Preprints, Joint Conference 8th IFAC Symposium on Mechatronic Systems (MECHATRONICS 2019), and 11th IFAC Symposium on Nonlinear Control Systems (NOLCOS 2019)
Vienna, Austria, Sept. 4-6, 2019

Stability of infinitely many interconnected systems ⋆

Sergey Dashkovskiy * **Andrii Mironchenko** **
Jochen Schmid ***,**** **Fabian Wirth** **

* *Institute for Mathematics, University of Würzburg, Germany (e-mail: sergey.dashkovskiy@uni-wuerzburg.de)*
** *Faculty of Computer Science and Mathematics, University of Passau (e-mail: {andrii.mironchenko,fabian.(lastname)}@uni-passau.de)*
*** *Institute for Mathematics, University of Würzburg, Germany*
**** *Fraunhofer Institute for Industrial Mathematics (ITWM), Kaiserslautern, Germany (e-mail: jochen.schmid@itwm.fraunhofer.de)*

Abstract: In this paper we consider countable couplings of finite-dimensional input-to-state stable systems. We consider the whole interconnection as an infinite-dimensional system on the ℓ_∞ state space. We develop stability conditions of the small-gain type to guarantee that the whole system remains ISS and highlight the differences between finite and infinite couplings by means of examples. We show that using our methodology it is possible to study uniform global asymptotic stability of nonlinear spatially invariant systems by solving a finite number of nonlinear algebraic inequalities.

Keywords: Stability analysis, nonlinear systems, interconnected systems, large-scale systems, infinite-dimensional systems, input-to-state stability

1. INTRODUCTION

Interconnected nonlinear systems appear in many modern applications, in particular related to control and stabilization problems. In such applications as vehicle platooning or formations of schools of fish or flocks of birds the number of agents is so large that it is reasonable to model such system as infinite couplings. This leads to consideration of a countable set of systems that can affect each other. We are interested in stability properties of such interconnections. It is already well known that the ISS framework Sontag [1989] is ultimately powerful in studying stability of nonlinear interconnected systems Karafyllis and Jiang [2011b]. This notion was first introduced for ordinary differential equations (ODEs) and then extended to infinite-dimensional ones, see for example Dashkovskiy and Mironchenko [2013], Jacob et al. [2018], Karafyllis and Krstic [2019]. However it is known that many ISS results developed for ODEs are not valid for infinite-dimensional systems Mironchenko and Wirth [2018b]. Recently different characterizations of ISS for this case were developed in Mironchenko and Wirth [2018a,b].

For finitely many interconnected ODE systems powerful stability conditions of small-gain type have been developed in the literature Jiang et al. [1994, 1996], Dashkovskiy et al. [2007, 2010], Rüffer [2010], Karafyllis and Jiang [2011a] by applying the ISS framework.

In view of importance of these results an extensive effort has been devoted to generalization of small-gain paradigm for analysis of infinite-dimensional systems. In particular, stability of finite couplings of infinite-dimensional systems has been studied in Dashkovskiy and Mironchenko [2013], Mironchenko and Ito [2015], Mironchenko [2019], Bao et al. [2018]. Challenges and obstacles appearing on this way have been reviewed in Mironchenko [2019].

The goal of this paper is to extend these small-gain results to the case of an interconnection of a countable set of subsystems. As we argue in this paper, in the case of infinite couplings a number of further challenges come to the foreground. In particular, usual operator and cyclic small-gain conditions are in most cases insufficient to guarantee the input-to-state stability of the whole network.

Stability of infinitely many interconnected systems was studied recently in several works, such as Heijmans et al. [2017], Besselink and Johansson [2017], Dashkovskiy and Pavlichkov [2017], Curtain et al. [2009], Bamieh and Voulgaris [2005]. Spatial invariance of the interconnection structure is assumed in most of these papers and either the gains or the system itself is assumed to be linear. None of these assumptions will be applied in this paper.

In this work we provide a new small-gain condition that guarantees stability of an interconnection of a countable set of finite-dimensional nonlinear systems (Theorem 5.1). This condition is stronger than the classical small-gain condition and it is motivated by several examples, discussed in Section 3. In Section 4 we derive several relations between various types of small-gain conditions. As an application of our new small-gain theorem we investigate stability properties of a spatially invariant nonlinear infinite-dimensional system at the end of the paper.

⋆ This research was supported by the DFG Grants No. DA 767/7-1, WI 1458/13-1 and MI 1886/2-1.

We omit the proofs of most results due to the page limitations. They can be found in the journal version of this paper, which is currently in preparation.

Notation. The set of positive integers is denoted by $\mathbb{N}$ and $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$. We set $\mathbb{R}_+ := [0, +\infty)$. For $x, y \in \mathbb{R}^n$ we define the relation "$\geq$" by: $x \geq y \Leftrightarrow x_i \geq y_i \; \forall i = 1, \dots, n$.

By ℓ_∞ we denote the Banach space of all infinite uniformly bounded sequences and ℓ_∞^+ denotes the positive cone in ℓ_∞ consisting of all vectors $s \in \ell_\infty$ with $s_i \geq 0$, $i \in \mathbb{N}$. ℓ_∞^+ endows ℓ_∞ with a natural order $\geq$, given by: for all $x, y \in \ell_\infty$ we say that $x \leq y$ if $x_i \leq y_i$ for all $i \in \mathbb{N}$. By "$\ngeq$" we understand the logical negation of "$\geq$". In particular, for $x, y \in \ell_\infty$ $x \ngeq y$ means that $\exists i \in \mathbb{N}$ so that $x_i < y_i$.

