is defined by replacing inputs with outputs in the ISS framework. In spite of a similar look, these concepts are very different as the roles of inputs and outputs are different; namely, outputs depend directly on states in OSS, whereas states are influenced by inputs in ISS. For linear ODE systems without inputs, OSS is equivalent to detectability. For nonlinear systems, OSS is equivalent to the existence of a proper OSS Lyapunov function (Sontag and Wang, 1996a). For systems with input and output, IOSS has been introduced in (Krichman et al., 2001). IOSS is equivalent to the existence of a smooth IOSS Lyapunov function (Krichman et al., 2001). ISS can be characterized as a combination of IOSS and input-to-output stability (IOS), see (Jiang et al., 1994, Proposition 3.1) for the ODE case and (Bachmann et al., 2025, Section V.B) for general infinite-dimensional systems.

Recently, the input-to-state stability framework has been extended to infinite-dimensional systems in (Jayawardhana et al., 2008; Karafyllis and Krstic, 2016, 2018; Jacob et al., 2018; Mironchenko and Wirth, 2018; Mironchenko and Prieur, 2020; Schwenninger, 2020; Zheng and Zhu, 2021) to cite a few. At the same time, despite the importance of detectability theory, the theory of output-to-state stability has remained completely unexplored for nonlinear infinite-dimensional systems (Mironchenko and Prieur, 2020), even in the special case of time-delay systems (Chaillet et al., 2023). Note, however, that for linear infinite-dimensional systems, various detectability concepts are well-studied; see, e.g., (Curtain and Zwart, 2020, Chapter 8).

In this paper, our contribution is threefold. First, we extend the concept of OSS to infinite-dimensional systems described by semilinear abstract Cauchy problems and establish a direct OSS Lyapunov theorem for both coercive OSS Lyapunov functions in dissipation form and in implication form. Second, we generalize the notion of uniform global asymptotic stability modulo output (UGASMO)

⋆ This work was partially supported by German Research Foundation (DFG), grants: MI 1886/2-2 and MI 1886/3-1.

2405-8963 Copyright © 2025 The Authors. This is an open access article under the CC BY-NC-ND license.
Peer review under responsibility of International Federation of Automatic Control.
10.1016/j.ifacol.2025.08.067

to our class of infinite-dimensional systems and prove that OSS systems are necessarily UGASMO. Third, we establish the relationships between OSS, UGASMO and the vanishing output vanishing state (VOVS) property for infinite-dimensional systems.

The remainder of this paper is organized as follows. In Section 2, the problem formulation is presented, which contains the explicit definitions of OSS and OSS Lyapunov functions. Section 3 discusses the direct OSS Lyapunov theorem. In Section 4, we introduce the concept of UGASMO and derive relationships between OSS, UGASMO and VOVS. Concluding remarks are given in Section 5. Several proofs are omitted due to page restrictions.

Notation: Let $\mathbb{R}$ be the set of real numbers, $\mathbb{R}_+ := [0,\infty)$ be the set of nonnegative real numbers and $\mathbb{N}$ be the set of natural numbers. Given topological spaces X, Y, the space of continuous functions from X to Y is denoted by $C(X,Y)$ and $C^2(0,1)$ denotes the space of real-valued functions that are twice continuously differentiable on the interval $(0,1)$. We use the following notation for the classes of comparison functions:

$$\begin{aligned}
\mathcal{P} &:= \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ |\ \ \gamma \text{ is continuous with } \gamma(0)=0 \text{ and } \gamma(r)>0 \text{ for } r>0\},\\
\mathcal{K} &:= \{\gamma \in \mathcal{P} |\ \ \gamma \text{ is strictly increasing}\},\\
\mathcal{K}_\infty &:= \{\gamma \in \mathcal{K} |\ \ \gamma \text{ is unbounded}\},\\
\mathcal{L} &:= \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ |\ \ \gamma \text{ is continuous, strictly decreasing with } \lim_{r\to\infty}\gamma(r)=0\},\\
\mathcal{KL} &:= \{\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ |\ \ \beta \text{ is continuous}, \beta(\cdot,t)\in\mathcal{K}, \forall t\ge 0, \beta(r,\cdot)\in\mathcal{L}, \forall r>0\}.
\end{aligned}$$

2. PROBLEM STATEMENT

Consider the system

$$\dot{x} = Ax(t) + f(x(t)), \quad y = h(x(t)), \tag{1}$$

where A is the generator of a strongly continuous semigroup T of bounded linear operators on a Banach space X, the output map $h : X \to Y$ is continuous with $h(0) = 0$. Here Y is a normed linear space of output values with norm $\|\cdot\|_Y$, $f : X \to X$ is Lipschitz continuous on bounded subsets of X, that is, for all $C > 0$, there is a constant $L(C) > 0$ such that for all $x_1, x_2 \in B_C(0) := \{x \in X : \|x\|_X < C\}$, it holds that

$$\|f(x_1) - f(x_2)\|_X \le L(C)\|x_1 - x_2\|_X.$$

We further assume that $f(0) = 0$.

