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Constructions of ISS-Lyapunov functions for interconnected impulsive systems

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Abstract— **In this paper we provide two small-gain theorems for impulsive systems. The first of them provides a construction of an ISS-Lyapunov function for interconnections of impulsive systems if ISS-Lyapunov functions for subsystems are given and a small-gain condition holds. If, in addition, these given ISS-Lyapunov functions are exponential then the second theorem provides a construction of an exponential ISS-Lyapunov function for the interconnection if the gains are power functions.**

Keywords: impulsive systems, input-to-state stability, dwell-time conditions.

I. Introduction

Impulsive systems are systems which behave like continuous systems for all times except for some given countable set of impulse times at which the state of a system jumps. The first monographs devoted entirely to impulsive systems are [19], [14]. Recent developments in this field one can find, in particular, in [9], [23].

Our aim is to study stability of impulsive systems with respect to external inputs. One of the main concepts in this theory is the notion of input-to-state stability (ISS), introduced in [21]. For a survey see e.g. [20], [2].

Input-to-state stability of impulsive systems has been investigated in the papers [10], [5] (finite-dimensional systems) and [1], [15], [22] (time-delay systems). The stability of interconnected impulsive systems has been studied in [4].

If both discrete and continuous dynamics taken separately from each other are ISS, then the impulsive system is ISS w.r.t. all admissible impulse time sequences, see [10] and [11], [12], where a small-gain theorem theorem for general classes of control systems has been developed. However, more interesting case is when either continuous or discrete dynamics destabilizes the system. In this case in order to achieve ISS of the system one has to impose restrictions on the density of impulse times, which are called dwell-time conditions.

In [5] the Lyapunov-type sufficient conditions for stability of the impulsive system have been developed. For the case, when the impulsive system possesses an ISS-Lyapunov function (in general nonexponential), it was proved that ISS of the system with respect to impulse time sequence is guaranteed if it satisfies certain nonlinear fixed dwell-time condition.

Also in [5] it was proved that the impulsive system, which has an exponential ISS Lyapunov function is uniform ISS for impulse time sequences, satisfying the generalized average dwell-time condition (gADT) condition. This theorem generalizes [10, Theorem 1], where this result has been proved for sequences, which satisfy more restrictive average dwell-time condition.

In this paper we are going to prove a small-gain theorem, which provides a construction of an ISS-Lyapunov function for an interconnection in the case, when ISS-Lyapunov functions for the subsystems are known, and the small-gain condition holds. ISS of the whole network can be then checked, using the sufficient condition, proved in [5]. Also we prove, that if all subsystems possess exponential ISS Lyapunov functions, and the gains are power functions, then the exponential ISS Lyapunov function for the whole system can be constructed. This generalizes [4, Theorem 4.2], where this result for linear gains has been proved.

The structure of the paper is as follows. In Section II we provide notation and main definitions. In Section III we recall the Lyapunov type results for single impulsive systems. In Section IV we investigate the ISS of interconnected systems via small-gain theorems.

II. Preliminaries

Let $T=\{t_1,\, t_2,\, t_3,\ldots\}$ be a strictly increasing sequence of impulse times without finite accumulation points. Consider a system of the form

$$\begin{cases} \dot{x}(t)=f(x(t),u(t)), & t\in[t_0,\infty)\backslash T,\\ x(t)=g(x^-(t),u^-(t)), & t\in T, \end{cases} \tag{1}$$

where $f,g:\mathbb{R}^n\times\mathbb{R}^m\to\mathbb{R}^n$.

The first equation of (1) describes the continuous dynamics of the system and the second describes the jumps of a state at the impulse times.

We assume that inputs belong to the class $L_\infty([t_0,\infty),\mathbb{R}^m)$ of essentially bounded Lebesgue measurable functions, and moreover, that left limits of u exist at all times (we denote $u^-(t)=\lim\limits_{s\to t-0}u(s)$ for all $t\geq t_0$) and that inputs are right-continuous. We endow this space with the supremum norm, which we denote by $\|\cdot\|_\infty$.

We suppose that f is locally Lipschitz w.r.t. the first argument uniformly w.r.t. the second in order to guarantee existence and uniqueness of solutions of the problem (1).

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We assume throughout the paper that $x \equiv 0$ is an equilibrium of the unforced system (1), that is $f(0,0) = g(0,0) = 0$.