The standard unit vectors in ℓ_∞ are denoted by $e_i, i \in \mathbb{N}$; that is e_i is the sequence of zeros with exception of position i, where the entry is 1. The ith position of a sequence $a \in \ell_\infty$ we will write as $\langle e_i, a\rangle$ where this is convenient. The vector of all ones in ℓ_∞ is denoted $\mathbf{1}$, it is defined by $\langle e_i, \mathbf{1}\rangle = 1$ for all $i \in \mathbb{N}$. For $a \in \mathbb{R}^n$ with $n \in \mathbb{N}$ we denote $|a| = \max_{i=1,\dots,n} |a_i|$.

We use the following classes of comparison functions

$$\begin{aligned}
\mathcal{K} &:= \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous, } \gamma(0) = 0 \\
&\qquad\qquad \text{and strictly increasing}\} \\
\mathcal{K}_\infty &:= \{\gamma \in \mathcal{K} \mid \gamma \text{ is unbounded}\} \\
\mathcal{L} &:= \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous and strictly} \\
&\qquad\qquad \text{decreasing with } \lim_{t\to\infty} \gamma(t) = 0\} \\
\mathcal{KL} &:= \{\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \mid \beta \text{ is continuous,} \\
&\qquad\qquad \beta(\cdot, t) \in \mathcal{K},\ \beta(r, \cdot) \in \mathcal{L},\ \forall t \geq 0, r > 0\}
\end{aligned}$$

2. PRELIMINARIES

Let $x_i \in \mathbb{R}^{n_i}$ be the state of the system Σ_i, $i \in \mathbb{N}$, $n_i \in \mathbb{N}$. For each $i \in \mathbb{N}$ let I_i be a finite subset of $\mathbb{N}$, which ennumerates the neighbors of Σ_i, i.e., those systems Σ_j that affect Σ_i. By definition we require that $i \notin I_i$, $i \in \mathbb{N}$. Let $\bar{x}_i \in \mathbb{R}^{N_i}$ denote the vector composed of the vectors $x_j \in \mathbb{R}^{n_j}$, $j \in I_i$ ordered by the index j and $N_i := \sum_{j\in I_i} n_j$.

The dynamics of Σ_i are given by the finite-dimensional ODE

$$\Sigma_i : \quad \dot{x}_i = f_i(x_i, \bar{x}_i, u_i), \quad i \in \mathbb{N}, \tag{1}$$

where $u_i \in L_{\text{loc}}^\infty(\mathbb{R}_+; \mathbb{R}^{m_i})$. For each $i \in \mathbb{N}$ the function f_i is assumed to be such that Σ_i has a unique solution which depends on the initial condition and the input functions $\bar{x}_i$ and u_i. From (1) we see that the dynamics of Σ_i is affected by some other systems numbered by elements of I_i and by the external signal u_i.

The whole interconnection of systems (1) is denoted by Σ and by a suitable definition of f collecting all f_i and $u := (u_1, u_2, \dots)$ it can be written as

$$\Sigma : \quad \dot{x} = f(x, u). \tag{2}$$

As a state space we choose ℓ_∞ as the largest from the ℓ_p-scale, i.e. we set

$$\begin{aligned}
X &:= \ell_\infty(\mathbb{N}, (\mathbb{R}^{n_i})_{i\in\mathbb{N}}) \\
&:= \{(x_i)_{i\in\mathbb{N}} \mid x_i \in \mathbb{R}^{n_i}, (|x_i|) \in \ell_\infty(\mathbb{N}, \mathbb{R})\}
\end{aligned}$$

with the norm $\|x\|_\infty := \sup_{i\in\mathbb{N}} |x_i|$, and as the input space we take

$$\mathcal{U} := L_{\text{loc}}^\infty\big(\mathbb{R}_+, \ell_\infty(\mathbb{N}, (\mathbb{R}^{m_i})_{i\in\mathbb{N}})\big)$$

We say that $x(\cdot) := (x_1(\cdot), x_2(\cdot), \dots) : [0, \tau] \to X$, is a solution of Σ on the interval $[0, \tau]$ if x_i is absolutely continuous in time for each $i \in \mathbb{N}$ and x solves the following set of the integral equations corresponding to Σ_i:

$$x_i(t) = x_i(0) + \int_0^t f\big(x_i(s), \bar{x}_i(s), u_i(s)\big) ds. \tag{3}$$

We assume that for Σ unique solutions exist on a certain interval in $\mathbb{R}_+$ for all initial conditions $x(0) \in X$ and all $u \in \mathcal{U}$. This assumption can be guaranteed for instance, if all the f_i are continuous and locally Lipschitz continuous in x locally uniformly in u, where the local Lipschitz constants and the moduli of continuity of the f_i can be chosen uniformly bounded in i.

In this work we aim to extend stability results known for finitely many interconnected systems to the case of infinitely large interconnections.