We consider mild solutions of (1), i.e., solutions of the following integral equation with $x(0) \in X$:

$$x(t) = T(t)x(0) + \int_0^t T(t-s)f(x(s))\mathrm{d}s \tag{2}$$

belonging to the space $C([0,\mu], X)$ for some $\mu > 0$.

Throughout the paper, we denote by $\phi(\cdot, x_0)$ the unique maximal solution of (2) with initial state $x(0) := x_0 \in X$, which is defined on an interval $[0, t_m(x_0))$ with $t_m(x_0) \in (0, +\infty]$ being the maximal existence time; see (Cazenave and Haraux, 1998, Proposition 4.3.3). The map ϕ is defined and continuous on the open set $\mathcal{D}_\phi := \cup_{x\in X}[0, t_m(x)) \times \{x\} \subset \mathbb{R}_+ \times X$; see (Cazenave and Haraux, 1998, Proposition 4.3.7). Thus, the output $y : (t, x_0) \mapsto h(\phi(t, x_0))$ is continuous from $\mathcal{D}_\phi$ to Y.

Definition 2.1. System (1) is called *forward complete*, if for every $x \in X$ and for all $t \ge 0$, the value $\phi(t,x) \in X$ is well-defined.

Outputs corresponding to forward complete trajectories belong to the space $\mathcal{Y} := C(\mathbb{R}_+, Y)$. We denote by $y(\cdot, x)$ the output corresponding to the initial condition $x(0) = x$ defined on $[0, t_m(x))$. The supremum norm of $y(\cdot; x)$ restricted to an interval $[0,t] \subset [0, t_m(x))$ is denoted by $\|y|_{[0,t]}\|_{\mathcal{Y}} := \max_{s\in[0,t]} \|y(s)\|_Y$.

The following definition formalizes the notion of OSS for system (1). Recall that $A \subset X$ is is said to be (forward) invariant for (1) if for any $x \in A$ it holds that $\phi(t,x) \in A$ for any $t \in [0, t_m(x))$.

Definition 2.2. System (1) is called *output-to-state stable (OSS)*, if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for all $x \in X$ and all $t \in [0, t_m(x))$ it holds that

$$\|\phi(t,x)\|_X \le \max\{\beta(\|x\|_X, t), \gamma(\|y(\cdot,x)|_{[0,t]}\|_{\mathcal{Y}})\}. \tag{3}$$

If $\tilde{X}$ is an invariant subset under (1) and (3) holds for all $\tilde{x} \in \tilde{X}$, then we say system (1) is OSS on $\tilde{X}$.

In this paper, we abbreviate both "output-to-state stability" and "output-to-state stable" as OSS.

Remark 2.3. From Definition 2.2, it follows readily that

(i) If the dynamic equation of (1)
$$\dot{x} = Ax(t) + f(x(t)),$$
is *uniformly globally asymptotically stable (UGAS)*, that is, if there exists $\beta \in \mathcal{KL}$ such that
$$\|\phi(t,x)\|_X \le \beta(\|x\|_X, t), \quad x \in X, \quad t \in \mathbb{R}_+,$$
then system (1) is OSS.

(ii) If the output satisfies $\|h(x)\|_Y \ge \xi(\|x\|_X)$ for some $\xi \in \mathcal{K}_\infty$ and all $x \in X$, then (1) is OSS with $\gamma = \xi^{-1}$.

By replacing the outputs in (3) with inputs, the definition of *input-to-state stability (ISS)* is obtained. While ISS may seem similar to OSS, the derivations of their corresponding characterizations are quite different. In OSS, the outputs depend directly on the states, whereas in ISS, the states are influenced by the inputs. Thus, while ISS focuses on how the inputs affect the states, OSS addresses the relationship between outputs and states, and is essentially a structural property of the system.

The inequality in (3) not only characterizes the property that "bounded outputs imply bounded states", but also shows that "small outputs imply asymptotically small states". Additionally, as we show in Corollary 4.8, OSS systems have the property that all forward complete trajectories with a vanishing output converge to zero, compare Sontag (2003) for the converging-input converging-state property.

As expected, Lyapunov functions are central to the analysis of OSS. To formulate Lyapunov conditions, we will require the notion of the (upper right-hand) Dini derivative, which we recall now. Given $V : X \to \mathbb{R}$, such that $t \mapsto V(\phi(t,x))$ is continuous, we define the (upper right-hand) *Dini derivative* of $V(\phi(\cdot,x))$ at t by:

$$D^+V(\phi(t,x)) = \overline{\lim_{\tau\to+0}} \frac{V(\phi(t+\tau,x)) - V(\phi(t,x))}{\tau}.$$

Further, let us define the *Lie derivative* of V at $x \in X$ by

$$\dot{V}(x) := D^+V(\phi(t,x))|_{t=0},$$
provided that $t \mapsto V(\phi(t,x))$ is continuous at $t = 0$.