From the assumptions on the inputs u it follows that $x(\cdot)$ is absolutely continuous between the impulses and $x^-(t) = \lim_{s \to t-0} x(s)$ exists for all t from the domain of definition of $x(\cdot)$.

Equations (1) together with the sequence of impulse times T define an impulsive system.

The system (1) is not time-invariant, i.e. the equality $\phi(t_2,t_1,x,u) = \phi(t_2+s,t_1+s,x,u)$ does not hold for all $s \geq -t_1$, where $\phi(t_2,t_1,x,u)$ denotes the state of the system (1) at time t_2 if its state at time t_1 was x and input u was applied. However, it holds

$$\phi(t_2,t_1,x,u) = \phi_s(t_2+s,t_1+s,x,u),$$

where ϕ_s has the same meaning as ϕ, but for a system (1) with shifted impulse time sequence $T_s := \{t_1+s,\ t_2+s,\ t_3+s, \dots\}$.

This means that the trajectory of the system (1) with initial time t_0 and impulse time sequence T is equal to the trajectory of (1) with zero initial time and impulse time sequence T_{-t_0}. Therefore we assume that t_0 is some fixed moment of time and will investigate the stability properties of the origin of the system (1) w.r.t. this initial time.

We will use the following classes of functions

$$\begin{aligned}
\mathcal{P} &:= \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous}, \\
&\qquad \gamma(0) = 0 \text{ and } \gamma(r) > 0 \text{ for } r > 0\} \\
\mathcal{K} &:= \{\gamma \in \mathcal{P} \mid \gamma \text{ is strictly increasing}\} \\
\mathcal{K}_\infty &:= \{\gamma \in \mathcal{K} \mid \gamma \text{ is unbounded}\} \\
\mathcal{L} &:= \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous and strictly} \\
&\qquad \text{decreasing with } \lim_{t \to \infty} \gamma(t) = 0\} \\
\mathcal{KL} &:= \{\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \mid \beta \text{ is continuous}, \\
&\qquad \beta(\cdot,t) \in \mathcal{K},\ \beta(r,\cdot) \in \mathcal{L},\ \forall t,r \geq 0\}
\end{aligned}$$

Denote the Euclidean norm in spaces $\mathbb{R}^k$ by $|\cdot|$ and $\mathbb{N} := \{1,2,3,\dots\}$.

We are interested in a study of stability of the system (1) w.r.t. external inputs. To this end we use the following notion:

Definition 1: For a given sequence T of impulse times we call system (1) *input-to-state stable (ISS)* if there exist $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}_\infty$, such that for all initial conditions x_0, for all inputs u, $\forall t \geq t_0$ it holds

$$|x(t)| \leq \max\{\beta(|x_0|, t-t_0), \gamma(\|u\|_\infty)\}. \tag{2}$$

The impulsive system (1) is *uniformly ISS* over a given set $\mathcal{S}$ of admissible sequences of impulse times if it is ISS for every sequence in $\mathcal{S}$, with β and γ independent of the choice of the sequence from the class $\mathcal{S}$.

In the next section we are going to find the sufficient conditions for an impulsive system (1) to be ISS.

III. ISS OF A SINGLE IMPULSIVE SYSTEM

For analysis of ISS of impulsive systems we exploit ISS-Lyapunov functions.

Definition 2: A Lipschitz-continuous function $V : \mathbb{R}^n \to \mathbb{R}_+$ is called an *ISS-Lyapunov function* for (1) if $\exists\ \psi_1, \psi_2 \in \mathcal{K}_\infty$, such that

$$\psi_1(|x|) \leq V(x) \leq \psi_2(|x|), \quad \forall x \in \mathbb{R}^n \tag{3}$$

holds and $\exists \chi \in \mathcal{K}_\infty$, $\alpha \in \mathcal{P}$ and continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}$ with $\varphi(0) = 0$ such that for almost all $x \in \mathbb{R}^n$, $\forall \xi \in \mathbb{R}^m$ it holds

$$V(x) \geq \chi(|\xi|) \Rightarrow \begin{cases} \dot{V}(x) = \nabla V \cdot f(x,\xi) \leq -\varphi(V(x)), \\ V(g(x,\xi)) \leq \alpha(V(x)). \end{cases} \tag{4}$$

If in addition $\varphi(s) = cs$ and $\alpha(s) = e^{-d}s$ for some $c,d \in \mathbb{R}$, then V is called *exponential ISS-Lyapunov function with rate coefficients* c,d.