Definition 2.1. System Σ in (2) is called *input-to-state stable (ISS)* from u to x if there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for any initial state $x(0) \in X$ and any input $u \in \mathcal{U}$ the corresponding solution satisfies

$$\|x(t)\|_\infty \leq \beta(\|x(0)\|_\infty, t) + \gamma(\sup_{k\in\mathbb{N}}(\sup_{t\geq 0}(|u_k(t)|))),\ t \geq 0. \tag{4}$$

We define also a weaker property:

Definition 2.2. System Σ in (2) is called *uniformly globally asymptotically stable at zero (0-UGAS)* if there exists $\beta \in \mathcal{KL}$ such that for any initial state $x(0) \in X$ and the input $u \equiv 0$ the corresponding solution satisfies

$$\|x(t)\|_\infty \leq \beta(\|x(0)\|_\infty, t),\ t \geq 0. \tag{5}$$

In particular, ISS implies 0-UGAS and furthermore, for any uniformly bounded input u the solution of an ISS system is bounded. *The aim of this paper is to establish stability conditions of the small-gain type that guarantee that the interconnection Σ of the ISS subsystems Σ_i is again an ISS system.*

A key tool to study ISS are ISS Lyapunov functions.

Definition 2.3. A continuous function $V : X \to \mathbb{R}_+$ is called a (coercive) *ISS Lyapunov function* for (2), if there exist $\psi_1, \psi_2 \in \mathcal{K}_\infty$, $\alpha \in \mathcal{P}$ and $\chi \in \mathcal{K}$ so that

$$\psi_1(\|x\|_\infty) \leq V(x) \leq \psi_2(\|x\|_\infty), \quad \forall x \in X. \tag{6}$$

and so that the Dini derivative of V along the trajectories of the system (2) satisfies the implication

$$\|x\|_\infty \geq \chi(\|u\|_{\mathcal{U}}) \quad \Rightarrow \quad \dot{V}_u(x) \leq -\alpha(\|x\|_\infty) \tag{7}$$

for all $x \in X$ and $u \in \mathcal{U}$, where

$$\dot{V}_u(x) = \overline{\lim_{t\to+0}} \frac{1}{t}\big(V(\phi(t, x, u)) - V(x)\big). \tag{8}$$

In [Dashkovskiy and Mironchenko, 2013, Theorem 1] it was shown that:

Proposition 2.1. If there is an ISS Lyapunov function for (2), then (2) is ISS.

In our main result we construct an ISS Lyapunov function for (2) from the knowledge of ISS-Lyapunov functions for subsystems (1) and of the interconnection structure.

We assume that for each $i \in \mathbb{N}$ system Σ_i is input-to-state stable with respect to the inputs x_j, $j \in I_i$ and u_i, and that there exists an *ISS-Lyapunov function* V_i, which is a continuous function satisfying

$$\alpha_{i1}(|x_i|) \leq V_i(x_i) \leq \alpha_{i2}(|x_i|), \quad x_i \in \mathbb{R}^{n_i} \tag{9}$$

so that for any $x_i, \bar{x}_i$ and u_i the following implication holds

$$V_i(x_i) \geq \max_{k \in I_i}\{\gamma_{ik}(V_k(x_k)), \gamma_i(|u_i|)\}$$
$$\Rightarrow \dot{V}_i(x_i) \leq -\alpha_i(V_i(x_i)) \quad (10)$$

where $\alpha_i, \alpha_{ij}, \gamma_i, \gamma_{ij} \in \mathcal{K}_\infty$ and the (right-hand upper) Dini derivative of V_i at x_i corresponding to the input $(\bar{x}_i, u_i)$ along the trajectories of Σ_i is defined by

$$\dot{V}_i(x_i) = \overline{\lim_{t \to +0}} \frac{1}{t}\Big(V_i\big(\tilde{\phi}_i(t, x_i, (\bar{x}_i, u_i))\big) - V_i(x_i)\Big), \quad (11)$$

where $\tilde{\phi}_i(t, x_i, (\bar{x}_i, u_i))$ is the trajectory of Σ_i for the initial condition x_i and the input $(\bar{x}_i, u_i)$.

The functions $\gamma_{ij} \in \mathcal{K}_\infty \cup \{0\}$ with $i \neq j$ in (10) are called *interconnection gains*. By convention we set $\gamma_{ii} = 0$ for all $i \in \mathbb{N}$. We collect all these functions to an infinite matrix $\Gamma = (\gamma_{ij})_{i,j=1}^\infty$ and define the nonlinear *gain operator*

$$\Gamma : \ell_\infty^+ \to \ell_\infty^+, \quad \langle e_i, \Gamma(v)\rangle = \sup_{j \in \mathbb{N}} \gamma_{ij}(v_j), \quad i \in \mathbb{N}. \quad (12)$$

This mapping is well defined under the following

Assumption 2.1. There exists a $\gamma \in \mathcal{K}_\infty$ such that $\gamma_{ij}(r) \leq \gamma(r)$ for all $i, j \in \mathbb{N}$ and $r \geq 0$.

To assure that the mapping Γ is continuous we additionally require the next

Assumption 2.2. For any $b \geq 0$ the set of all interconnection gains $\{\gamma_{ij},\ i, j \in \mathbb{N}\}$ is uniformly equicontinuous in $[0, b]$, that is for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $r_1, r_2 \in [0, b]$ with $|r_1 - r_2| < \delta$ it follows that $|\gamma_{ij}(r_1) - \gamma_{ij}(r_2)| < \varepsilon$ for all $i, j \in \mathbb{N}$.