Definition 2.4. A function $V : X \to \mathbb{R}_+$ is called a *coercive OSS Lyapunov function in dissipation form* for (1) if:

(i) There exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that for every $x \in X$:
$$\alpha_1(\|x\|_X) \le V(x) \le \alpha_2(\|x\|_X), \tag{4}$$
(ii) For every $x \in X$, the map $t \mapsto V(\phi(t,x))$, $t \in [0, t_m(x))$, is continuous.

(iii) There exist $\alpha, \sigma \in \mathcal{K}_\infty$ such that for every $x \in X$, the following dissipation inequality holds:
$$\dot{V}(x) \le -\alpha(\|x\|_X) + \sigma(\|h(x)\|_Y). \tag{5}$$

If $V(0) = 0$ and (4) is replaced by
$$0 < V(x) \le \alpha_2(\|x\|_X) \quad \forall x \in X\backslash\{0\}, \tag{6}$$
then V is called a *non-coercive OSS Lyapunov function in dissipation form.* If $\tilde{X}$ is a subset of X, which is invariant under the dynamics (1) and (i)-(iii) hold for every $\tilde{x} \in \tilde{X}$, then we say that V is an OSS Lyapunov function in dissipation form on $\tilde{X}$.

A slightly different concept of OSS Lyapunov function is defined as follows.

Definition 2.5. A function $V : X \to \mathbb{R}_+$ is called a *coercive OSS Lyapunov function in implication form* for system (1), if items (i) and (ii) of Definition 2.4 hold and

(iii') there exist $\alpha_3, \chi \in \mathcal{K}_\infty$ such that for every $x \in X$, the following implication holds:
$$\|x\|_X \ge \chi(\|h(x)\|_Y) \quad \Rightarrow \quad \dot{V}(x) \le -\alpha_3(\|x\|_X). \tag{7}$$

If $V(0) = 0$ and (4) is replaced by (6), then V is called a *non-coercive OSS Lyapunov function in implication form.* If $\tilde{X}$ is a subset of X, which is invariant under the dynamics (1) and properties (i), (ii) and (iii') hold for every $\tilde{x} \in \tilde{X}$, then we call V an OSS Lyapunov function in implication form on $\tilde{X}$.

We begin with the following result.

Proposition 2.6. If V is a coercive (non-coercive) OSS Lyapunov function in dissipation form, then V is a coercive (non-coercive) OSS Lyapunov function in implication form respectively. The same is true when OSS Lyapunov functions on invariant subsets are considered.

Remark 2.7. It is not known whether the converse implication to Proposition 2.6 holds. For ODE systems the converse holds: while the result has not been explicitly stated in Krichman et al. (2001), it is obvious from the results of Lemma 4.23 and inequality (4.68) obtained in that reference.

3. DIRECT LYAPUNOV THEOREM

It is hard to determine, using the definition, whether a given system is OSS. The construction of an OSS Lyapunov function is a more realistic way to verify that a nonlinear control system is OSS. In this section, we prove the direct OSS Lyapunov theorem for system (1).

Recall the following comparison principle:

Lemma 3.1. (Comparison principle). For any $\tilde{\rho} \in \mathcal{P}$, there is $\beta \in \mathcal{KL}$ so that for any $y \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfying
$$D^+y(t) \le -\tilde{\rho}(y(t)) \quad \forall t > 0, \tag{8}$$
it holds that
$$y(t) \le \beta(y(0), t) \quad \forall t \ge 0. \tag{9}$$

Proof: See (Mironchenko, 2023, Proposition A.35). ■

Utilizing Lemma 3.1, we are able to get the first main result in this paper.

Theorem 3.2. (Direct OSS Lyapunov theorem). Let $\tilde{X}$ be a subset of X that is invariant for system (1). If system (1) admits a coercive OSS Lyapunov function (in dissipation form or in implication form) on $\tilde{X}$, then it is OSS on $\tilde{X}$.

Proof: By Proposition 2.6, it is sufficient to prove the statement for coercive OSS Lyapunov functions in implication form.

Let system (1) admit a coercive OSS Lyapunov function in implication form on $\tilde{X}$. Then inequalities (4) and (7) hold for all $\tilde{x} \in \tilde{X}$.

The proof is divided into two steps. In the first step, we establish an upper bound on the state norm in terms of the output norm via a class $\mathcal{K}$ function. In the second step, we derive a separate upper bound in terms of the initial state and time using a class $\mathcal{KL}$ function. Since these estimates are valid over complementary time intervals, we combine them to obtain a global bound on the state norm given by the maximum of the two expressions. This results in the OSS estimate as stated in Definition 2.2.