Remark 1: Note that we do not assume that $\varphi \in \mathcal{P}$ and $\alpha < id$. If both these conditions hold, then, according to [10, Theorem 2] the system is strongly uniformly ISS. If neither of them holds, then an existence of an ISS-Lyapunov function does not imply ISS of the system w.r.t. any impulse time sequence. In this paper we are interested mostly in the case, when only one of these conditions hold, that is, either continuous or discrete part of the system is ISS. We are going to find conditions, which guarantee ISS of the impulsive system w.r.t. certain classes of impulse time sequences.

Remark 2: Note that an ISS-Lyapunov function for an impulsive system (1) does not depend on the sequence of impulse times. Consequently in the case when either continuous or discrete dynamics destabilizes the system, the existence of ISS-Lyapunov function is not enough to prove stability of the system (1). One needs to restrict the sets of admissible impulse time sequences.

Our definition of ISS-Lyapunov function is given in an implication form. The next proposition shows another way to introduce an ISS Lyapunov function, which is frequently used in the literature on hybrid systems, see e.g. [17]. We will use it for the formulation of the small-gain theorem in Section IV. This proposition is a counterpart of [13, Proposition 2.2.19] where analogous result for hybrid systems has been proved.

Recall that a function $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is called locally bounded, if for all $\rho > 0$ there exists $K > 0$, so that $\sup_{x \in \mathbb{R}^n : |x| \leq \rho,\ u \in \mathbb{R}^m : |u| \leq \rho} |g(x,u)| \leq K$.

Proposition 1: Let $V : \mathbb{R}^n \to \mathbb{R}_+$ be Lipschitz continuous. Assume that there exist $\psi_1, \psi_2 \in \mathcal{K}_\infty$, such that (3) holds and there exist $\gamma \in \mathcal{K}_\infty$, $\alpha \in \mathcal{P}$ and continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}$, $\varphi(0) = 0$ such that for almost all $x \in \mathbb{R}^n$ and all $\xi \in \mathbb{R}^m$ it holds

$$V(x) \geq \gamma(|\xi|) \quad \Rightarrow \quad \nabla V \cdot f(x,\xi) \leq -\varphi(V(x)) \tag{5}$$

and $\forall x \in \mathbb{R}^n, \xi \in \mathbb{R}^m$ it holds

$$V(g(x,\xi)) \leq \max\{\alpha(V(x)), \gamma(|\xi|)\}. \tag{6}$$

Then V is an ISS-Lyapunov function. If g is locally bounded, then also the converse implication holds.

Proof: "$\Rightarrow$" Pick any $\rho \in \mathcal{K}_\infty$ such that $\alpha(r) < \rho(r)$ for all $r > 0$.

Then for all $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^m$ from (6) we have

$$V(g(x,\xi)) \le \max\{\rho(V(x)), \gamma(|\xi|)\}.$$

Define $\chi := \max\{\gamma, \rho^{-1} \circ \gamma\} \in \mathcal{K}_\infty$. For all $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^m$ such that $V(x) \ge \chi(|\xi|)$ it follows $\rho(V(x)) \ge \gamma(|\xi|)$ and hence

$$V(g(x,\xi)) \le \rho(V(x)).$$

Since $\chi(r) \ge \gamma(r)$ for all $r > 0$, it is clear, that (4) holds. Thus, V is an ISS-Lyapunov function.

"$\Leftarrow$" Let g be locally bounded and let V be an ISS-Lyapunov function for the system (1). Then $\exists \chi \in \mathcal{K}$ and $\alpha \in \mathcal{P}$ such that for all $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^m$ from $V(x) > \chi(|\xi|)$ it follows $V(g(x,\xi)) \le \alpha(V(x))$.