Lemma 2.1. Let Γ be defined by (12) with interconnection gains $\{\gamma_{ij},\ i, j \in \mathbb{N}\}$ satisfying the Assumptions 2.1 and 2.2. Then Γ is a continuous mapping on ℓ_∞^+.

For finitely many interconnected systems $i \in \{1, \ldots, n\}$ the mapping Γ was defined in the same way and it was proved that if $\Gamma(s) \not\geq s$ for all nonzero $s \in \mathbb{R}_+^n$ then the finite interconnection of such systems is again ISS. In the next section we will see that this condition is not suitable in case of infinitely many interconnected systems and needs to be strengthened essentially.

3. MOTIVATING EXAMPLES

In this section we demonstrate by means of examples that many properties known for finitely many interconnections are not satisfied for couplings of an infinite number of systems. It is known that (finite) cascades of ISS systems are ISS. The situation is different for infinite cascades as the following examples show.

Example 3.1. For $x_i \in \mathbb{R}$, $i \in \mathbb{N}$ consider the cascade

$$\dot{x}_1 = -x_1 + u, \quad \dot{x}_{k+1} = -x_{k+1} + 2x_k, \quad k \in \mathbb{N},$$

where each one-dimensional subsystem is ISS, but the whole system is not ISS because for the constant input $u = 1$ and for the initial condition $x(0) = \mathbf{1}$ the derivatives $\dot{x}_2, \dot{x}_3, \ldots$ are strictly positive for all times and for each $k \in \mathbb{N}$ it holds that $x_k(t) \to 2^{k-1}$, $t \to \infty$ and in particular $\lim_{t\to\infty} \|x(t)\|_\infty = \infty$ contradicting ISS.

Taking $V_i(x_i) = |x_i|$ as a Lyapunov function for Σ_i we get for each $i \geq 2$ that $V_i(x_i) \geq \gamma_{i,i-1}(|x_{i-1}|)$ implies

$$\dot{V}_i(x_i) \leq -V_i(x_i) + 2V_{i-1}(x_{i-1}) \leq -\frac{\varepsilon}{1+\varepsilon}|x_i|, \quad x_i \in \mathbb{R}^{N_i},$$

that is the ISS gains in the above subsystems can be calculated as $\gamma_{i+1,i} = (2+\varepsilon)\,\mathtt{id}$ for any $\varepsilon > 0$ and $i \geq 2$. All other interconnection gains are $\gamma_{ij} = 0$. ⋄

Example 3.2. Consider the cascade of ISS systems

$$\dot{x}_1 = -x_1 + u, \quad \dot{x}_{k+1} = -x_{k+1} + x_k, \quad k \in \mathbb{N}.$$

Taking $u \equiv 0$ and the initial state $x(0) = \mathbf{1}$, the solution is given by $x_i(t) = e^{-t}\Big(\sum_{k=0}^{i} \frac{t^k}{k!}\Big)$, $i \in \mathbb{N}$ for which $\lim_{t\to\infty} \|x(t)\|_\infty = 1 \neq 0$ contradicting the ISS property. Using the same ISS-Lyapunov functions $V_i = |x_i|$ it can be verified that the non-zero ISS gains can be calculated as $\gamma_{i+1,i} := (1+\varepsilon)\,\mathtt{id}$ for arbitrarily small $\varepsilon > 0$ and all $i \in \mathbb{N}$. Moreover one can construct a similar example where all subsystems gains are strictly less then identity, but the cascade of such systems is not ISS: ⋄

Example 3.3. Consider the cascade

$$\dot{x}_k = -\frac{k+1}{k}x_k + x_{k+1} + u_k, \quad k \in \mathbb{N}. \quad (13)$$

Again taking the derivative of $V_i(x_i) = |x_i|$ along solutions of Σ_i we get the interconnection gains $\gamma_{ij} = 0 \Leftrightarrow j \neq i+1$ and $\gamma_{k,k+1} = \frac{k+\varepsilon}{k+1} < \mathtt{id}$, $k \in \mathbb{N}$ with any $\varepsilon \in (0,1)$.

The bounded linear operator $A : \ell_\infty \to \ell_\infty$ describing the uncontrolled system $\dot{x} = Ax$ is defined as a multiplication on an infinite matrix $A = (a_{ij})_{i,j\in\mathbb{N}}$ with $a_{ii} = -\frac{i+1}{i}$, $a_{i,i+1} = 1$, $i \in \mathbb{N}$ and all other entries of A are zero. It is easy to see that A does not have a bounded inverse A^{-1}, i.e. $\lambda = 0 \in \sigma(A)$. Hence (13) is not 0-UGAS and thus not ISS. For the gain matrix $\Gamma = (\gamma_{ij})$ with gains calculated above the condition $\Gamma(s) \not\geq s$ is satisfied for all $s \in l_\infty^+ \setminus \{0\}$. On the other hand, we may interpret Γ as a bounded linear operator on ℓ_∞ and then it is not difficult to check that $1 \in \sigma(\Gamma)$. In particular, the discrete time system $s(k+1) = \Gamma s_k$ is not exponentially stable, in fact it even has trajectories that do not converge to 0.