Step 1. Fix $\tilde{x} \in \tilde{X}$ and consider the trajectory $\phi(\cdot) := \phi(\cdot, \tilde{x})$ and note that $\phi(t) \in \tilde{X}$ for all $t \in [0, t_m(\tilde{x}))$ as $\tilde{X}$ is invariant. Define
$$a(t) := \alpha_2 \circ \chi\left(\|y|_{[0,t]}\|_{\mathcal{Y}}\right) \quad \forall t \in [0, t_m(\tilde{x}))$$
and
$$S(t) := \{w \in \tilde{X} |\ V(w) \le a(t)\}, \quad t \in [0, t_m(\tilde{x})),$$
where $\alpha_2, \chi \in \mathcal{K}_\infty$ are as in Definition 2.5. We first prove that if there exists $T \ge 0$ such that $\phi(T) \in S(T)$, then $\phi(t) \in S(t)$ for all $t \in [T, t_m(\tilde{x}))$.

Let $T \ge 0$ be such that $\phi(T) \in S(T)$. Seeking a contradiction, assume that there exist $\varepsilon > 0$ and $T_1 \ge T$ such that $V(\phi(T_1)) > a(T_1) + \varepsilon$. Without loss of generality, let
$$T_1 := \inf\{t \in [T, t_m(\tilde{x})) : V(\phi(t)) > a(t) + \varepsilon\}.$$
By the continuity of the map $t \mapsto V(\phi(t))$ and $a(\cdot)$, there is an open interval $\mathcal{N}(T_1)$ containing T_1 such that $V(\phi(t)) \ge a(t)$ for all $t \in \mathcal{N}(T_1)$. Therefore, by (4), we obtain $\alpha_2(\|\phi(t)\|_X) \ge a(t)$ for $t \in \mathcal{N}(T_1)$. Then $\|\phi(t)\|_X \ge \chi\left(\|y|_{[0,t]}\|_{\mathcal{Y}}\right)$, and from (7) we have $\dot{V}(\phi(t)) \le -\alpha_3(\|\phi(t)\|_X)$, $t \in \mathcal{N}(T_1)$.

It follows from (Bruckner, 1978, Theorem XI.4.1) that the negativity of the Dini derivative implies that $t \mapsto V(\phi(t))$ is decreasing on the interval $\mathcal{N}(T_1)$. Thus
$$V(\phi(t)) \ge V(\phi(T_1)) \ge a(T_1) + \varepsilon \ge a(t) + \varepsilon$$
for $t \in \mathcal{N}(T_1) \cap (T, T_1)$, contradicting to the definition of T_1. The claimed invariance property of $S(\cdot)$ is proved.

Let $T^*(\tilde{x}) := \inf\{t \in [0, t_m(\tilde{x})) : \phi(t) \in S(t)\}$. By (4),
$$\|\phi(t)\|_X \le \alpha_1^{-1} \circ \alpha_2 \circ \chi(\|y|_{[0,t]}\|_{\mathcal{Y}}),\ t \in [T^*(\tilde{x}), t_m(\tilde{x})). \tag{10}$$

Step 2. By definition of $T^*(\tilde{x})$, (4), and (7), it follows that
$$\dot{V}(\phi(t)) \le -\alpha_3 \circ \alpha_2^{-1}(V(\phi(t))), \quad t \in [0, T^*(\tilde{x})). \tag{11}$$

By Lemma 3.1, there exists $\beta_0 \in \mathcal{KL}$ (independent of $\tilde{x}$, as the decay estimate in (11) is independent of $\tilde{x}$) such that

$$V(\phi(t)) \le \beta_0(V(\phi(0)), t), \quad t \in [0, T^*(\tilde{x})), \tilde{x} \in \tilde{X}.$$

Then it follows from (4) that

$$\|\phi(t)\|_X \le \alpha_1^{-1} \circ \beta_0(\alpha_2(\|\tilde{x}\|_X), t) \quad \forall t \in [0, T^*(\tilde{x})). \tag{12}$$

Using (10) and (12), we obtain (3) with $\beta(r,t) = \alpha_1^{-1} \circ \beta_0(\alpha_2(r), t)$ and $\gamma(r) = \alpha_1^{-1} \circ \alpha_2(\chi(r))$. Thus, (1) is OSS on $\tilde{X}$. ■

When verifying whether a given Lyapunov candidate is an OSS Lyapunov function, it is necessary to compute its Dini derivative on the given state space. The following density argument simplifies these computations in many cases. The proof is omitted for reasons of space.

Lemma 3.3. (Density argument). Let $\hat{X}$ be a norm-dense subset of X. Suppose there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for all $\hat{x} \in \hat{X}$ and all $t \in [0, t_m(\hat{x}))$,

$$\|\phi(t, \hat{x})\|_X \le \beta(\|\hat{x}\|_X, t) + \gamma(\|y(\cdot, \hat{x})|_{[0,t]}\|_{\mathcal{Y}}).$$

Then the system (1) is OSS on X with the same β and γ.

Next, we illustrate the application of Theorem 3.2 by an example.