Let $V(x) \le \chi(|\xi|)$. Then $|x| \le \psi_1^{-1} \circ \chi(|\xi|)$. Define $S(r) := \{x \in \mathbb{R}^n : |x| \le \psi_1^{-1} \circ \chi(r)\}$ and $\omega(r) := \sup_{|\xi| \le r,\ x \in S(r)} \psi_2(|g(x,\xi)|)$. This supremum exists since g is locally bounded. Clearly, ω is nondecreasing and $\omega(0) = \psi_2(|g(0,0)|) = 0$. Pick any $\gamma \in \mathcal{K}_\infty$: $\gamma \ge \max\{\omega, \chi\}$. Then for all $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^m$ the inequality (6) holds. Clearly, for all $x : |x| \ge \gamma(|\xi|)$ the implication (5) holds. ■

Similarly one can prove the following proposition (which is not a consequence of Proposition 1):

Proposition 2: Let $V : \mathbb{R}^n \to \mathbb{R}_+$ be Lipschitz continuous. Assume that there exist $\psi_1, \psi_2 \in \mathcal{K}_\infty$, such that (3) holds and there exist $\gamma \in \mathcal{K}_\infty$ and $c, d \in \mathbb{R}$ such that for almost all $x \in \mathbb{R}^n$ and for all $\xi \in \mathbb{R}^m$ it holds

$$V(x) \ge \gamma(|\xi|) \Rightarrow \nabla V \cdot f(x,\xi) \le -cV(x)$$

and $\forall x \in \mathbb{R}^n, \xi \in \mathbb{R}^m$ it holds

$$V(g(x,\xi)) \le \max\{e^{-d}V(x), \gamma(|\xi|)\}.$$

Then V is an exponential ISS-Lyapunov function (with the rate coefficients c and d). If g is locally bounded, then also the converse implication holds.

In [5] there have been proved two Lyapunov-type sufficient conditions for the system (1) to be ISS. The first of them provides a sufficient condition for the impulsive system (1), which possesses an ISS-Lyapunov function to be ISS w.r.t. certain impulse time sequences.

Define $S_\theta := \{\{t_i\}_1^\infty \subset [t_0, \infty) \ : \ t_{i+1} - t_i \ge \theta, \ \forall i \in \mathbb{N}\}$.

Theorem 3: Let V be an ISS-Lyapunov function for (1) and let φ, α be as in the Definition 2 with $\varphi \in \mathcal{P}$. If for some $\theta, \delta > 0$ and all $a > 0$ it holds

$$\int_a^{\alpha(a)} \frac{ds}{\varphi(s)} \le \theta - \delta, \qquad (7)$$

then (1) is ISS for all impulse time sequences $T \in S_\theta$.

For the systems, which possess exponential ISS-Lyapunov functions a stronger result is provided by the following theorem.

For a given sequence of impulse times denote by $N(t,s)$ the number of jumps within time-span $(s,t]$.

Theorem 4: Let V be an exponential ISS-Lyapunov function for (1) with corresponding coefficients $c \in \mathbb{R}$, $d \ne 0$. For arbitrary function $h : \mathbb{R}_+ \to (0, \infty)$, for which there exists $g \in \mathcal{L}$: $h(x) \le g(x)$ for all $x \in \mathbb{R}_+$ consider the class $\mathcal{S}[h]$ of impulse time-sequences, satisfying the generalized average dwell-time (gADT) condition:

$$-dN(t,s) - c(t-s) \le \ln h(t-s), \quad \forall t \ge s \ge t_0. \qquad (8)$$

Then the system (1) is uniformly ISS over $\mathcal{S}[h]$.

For a discussion of the relations between the fixed dwell-time condition (7) and gADT condition (8) see [16, Section 3.2.2].

If an ISS-Lyapunov function has been already constructed, then the above theorems provide us with the conditions, which allow us to verify the ISS of impulsive system. However, there exist no general method for construction of ISS-Lyapunov functions.

In the next section we prove small-gain theorems, which provide the construction of the ISS-Lyapunov function for an interconnection of impulsive subsystems, for which ISS-Lyapunov functions are given and the small-gain condition holds.

IV. ISS OF INTERCONNECTED IMPULSIVE SYSTEMS

Let $T = \{t_1, \ldots, t_k, \ldots\}$ be a sequence of impulse times for all subsystems (we assume, that all systems jump at the same time).