In contrast, in the finite-dimensional case it holds that the discrete dynamical system defined by

$$s(k+1) := \Gamma(s(k)), \quad s_0 \in \mathbb{R}_n^+, \quad k \in \mathbb{N}$$

is GAS if and only if $\Gamma(s) \not\geq s$ for all $s \in \mathbb{R}_+^n \setminus \{0\}$, [Rüffer, 2010, Theorem 6.4]. We stress that this equivalence relies on the fact that the ISS condition is formulated in the maximum case.

Finally, we point out that the above examples do not essentially rely on the fact that there is an infinite cascade present. Similar examples can be built for systems in which cascades of arbitrary but finite length appear. Thus it is not even sufficient to add the consideration of infinite chains of gains to the usual finite-dimensional analysis of finite cycles.

Example 3.4. Consider a gain matrix Γ_0, where all entries are zero except for

$$\gamma_{i,i+1} = 1, \quad i \in \mathbb{N} \setminus \{2^k \,|\, k \in \mathbb{N}\}. \quad (14)$$

Again here it is easy to see that $\Gamma_0^k(\mathbf{1})$ does not converge to zero, even though $\Gamma_0(s) \not\geq s$. ⋄

We can extend the previous example to have the example of a system in which every cycle is a contraction, but the convergence of $\Gamma^k(\mathbf{1})$ to 0 still fails.

Example 3.5. Fix some $\delta > 0$. Again, all entries of Γ_δ that are not explicitly specified are assumed to be zero. We set

$$\gamma_{i,i+1} = 1, \quad i \in \mathbb{N}\backslash\{2^k \,|\, k \in \mathbb{N}\},$$
$$\gamma_{2,1} = \delta,$$
$$\gamma_{2^{k+1},2^k+1} = \delta, \quad k \in \mathbb{N}\,.$$

As $\Gamma_\delta(s) \geq \Gamma_0(s)$ it follows that $\Gamma_\delta^k(\mathbf{1}) \not\to 0$. However, in this case there are only finite cycles and each cycle product is equal to δ, which may be chosen arbitrarily small. Note that the length of these cycles is unbounded and this is enough to obstruct the uniform decay estimates required by the definition of ISS. ⋄

The above examples show that basic equivalences fail in the infinite dimensional case. We will add two requirements to the small-gain conditions. A linear variant of the strong small-gain condition introduced in Dashkovskiy et al. [2010] is to require the existence of an $\varepsilon > 0$ such that Γ satisfies

$$\Gamma(s) \not\geq (1-\varepsilon)s, \quad s \in \ell_\infty^+ \setminus \{0\}.$$

It is easy to see that this rules out the counterexamples of the form Γ_δ for arbitrary $\delta > 0$. However, the condition does not rule out the case of Γ_0.

We will therefore require the robust strong small gain condition, defined as follows.

Definition 3.1. [Robust strong small-gain condition] A gain-operator $\Gamma : \ell_\infty^+ \to \ell_\infty^+$ is said to satisfy the *robust strong small-gain condition*, if there exist constants $\eta, \varepsilon > 0$ such that for all $i, j \in \mathbb{N}$ the operator

$$\Gamma_{i,j}(s) := \Gamma(s) + \eta\langle e_j, s\rangle e_i, \quad s \in \ell_\infty^+$$

satisfies

$$\Gamma_{i,j}(s) \not\geq (1-\varepsilon)s, \quad s \in \ell_\infty^+ \setminus \{0\}. \tag{15}$$

In case of feedback interconnections the situation becomes even more tricky. It is well known that feedback interconnections of ISS systems are not necessarily stable. Stability conditions of the small-gain type were developed for interconnections of finitely many systems. One such condition is the small gain condition mentioned above. For the maximum formulation of ISS, this condition can be equivalently stated in form of cycle compositions of gains (see the definition below), namely requiring that all cycle compositions of gain functions are contractions. In case of infinite interconnections one can expect that infinite compositions should be considered. However in contrast to a finite composition of $\mathcal{K}_\infty$-functions an infinite composition of $\mathcal{K}_\infty$-gains does not necessarily define a function from $\mathbb{R}_+$ to itself.

The above considerations show that stability of infinite interconnections needs to be studied carefully and some new stability conditions should be developed for this case.

4. PROPERTIES OF Γ

In the rest of the paper we always assume that Assumptions 2.1 and 2.2 are satisfied, which in particular implies by Lemma 2.1 that the mapping Γ is continuous on ℓ_∞^+.

Definition 4.1. A finite composition

$$\gamma_{i_1i_2} \circ \gamma_{i_2i_3} \circ \cdots \circ \gamma_{i_{k-1}i_k}, \quad k \in \mathbb{N} \tag{16}$$

is called *(finite) cycle* if all $i_1, \ldots, i_{k-1}$ indices are pairwise different and $i_1 = i_k$. In case all indices are pairwise different in (16) then it is called a *finite chain*.

An *infinite chain* $\chi = (\gamma_{i_k,i_{k+1}})_{k\in\mathbb{N}}$ is a composition of an infinite number of functions

$$\gamma_{i_1i_2} \circ \gamma_{i_2i_3} \circ \cdots \circ \gamma_{i_{k-1}i_k} \circ \ldots \tag{17}$$

where all indices $i_k,\ k \in \mathbb{N}$ are pairwise different.