3.1 Example

Let $a > 0$, and $c, q \in \mathbb{R}$. Consider the linear parabolic equation with output

$$x_t(z,t) = a x_{zz}(z,t) + c x(z,t), \tag{13a}$$

$$y(t) = \int_0^1 x^2(z)\mathrm{d}z, \tag{13b}$$

where $x_t(z,t), x_{zz}(z,t)$ denote the first, resp. second partial derivatives of $x(z,t)$ with respect to t resp. z. The equation is defined for $z \in (0,1), t > 0$, endowed with boundary conditions (namely a Dirichlet condition at the right end and a Robin condition on the left end)

$$x_z(0,t) = -q x(0,t), \tag{14a}$$

$$x(1,t) = 0, \tag{14b}$$

and an initial condition

$$x(z,0) = x_0(z).$$

This system fits into our framework because there is an associated analytic semigroup; see Engel and Nagel (2000, Chapter VI, Exercise 4.7), and so there is an equivalent formulation in the form of (1) with nonlinearity $f = 0$. The generator of the corresponding analytic semigroup is given by the operator $A = a\frac{\mathrm{d}^2}{\mathrm{d}z^2} + c$ on $X = L^2(0,1)$ with domain

$$D(A) = \{x \in H^2(0,1) \; ; \; x_z(0) = -q x(0), x(1) = 0\}, \tag{15}$$

where $H^2(0,1)$ is the standard Sobolev space of square integrable functions with (weak) first and second derivative that are again square integrable.

System (13) with boundary conditions (14) is unstable if c or q are positive and sufficiently large, see (Smyshlyaev and Krstic, 2010, p. 13). We now show that the system is always OSS. In some cases, an OSS Lyapunov function is easy to find.

Proposition 3.4. Let $X = L^2(0,1)$ and $Y = \mathbb{R}$. Then:

(i) System (13) with (14) is OSS for any $a > 0$ and any $c, q \in \mathbb{R}$.

(ii) If $q < 1$, then

$$V(x) = \frac{1}{2}\int_0^1 x^2(z)\mathrm{d}z = \frac{1}{2}\|x\|_X^2 \tag{16}$$

is a coercive OSS Lyapunov function in a dissipative form for (13)–(14).

Proof: The item (i) follows from Remark 2.3(ii).

(ii) Consider the OSS Lyapunov function candidate from (16), which is obviously coercive. Assume that $x \in D(A)$ is a twice continuously differentiable function on $(0,1)$. Using integration by parts and the boundary conditions (14) yields

$$\begin{aligned}
\dot{V}(x) &= \int_0^1 x(z)x_t(z)\mathrm{d}z \\
&= \int_0^1 x(z)(a x_{zz}(z) + c x(z))\mathrm{d}z \\
&= -a x(0)x_z(0) - \int_0^1 a x_z^2(z)\mathrm{d}z + \int_0^1 c x^2(z)\mathrm{d}z \\
&= aqx^2(0) - a\int_0^1 x_z^2(z)\mathrm{d}z + c\int_0^1 x^2(z)\mathrm{d}z. \qquad (17)
\end{aligned}$$

By a variation of Wirtinger's inequality (Smyshlyaev and Krstic, 2010, Remark B.1) and $x(1) = 0$, we deduce that

$$-\int_0^1 x_z^2(z)\mathrm{d}z \le -\frac{\pi^2}{4}\int_0^1 x^2(z)\mathrm{d}z.$$

For $q \le 0$, we have

$$\begin{aligned}
\dot{V}(x) &\le -a\int_0^1 x_z^2(z)\mathrm{d}z + c\int_0^1 x^2(z)\mathrm{d}z \\
&\le -\frac{a\pi^2}{4}\int_0^1 x^2(z)\mathrm{d}z + c\int_0^1 x^2(z)\mathrm{d}z.
\end{aligned}$$

For $0 < q < 1$, applying Agmon's inequality (Smyshlyaev and Krstic, 2010, Lemma B.2) and $x(1) = 0$, we have

$$\begin{aligned}
\max_{z\in[0,1]} |x(z)|^2 &\le 2\left(\int_0^1 x^2(z)\mathrm{d}z\right)^{\frac{1}{2}}\left(\int_0^1 x_z^2(z)\mathrm{d}z\right)^{\frac{1}{2}} \\
&\le \int_0^1 x^2(z)\mathrm{d}z + \int_0^1 x_z^2(z)\mathrm{d}z,
\end{aligned}$$

and (17) becomes

$$\begin{aligned}
\dot{V}(x) \le& aq\int_0^1 x^2(z)\mathrm{d}z + aq\int_0^1 x_z^2(z)\mathrm{d}z \\
&- a\int_0^1 x_z^2(z)\mathrm{d}z + c\int_0^1 x^2(z)\mathrm{d}z \\
=& (aq + c)\int_0^1 x^2(z)\mathrm{d}z - a(1-q)\int_0^1 x_z^2(z)\mathrm{d}z \\
\le& -\frac{a\pi^2(1-q)}{4}\int_0^1 x^2(z)\mathrm{d}z + (aq + c)\int_0^1 x^2(z)\mathrm{d}z.
\end{aligned}$$

Now $\tilde{X} := D(A) \cap C^2(0,1)$ in invariant under the dynamics (13) with (14), see (Ladyzhenskaya et al., 1968, Chapter IV, Theorem 5.3). Thus, if $q < 1$, then the previous estimate shows that V is an OSS Lyapunov function in dissipation form on $\tilde{X}$ and Theorem 3.2 implies that (13) and (14) is OSS on $\tilde{X}$.