Consider the system consisting of n interconnected impulsive subsystems:

$$\begin{cases} \dot{x}_i(t) = f_i(x_1(t), \ldots, x_n(t), u(t)), \ t \notin T, \\ x_i(t) = g_i(x_1^-(t), \ldots, x_n^-(t), u^-(t)), \ t \in T, \\ i = \overline{1,n}, \end{cases} \qquad (9)$$

where the state x_i of the i-th subsystem is absolutely continuous between impulses; u is a locally bounded, Lebesgue-measurable input and x_j, $j \ne i$ can be interpreted as internal inputs of the i-th subsystem.

Furthermore, $f_i : \mathbb{R}^{N_1} \times \ldots \times \mathbb{R}^{N_n} \times \mathbb{R}^m \to \mathbb{R}^{N_i}$ and $g_i : \mathbb{R}^{N_1} \times \ldots \times \mathbb{R}^{N_n} \times \mathbb{R}^m \to \mathbb{R}^{N_i}$, where we assume that the f_i are locally Lipschitz for all $i = 1, \ldots, n$. We continue to assume that all signals (x_i, $i = 1, \ldots, n$ and inputs u) are right-continuous and have left limits at all times.

We define $N := N_1 + \ldots + N_n$, $x := (x_1^T, \ldots, x_n^T)^T$, $f := (f_1^T, \ldots, f_n^T)^T$ and $g := (g_1^T, \ldots, g_n^T)^T$ such that the interconnected system (9) is of the form

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)), \quad t \notin T, \\ x(t) = g(x^-(t), u^-(t)), \quad t \in T. \end{cases} \qquad (10)$$

Remark 3: Let us suppose for a while, that the assumption, that impulse time sequence is the same for all subsystems, is dropped, and consider the interconnections of the following form:

$$\begin{cases} \dot{x}_i(t) = f_i(x_1(t), \ldots, x_n(t), u(t)), \ t \notin T_i, \\ x_i(t) = g_i(x_1^-(t), \ldots, x_n^-(t), u^-(t)), \ t \in T_i, \\ i = \overline{1,n}, \end{cases} \qquad (11)$$

where T_i can be different for different i. In this case the interconnected system cannot be rewritten in the form (10), and thus (11) defines a more general class of systems than

(1). For such systems the input-to-state stability theory is not developed at the time.

According to the Proposition 1 for the i-th subsystem of (9) the definition of an ISS-Lyapunov function can be written as follows. A Lipschitz continuous function $V_i:\mathbb{R}^{N_i}\to\mathbb{R}_+$ is an ISS-Lyapunov function for i-th subsystem of (9), if three properties hold:

1) There exist functions $\psi_{i1},\psi_{i2}\in\mathcal{K}_\infty$, such that:
$$\psi_{i1}(|x_i|)\le V_i(x_i)\le\psi_{i2}(|x_i|),\quad \forall x_i\in\mathbb{R}^{N_i}$$

2) There exist $\gamma_{ij},\gamma_i\in\mathcal{K}$, $j=1,\dots,n$, $\gamma_{ii}:=0$ and $\varphi_i\in\mathcal{P}$, so that for almost all $x_i\in\mathbb{R}^{N_i}$, $i=1,\dots,n$, for all $\xi\in\mathbb{R}^m$ from
$$V_i(x_i)\ge\max\{\max_{j=1}^{n}\gamma_{ij}(V_j(x_j)),\gamma_i(|\xi|)\},\tag{12}$$
it follows
$$\nabla V_i\cdot f_i(x,\xi)\le-\varphi_i\left(V_i(x_i(t))\right).\tag{13}$$

3) There exist $\alpha_i\in\mathcal{P}$, such that for gains $\gamma_{ij},\gamma_i\in\mathcal{K}$ defined above and for all $x\in\mathbb{R}^N$ and for all $\xi\in\mathbb{R}^m$ it holds
$$V_i(g_i(x,\xi))\le\max\{\alpha_i(V_i(x_i)),\max_{j=1}^{n}\gamma_{ij}(V_j(x_j)),\gamma_i(|\xi|)\}.\tag{14}$$

If $\varphi_i(y)=c_iy$ and $\alpha_i(y)=e^{-d_i}y$ for all $y\in\mathbb{R}_+$, then V_i is an exponential ISS-Lyapunov function for the i-th subsystem of (9) with rate coefficients $c_i,d_i\in\mathbb{R}$.