We note that at this stage the definition (17) is purely formal. While the composition of any finite number of $\mathcal{K}_\infty$ functions is again in $\mathcal{K}_\infty$, an infinite composition of $\mathcal{K}_\infty$ functions does not necessarily define a function.

Definition 4.2. A finite cycle (16) with $i_1 = i_k$ is called a *contraction* if for any $r > 0$ it holds that

$$\gamma_{i_1i_2} \circ \gamma_{i_2i_3} \circ \cdots \circ \gamma_{i_{k-1}i_k}(r) < r.$$

An infinite chain $\chi = (\gamma_{i_k,i_{k+1}})_{k\in\mathbb{N}}$ is called *well-posed*, if it defines a function f_χ by

$$f_\chi(r) := \limsup_{k\to\infty} \gamma_{i_1i_2} \circ \gamma_{i_2i_3} \circ \cdots \circ \gamma_{i_{k-1}i_k}(r), r \geq 0.$$

A well-posed infinite chain χ is called a *contraction*, if $f_\chi(r) < r$ for all $r > 0$.

For infinite chains it will be convenient to consider the (left) shift θ defined by $\theta\chi = (\gamma_{i_k,i_{k+1}})_{k=2}^\infty = (\gamma_{i_2,i_3}, \gamma_{i_3,i_4}, \ldots)$. We note that χ is well-posed if and only if all shifts $\theta^k\chi, k \in \mathbb{N}$ are well posed, because $f_{\theta^k\chi} = \gamma_{i_{k-1}i_k}^{-1} \circ \cdots \circ \gamma_{i_1i_2}^{-1} \circ f_\chi$. We note that in the well-posed case f_χ is nondecreasing but not necessarily in $\mathcal{K}_\infty$, indeed in many examples $f_\chi \equiv 0$.

Lemma 4.1. Let Γ satisfy $\Gamma(s) \not\geq s,\ s \in \ell_\infty^+, s \neq 0$, then all finite cycles are contractions.

Remark 4.1. As was shown by Examples in Section 3, the implication in the claim of Lemma 4.1 cannot be reversed: even if all finite cycles are identical zero functions, the small-gain condition does not necessarily hold.

Remark 4.2. The condition $\Gamma(s) \not\geq s,\ s \in \ell_\infty^+\backslash\{0\}$ does not guarantee that all infinite chains are well-posed. Examples are rather simple to build by considering a weighted shift with weights 1/2 and 2, that alternate on increasingly long intervals. We omit the details for reasons of space.

In any case such examples show that the small-gain condition $\Gamma(s) \not\geq s,\ s \in \ell_\infty^+\backslash\{0\}$ is inherently nonrobust in the case of infinite couplings. The reason is that the condition may hold but can be destroyed by adding arbitrarily small additional couplings. This is a further justification of considering the robust strong small-gain condition (15).

The following lemma shows that for any given $s \in \ell_\infty^+$ the components of $\Gamma^m(s)$ can be approximated by finite compositions of gains of the length m.

Lemma 4.2. For every $s \in \ell_\infty^+$, $i \in \mathbb{N}$, $m \in \mathbb{N}$, $\varepsilon > 0$ there is an index $j \in \mathbb{N}$ and a path $i_0 = j, i_1, \ldots, i_m = i$ so that

$$\gamma_{i,i_{m-1}} \circ \cdots \circ \gamma_{i_1,j}(s_j) \leq \langle e_i, \Gamma^m(s)\rangle \leq \gamma_{i,i_{m-1}} \circ \cdots \circ \gamma_{i_1,j}(s_j) + \varepsilon.$$

In the next lemma we introduce an operator Q, which is an infinite-dimensional version of the operator, introduced in [Karafyllis and Jiang, 2011a, Prop. 2.7, Remark 2.8].

Lemma 4.3. If there exist $\eta, \varepsilon \in (0,1)$ such that Γ satisfies the robust strong small-gain condition for η, ε, then

(i) The next operator is well defined

$$Q : \ell_\infty^+ \to \ell_\infty^+, \quad s \mapsto Q(s) := \sup_{k\in\mathbb{N}_0} \{\Gamma^k(s)\}, \tag{18}$$

where the supremum is taken component-wise.

(ii) It holds that

$$\Gamma(Q(s)) \leq Q(s), \quad s \in \ell_\infty^+. \tag{19}$$

Lemma 4.4. Assume Γ has no zero rows and

$$\lim_{k\to\infty} \Gamma^k(s) = 0, \quad s \in \ell_\infty^+.$$

If there exist $\eta, \varepsilon \in (0,1)$ such that Γ satisfies the robust strong small-gain condition for η, ε, then for any $i \in \mathbb{N}$ the function

$$\sigma_i(r) := \langle e_i, Q(r\mathbf{1})\rangle, \quad r \geq 0 \tag{20}$$

is continuous, non-decreasing, unbounded and $\sigma_i(0) = 0$.