In particular, there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for all $x \in D(A) \cap C^2(0,1)$ and all $t \in \mathbb{R}_+$,

$$\|\phi(t,x)\|_X \leq \beta(\|x\|_X, t) + \gamma(\|y|_{[0,t]}\|_{\mathcal{Y}}).$$

As $D(A) \cap C^2(0,1)$ is also dense in X in the norm topology, the assertion for the entire state space X follows by an application of Lemma 3.3. ■

4. UNIFORM GLOBAL ASYMPTOTIC STABILITY MODULO OUTPUT

Following (Krichman et al., 2001, Definition 2.9), we first recall the following notion:

Definition 4.1. System (1) is called *uniformly globally asymptotically stable modulo output (UGASMO)*, if there exist $\rho \in \mathcal{K}_\infty$ and $\theta \in \mathcal{KL}$ such that for all $x \in X$ and any $T \in [0, t_m(x))$, if

$$\|\phi(t,x)\|_X \geq \rho(\|y(t,x)\|_Y) \quad \forall t \in [0,T],$$

then

$$\|\phi(t,x)\|_X \leq \theta(\|x\|_X, t) \quad \forall t \in [0,T].$$

This means that either the states are bounded by a properly scaled output, or they uniformly converge to zero.

Definition 4.2. We say that (1) has the *vanishing output vanishing state (VOVS) property* if for all $x \in X$ with $t_m(x) = \infty$ we have

$$\lim_{t\to\infty} y(t,x) = 0 \quad \Rightarrow \quad \lim_{t\to\infty} \phi(t,x) = 0.$$

Let us show that UGASMO systems are VOVS.

Proposition 4.3. If system (1) is UGASMO, then it has the VOVS property.

Proof: Let $x \in X$ be such that $t_m(x) = \infty$ and $y(t,x) \to 0$ as $t \to \infty$. Thus, for any $\varepsilon > 0$ there is $\tau_1 = \tau_1(\varepsilon)$ such that $\|y(t,x)\|_Y \leq \varepsilon$ for all $t \geq \tau_1$. Define

$$\tau_2(\varepsilon) := \inf\{t \geq \tau_1 : \|\phi(t,x)\|_X \leq \rho(\|y(t,x)\|_Y)\},$$

provided that the set over which the infimum is taken is nonempty. Otherwise, we set $\tau_2(\varepsilon) := +\infty$.

If $\tau_2(\varepsilon) = +\infty$, then $\phi(t,x) \to 0$ as $t \to \infty$ by UGASMO and the cocycle property. Otherwise, by continuity of trajectories and outputs, we have that

$$\|\phi(\tau_2(\varepsilon),x)\|_X \leq \rho(\|y(\tau_2(\varepsilon),x)\|_Y). \tag{18}$$

Now pick any $\hat{t} > \tau_2(\varepsilon)$. Then either

$$\|\phi(\hat{t},x)\|_X \leq \rho(\|y(\hat{t},x)\|_Y) \leq \rho(\varepsilon), \tag{19}$$

or $\|\phi(\hat{t},x)\|_X > \rho(\|y(\hat{t},x)\|_Y)$. In the latter case, by continuity of trajectories and outputs, there are $t_1 \geq \tau_2(\varepsilon)$ and $t_2 \in (t_1, +\infty]$ such that $\hat{t} \in (t_1,t_2)$ and (t_1,t_2) is the largest interval in the sense of set inclusion such that

$$\|\phi(t,x)\|_X > \rho(\|y(t,x)\|_Y), \quad t \in (t_1,t_2). \tag{20}$$

By the cocycle property,

$$\|\phi(t,\phi(t_1,x))\|_X \geq \rho(\|y(t,\phi(t_1,x))\|_Y), \quad t \in [0, t_2 - t_1).$$

Using UGASMO and the cocycle property, we obtain for $t \in [0, t_2 - t_1)$ that

$$\|\phi(t+t_1,x)\|_X = \|\phi(t,\phi(t_1,x))\|_X \leq \theta(\|\phi(t_1,x)\|_X, t).$$

If $t_1 > \tau_2(\varepsilon)$, then as (t_1,t_2) is the largest set satisfying (20), and we have that $\|\phi(t_1,x)\|_X = \rho(\|y(t_1,x)\|_Y)$. Otherwise, if $t_1 = \tau_2(\varepsilon)$, then (18) holds, and in any of these cases

$$\|\phi(t_1,x)\|_X \leq \rho(\|y(t_1,x)\|_Y) \leq \rho(\varepsilon),$$

and we proceed to

$$\|\phi(t+t_1,x)\|_X \leq \theta(\rho(\varepsilon),t) \leq \theta(\rho(\varepsilon),0), \quad t \in [0, t_2 - t_1).$$

Together with (19), we obtain that

$$\|\phi(t,x)\|_X \leq \max\{\theta(\rho(\varepsilon),0), \rho(\varepsilon)\}, \quad t \geq \tau_2(\varepsilon).$$

As $\max\{\theta(\rho(\varepsilon),0),\rho(\varepsilon)\} \to 0$ as $\varepsilon \to 0$, and since ε can be chosen arbitrarily small, the claim follows. ■

In (Krichman et al., 2001, Proposition 2.10), an equivalent criterion of UGASMO has been shown. Here, we extend it to the infinite-dimensional setting. We need the following proposition, see Mironchenko (2025):

Proposition 4.4. Let $g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ satisfy the following properties:

(i) For all $r > 0$ and all $\varepsilon > 0$ there is $\tau = \tau(\varepsilon, r) > 0$ such that

$$s \leq r \ \wedge \ t \geq \tau \quad \Rightarrow \quad g(s,t) \leq \varepsilon. \tag{21}$$

(ii) There is $\sigma_1 \in \mathcal{K}_\infty$ and $\delta > 0$, such that

$$s \leq \delta \ \wedge \ t \geq 0 \quad \Rightarrow \quad g(s,t) \leq \sigma_1(s). \tag{22}$$

(iii) g is bounded on bounded sets.

Then there is $\beta \in \mathcal{KL}$ such that

$$g(s,t) \leq \beta(s,t) \quad \forall s,t \in \mathbb{R}_+. \tag{23}$$

Based on Proposition 4.4, we have

Proposition 4.5. System (1) is UGASMO if and only if there exists $\rho \in \mathcal{K}_\infty$ so that the following statements hold:

(i) There exists $\kappa \in \mathcal{K}$ such that for any $x \in X$ and any $T \in [0, t_m(x))$ such that

$$\|\phi(t,x)\|_X \geq \rho(\|y(t,x)\|_Y) \quad \forall t \in [0,T],$$

the following estimate holds:

$$\|\phi(t,x)\|_X \leq \kappa(\|x\|_X) \quad \forall t \in [0,T].$$

(ii) For any $\varepsilon > 0$, any $r > 0$, there exists $T(r,\varepsilon) \geq 0$ such that for any $\|x\|_X \leq r$ and any $T \in [0, t_m(x))$, if

$$\|\phi(t,x)\|_X \geq \rho(\|y(t,x)\|_Y) \quad \forall t \in [0,T],$$

then

$$\|\phi(t,x)\|_X < \varepsilon \quad \forall t \in [T(r,\varepsilon), T].$$

Proof: The necessity is obvious. To prove sufficiency, assume that statements (i) and (ii) hold. Define the function $g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ by

$$g(s,t) := \sup\{\|\phi(\tau,x)\|_X : \|x\|_X \leq s, \tau < \min\{t, t_m(x)\},$$
$$\|\phi(\hat{t},x)\|_X \geq \rho(\|y(\hat{t},x)\|_Y) \quad \forall \hat{t} \in [0,\tau]\}.$$

Note that as $\phi(\cdot,0) \equiv 0$ we always have $g(s,t) \geq 0$. The finiteness of g follows from statement (i), which ensures $\|\phi(\tau,x)\|_X \leq \kappa(s)$. Furthermore, statement (ii) guarantees that for sufficiently large time, $\|\phi(t,x)\|_X$ eventually becomes arbitrarily small. This function g satisfies the three conditions in Proposition 4.4, which completes the proof. ■

Remark 4.6. If the outputs are identically zero, then the statement (i) of Proposition 4.5 becomes uniform global stability and the statement (ii) of Proposition 4.5 is uniform global attractivity, see Mironchenko (2023, Definition B.21).

Proposition 4.7. If system (1) is OSS, then it is UGASMO.

As a consequence of Propositions 4.7 and 4.3, we obtain exists coercive OSS LF in dissipation form $\xrightarrow{\text{Prop. 2.6}}$ exists coercive OSS LF in implication form $\xrightarrow{\text{Thm. 3.2}}$ OSS $\xrightarrow{\text{Prop. 4.7}}$ UGASMO $\xrightarrow{\text{Prop. 4.3}}$ VOVS

Fig. 1. Relations between key properties of system (1).

Corollary 4.8. If system (1) is OSS, then it has the VOVS property.

We summarize our main results and relationships discussed in this paper in Figure 1.

5. CONCLUSION

The primary contribution of this article is the formulation of OSS for infinite-dimensional systems. By employing the comparison principle, we present the direct OSS Lyapunov theorem and provide an illustrative example to validate the result. Furthermore, we investigate the relationships between OSS, the existence of coercive OSS Lyapunov functions (in dissipation form or in implication form), VOVS, and UGASMO.

In future research, we aim to extend the current framework by proving that UGASMO implies the existence of OSS Lyapunov functions. By combining this result with the findings of our current paper, we plan to derive the converse OSS Lyapunov theorem for such systems. Moreover, we intend to explore IOSS in this context.