Lemma 4.5. Assume Γ has no zero rows and

$$\lim_{k\to\infty} \Gamma^k(s) = 0, \quad s \in \ell_\infty^+.$$

If there exist $\eta, \varepsilon \in (0,1)$ such that Γ satisfies the robust strong small-gain condition for η, ε, then there exists a function $\sigma : \mathbb{R}_+ \to \ell_\infty^+$ with $\sigma_i(\cdot) := \langle e_i, \sigma(\cdot)\rangle \in \mathcal{K}_\infty$, $i \in \mathbb{N}$ and $\varepsilon_2 > 0$ such that

$$\Gamma(\sigma(r)) \leq (1-\varepsilon_2)\sigma(r), \quad r \geq 0. \tag{21}$$

Remark 4.3. The property, given by (21) is the property which we would like to use for the proof of the small-gain theorem. Lemma 4.5 gives us a sufficient condition for the validity of (21) for σ, constructed as in (20). At the same time, in some cases it is possible to construct such a path using alternative methods, see Section 6.

5. SMALL-GAIN CONDITION FOR INFINITE INTERCONNECTIONS

The main result is the following.

Theorem 5.1. For each $i \in \mathbb{N}$ let the system Σ_i, defined via (1) be ISS from $(\bar{x}_i, u_i)$ to x_i and V_i be an ISS-Lyapunov function for Σ_i satisfying (9) and (10). Assume that

(i) there exists an $\alpha \in \mathcal{K}_\infty$ such that for all $i \in \mathbb{N}$ we have $\alpha_i(r) \geq \alpha(r)$, $r \geq 0$,
(ii) there exist $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ such that for all $i \in \mathbb{N}$ we have $\underline{\alpha}(r) \leq \alpha_{i1}(r) \leq \alpha_{i2}(r) \leq \overline{\alpha}(r)$ for all $r \geq 0$,
(iii) there exists a $\tilde{\gamma} \in \mathcal{K}_\infty$ such that $\gamma_i(r) \leq \tilde{\gamma}(r)$ for all $i \in \mathbb{N}$ and $r \geq 0$.
(iv) there are linear functions given by constants $\sigma_i > 0$, $i \in \mathbb{N}$, satisfying (21) and such that for some $\overline{\sigma} > \underline{\sigma} > 0$ it holds $\overline{\sigma} \geq \sigma_i \geq \underline{\sigma}$ for all $i \in \mathbb{N}$.
(v) f defining the interconnected system is bounded on bounded balls, i.e. for each $R > 0$ there is a constant M_R such that

$$|f(x,u)|_\infty \leq M_R \quad \forall x, u: \ \|x\|_\infty < R, \ \|u\|_\infty < R.$$

Consider the function $V : X \to \mathbb{R}_+$, defined by:

$$V(x) = \sup_{i\in\mathbb{N}} \{\sigma_i^{-1} V_i(x_i)\}. \tag{22}$$

If V is a continuous function, then V is also a coercive ISS Lyapunov function for the system Σ, and thus the interconnected system Σ is ISS from u to x.

6. APPLICATION: STABILITY OF SPATIALLY INVARIANT SYSTEMS

In this example we show how our methodology can be used to analyze uniform global asymptotic stability (UGAS) of nonlinear interconnected spatially invariant systems. Consider the following system:

$$\frac{d}{dt}\begin{pmatrix} x_i \\ y_i \end{pmatrix} = \begin{pmatrix} -x_i + y_i^3 \\ -x_i^{1/3} - y_i \end{pmatrix} + \lambda \begin{pmatrix} x_{i+1} \\ y_{i+1} \end{pmatrix}, \quad i \in \mathbb{Z}. \tag{23}$$

Defining $z_i := (x_i, y_i)^T$, and denoting the ODE system governing the dynamics of z_i by Σ_i we can interpret (23) as a spatially invariant system consisting of identical components with matched nonlinearities Σ_i with a linear coupling between them.

We would like to analyze for which λ this system is UGAS. With our methodology, we would like first to analyze ISS properties of the subsystems Σ_i, $i \in \mathbb{Z}$.

Pick the following Lyapunov function for each Σ_i:

$$V_i(z_i) := x_i^{\frac{4}{3}} + b y_i^4,$$

where the coefficient $b > 0$ is to be defined later.

The Dini derivative of V_i with respect to Σ_i

$$\begin{aligned}\dot{V}(z_i) &= \frac{4}{3}x_i^{\frac{1}{3}}(-x_i + y_i^3 + \lambda x_{i+1}) + 4by_i^3(-x_i^{\frac{1}{3}} - y_i + \lambda y_{i+1}) \\ &= -\frac{4}{3}x_i^{\frac{4}{3}} + \Big(\frac{4}{3} - 4b\Big)x_i^{\frac{1}{3}}y_i^3 + \frac{4}{3}\lambda x_i^{\frac{1}{3}}x_{i+1} \\ &\qquad - 4by_i^4 + \lambda 4by_i^3 y_{i+1}.\end{aligned}$$

can be estimated applying Young's inequality $ab \leq \frac{\omega}{p}a^p + \frac{1}{\omega^{\frac{1}{p-1}}}\frac{p-1}{p}b^{\frac{p}{p-1}}$ for all $a, b \geq 0$ and all $\omega, p > 0$, as follows

$$\dot{V}_i(z_i) \leq -\frac{4}{3}x_i^{\frac{4}{3}} + \Big(\frac{4}{3} - 4b\Big)\Big(\frac{\omega_1}{4}x_i^{\frac{4}{3}} + \frac{1}{\omega_1^{\frac{1}{3}}}\frac{3}{4}y_i^4\Big)$$