REFERENCES

Allan, D.A., Rawlings, J., and Teel, A.R. (2021). Nonlinear detectability and incremental input/output-to-state stability. *SIAM J. Control Optim.*, 59(4), 3017–3039.

Bachmann, P., Dashkovskiy, S., and Mironchenko, A. (2025). Characterization of input-to-output stability for infinite dimensional systems. *Provisionally accepted to IEEE Transactions on Automatic Control.*

Bruckner, A.M. (1978). *Differentiation of Real Functions*, volume 659 of *Lecture Notes in Mathematics*. Springer.

Cazenave, T. and Haraux, A. (1998). *An Introduction to Semilinear Evolution Equations*. Oxford University Press, New York.

Chaillet, A., Karafyllis, I., Pepe, P., and Wang, Y. (2023). The ISS framework for time-delay systems: a survey. *Math. Control, Signals, Syst.*, 35(2), 237–306.

Curtain, R. and Zwart, H. (2020). *Introduction to Infinite-Dimensional Systems Theory: A State-Space Approach.* Springer.

Dashkovskiy, S., Rüffer, B., and Wirth, F. (2010). Small gain theorems for large scale systems and construction of ISS Lyapunov functions. *SIAM J. Control Optim.*, 48(6), 4089–4118.

Engel, K.J. and Nagel, R. (2000). *One-parameter Semigroups for Linear Evolution Equations*, volume 194 of *Graduate Texts in Math.* Springer-Verlag, New York.

Jacob, B., Nabiullin, R., Partington, J.R., and Schwenninger, F.L. (2018). Infinite-dimensional input-to-state stability and Orlicz spaces. *SIAM J. Control Optim.*, 56(2), 868–889.

Jayawardhana, B., Logemann, H., and Ryan, E.P. (2008). Infinite-dimensional feedback systems: the circle criterion and input-to-state stability. *Communications in Information and Systems*, 8(4), 413–444.

Jiang, Z., Teel, A., and Praly, L. (1994). Small-gain theorem for ISS systems and applications. *Math. Control, Signals, Syst.*, 7(2), 95–120.

Karafyllis, I. and Krstic, M. (2016). ISS with respect to boundary disturbances for 1-D parabolic PDEs. *IEEE Transactions on Automatic Control*, 61(12), 3712–3724.

Karafyllis, I. and Krstic, M. (2018). *Input-to-State Stability for PDEs.* Springer-Verlag, London.

Krichman, M., Sontag, E.D., and Wang, Y. (2001). Input-output-to-state stability. *SIAM J. Control Optim.*, 39(6), 1874–1928.

Ladyzhenskaya, O.A., Solonnikov, V.A., and Ural'tseva, N.N. (1968). *Linear and Quasi-linear Equations of Parabolic Type.* AMS, Providence, RI.

Mironchenko, A. (2023). *Input-to-State Stability: Theory and Applications.* Springer Nature.

Mironchenko, A. and Prieur, C. (2020). Input-to-state stability of infinite-dimensional systems: recent results and open questions. *SIAM Review*, 62(3), 529–614.

Mironchenko, A. and Wirth, F. (2018). Characterizations of input-to-state stability for infinite-dimensional systems. *IEEE Trans. Aut. Control*, 63(6), 1692–1707.

Mironchenko, A. (2025). Modeling and stability analysis of live systems with time-varying dimension. *Submitted. arxiv.org/abs/2501.15991.*

Schwenninger, F.L. (2020). Input-to-state stability for parabolic boundary control: Linear and semilinear systems. In *Control Theory of Infinite-Dimensional Systems*, 83–116. Springer.

Smyshlyaev, A. and Krstic, M. (2010). *Adaptive Control of Parabolic PDEs.* Princeton Univ. Press, Princeton, NJ.

Sontag, E.D. (1989). Smooth stabilization implies coprime factorization. *IEEE Trans. Aut. Control*, 34(4), 435–443.

Sontag, E.D. (1998). *Mathematical Control Theory: Deterministic Finite Dimensional Systems.* Springer-Verlag, New York, 2nd edition.

Sontag, E.D. (2008). Input to state stability: Basic concepts and results. In *Nonlinear and Optimal Control Theory*, chapter 3, 163–220. Springer, Heidelberg.

Sontag, E.D. and Wang, Y. (1996a). Detectability of nonlinear systems. In *Proc. Conf. Information Science and Systems (CISS 96)*, 1031–1036. Princeton, NJ.

Sontag, E. and Wang, Y. (1996b). New characterizations of input-to-state stability. *IEEE Trans. Aut. Control*, 41(9), 1283–1294.

Sontag, E.D. (2003). A remark on the converging-input converging-state property. *IEEE Trans. Aut. Control*, 48(2), 313–314.

Wonham, W.M. (1968). On a matrix Riccati equation of stochastic control. *SIAM J. Control*, 6(4), 681–697.

Zheng, J. and Zhu, G. (2021). Approximations of Lyapunov functionals for ISS analysis of a class of higher dimensional nonlinear parabolic PDEs. *Automatica*, 125, 109414.