$$+\frac{4}{3}\lambda\Big(\frac{\omega_2}{4}x_i^{\frac{4}{3}} + \frac{1}{\omega_2^{\frac{1}{3}}}\frac{3}{4}x_{i+1}^{\frac{4}{3}}\Big) - 4by_i^4 + 4\lambda b\Big(\frac{\omega_3}{4}y_i^4 + \frac{1}{\omega_3^{\frac{1}{3}}}\frac{3}{4}y_{i+1}^4\Big)$$

$$= \Big(-\frac{4}{3} + \omega_1\Big(\frac{1}{3} - b\Big) + \frac{\lambda\omega_2}{3}\Big)x_i^{\frac{4}{3}}$$

$$+\Big(\frac{\omega_1^{\frac{1}{3}}}{b}(1-3b) - 4 + \lambda\omega_3\Big)by_i^4 + \frac{3\lambda}{\omega_3^{\frac{1}{3}}}by_{i+1}^4 + \frac{\lambda}{\omega_2^{\frac{1}{3}}}x_{i+1}^{\frac{4}{3}}.$$

We have the freedom to choose positive coefficients b, ω_1, ω_2 and ω_3 to obtain the optimal estimate for $\dot{V}_i(z_i)$.

First of all, for small-gain theorems we need a dissipation inequality containing Lyapunov functions V_i and V_{i+1}. To avoid the conservative transitions, it is natural to require the equality between the coefficients before the terms by_i^4 and $x_i^{\frac{4}{3}}$ as well as before the terms by_{i+1}^4 and $x_{i+1}^{\frac{4}{3}}$. This leads to the following constraints on b, ω_1, ω_2, ω_3:

$$R_1 := -\frac{4}{3} + \omega_1\Big(\frac{1}{3} - b\Big) + \frac{\lambda\omega_2}{3} = \frac{1}{b}\omega_1^{\frac{1}{3}}\Big(1 - 3b\Big) - 4 + \lambda\omega_3 \tag{24}$$

$$R_2 := \frac{3\lambda}{\omega_3^{\frac{1}{3}}} = \frac{\lambda}{\omega_2^{\frac{1}{3}}}. \tag{25}$$

With (24) and (25) we obtain the dissipation inequality:

$$\dot{V}_i(z_i) \leq R_1 V_i(z_i) + R_2 V_{i+1}(z_{i+1}). \tag{26}$$

It is natural to choose linear Lyapunov gains. As all Σ_i have identical internal type and interconnection style, it is enough to choose the same gains $\gamma_{i,i+1} = \gamma$, $i \in \mathbb{Z}$. Hence

$$V_i(z_i) \geq \gamma V_{i+1}(z_{i+1}) \quad \Rightarrow \quad \dot{V}_i(z_i) \leq \Big(R_1 + \frac{R_2}{\gamma}\Big)V_i(z_i).$$

V_i is an ISS Lyapunov function, provided

$$R_1 + \frac{R_2}{\gamma} < 0. \tag{27}$$

In this case, the operator Γ is γL, where L is the left-shift operator, which has and infinite-matrix representation given by ones on the first diagonal above the main diagonal, with all other entries being zero. Clearly, this operator satisfies the robust strong small-gain condition, and furthermore,

$$\Gamma(\mathbf{1}r) = \gamma \mathbf{1}r,$$

and thus choosing $\gamma < 1$ we see that (21) holds with $\varepsilon := 1 - \gamma$ and $\sigma(r) := \mathbf{1}r$. Now imposing the condition

$$R_1 < -R_2, \tag{28}$$

we can always find $\gamma < 1$ so that the condition (27) holds and at the same time all the assumptions of Theorem 5.1 are fulfilled. Thus, the application of Theorem 5.1 guarantees UGAS of the whole interconnection if (28) holds.

Overall, according to our small-gain based method the system (23) is UGAS for those λ, for which there exist $\omega_1, \omega_2, \omega_3, b > 0$ so that (24), (25) and (28) hold.

Hence the question about stability of the spatially invariant system with infinitely many nonlinear components and linear couplings can be resolved by the analysis of a finite number of nonlinear algebraic equations and inequalities.

Our method can be generalized to spatially invariant systems consisting of an arbitrary finite number of types of nonlinear components. In this case it is possible again to obtain a system of a finite number of nonlinear algebraic equations and inequalities which have to be solved to find conditions for stability of the interconnection.

Remark 6.1. Note that if the system (23) were a finite interconnection, then as it is a cascade interconnection, the stability of the whole system would be equivalent to all subsystems being ISS. As for finite interconnections one could choose the gains arbitrarily large, a 0-UGAS property of Σ_1 (and thus of all Σ_i, $i = 1, \ldots, n$) would be sufficient for UGAS of the whole interconnection. And the 0-UGAS property of Σ_1 can be shown using e.g. the Lyapunov function $V_1(z_1) := x_1^{\frac{4}{3}} + \frac{1}{3}y_1^4$.

7. CONCLUSIONS

We have studied interconnections of infinitely many finite-dimensional systems and derived an ISS small-gain result. A novel feature is the robustness property required in the small-gain condition, which is automatic in the case of finitely many couplings. It will be of interest to study interconnections with a more regular structure which can be expected to require less stringent conditions.

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