The internal Lyapunov gains γ_{ij} characterize the interconnection structure of subsystems. As we will see, the question, whether the interconnection (9) is ISS, depends on the properties of the gain operator $\Gamma:\mathbb{R}^n_+\to\mathbb{R}^n_+$ defined by
$$\Gamma(s):=\left(\max_{j=1}^{n}\gamma_{1j}(s_j),\dots,\max_{j=1}^{n}\gamma_{nj}(s_j)\right),\ s\in\mathbb{R}^n_+.\tag{15}$$

To construct an ISS-Lyapunov function for the whole interconnection we will use the notion of Ω-path (see [8], [18]).

Definition 3: A function $\sigma=(\sigma_1,\dots,\sigma_n)^T:\mathbb{R}^n_+\to\mathbb{R}^n_+$, where $\sigma_i\in\mathcal{K}_\infty$, $i=1,\dots,n$ is called an Ω*-path* (with respect to operator Γ), if it possesses the following properties:

1) σ_i^{-1} is locally Lipschitz continuous on $(0,\infty)$;
2) for every compact set $P\subset(0,\infty)$ there are finite constants $0<K_1<K_2$ such that for all points of differentiability of σ_i^{-1} we have
$$0<K_1\le(\sigma_i^{-1})'(r)\le K_2,\quad\forall r\in P;$$
3)
$$\Gamma(\sigma(r))\le\sigma(r),\ \forall r>0.\tag{16}$$

Remark 4: Note that usually in the definition of Ω-path it is assumed that (16) holds with "$<$" instead of "$\le$" [7]. However, this small weakening of requirements doesn't change the proofs of our small-gain theorems and provides us with more flexibility.

We say that Γ satisfies *the small-gain condition* if the following inequality holds
$$\Gamma(s)\not\ge s,\ \forall\ s\in\mathbb{R}^n_+\backslash\{0\}.\tag{17}$$

Now we prove a small-gain theorem for nonlinear impulsive systems. The technique for treatment of the discrete dynamics was adopted from [17] and [3].

Theorem 5: Consider the system (9). Let V_i be the ISS-Lyapunov function for i-th subsystem of (9) with corresponding gains γ_{ij} from (12)-(14). If the operator Γ defined by (15) satisfies the small-gain condition (17), then an ISS-Lyapunov function V for the whole system can be constructed as
$$V(x):=\max_i\{\sigma_i^{-1}(V_i(x_i))\},\tag{18}$$
where $\sigma=(\sigma_1,\dots,\sigma_n)^T$ is an Ω-path.

Proof: Define the function γ by
$$\gamma(r):=\max_i\sigma_i^{-1}(\gamma_i(r)).\tag{19}$$
In [6] it was proved, that for almost all $x\in\mathbb{R}^N$ and all $\xi\in\mathbb{R}^m$ from $V(x)\ge\gamma(|\xi|)$ it follows
$$\frac{d}{dt}V(x)\le-\varphi(V(x)),$$
for
$$\varphi(r):=\min_{i=1}^{n}\left\{\left(\sigma_i^{-1}\right)'(\sigma_i(r))\varphi_i(\sigma_i(r))\right\}.\tag{20}$$
Function φ is positive definite, because $\sigma_i^{-1}\in\mathcal{K}_\infty$ and all φ_i are positive definite functions.

Thus, implication (5) is verified and it remains to check the estimation (6). With the help of inequality (14) we make for all $x\in\mathbb{R}^N$ and $\xi\in\mathbb{R}^m$ the following estimations
$$\begin{aligned}V(g(x,\xi)) &= \max_i\{\sigma_i^{-1}(V_i(g_i(x,\xi)))\}\\ &\le \max_i\{\sigma_i^{-1}(\max\{\alpha_i(V_i(x_i)),\\ &\qquad \max_{j=1}^{n}\gamma_{ij}(V_j(x_j)),\gamma_i(|\xi|)\})\}\\ &= \max\{\max_i\{\sigma_i^{-1}\circ\alpha_i(V_i(x_i))\},\\ &\qquad \max_{i,j\ne i}\{\sigma_i^{-1}\circ\gamma_{ij}(V_j(x_j))\},\max_i\{\sigma_i^{-1}\circ\gamma_i(|\xi|)\}\}\\ &= \max\{\max_i\{\sigma_i^{-1}\circ\alpha_i\circ\sigma_i\circ\sigma_i^{-1}(V_i(x_i))\},\\ &\qquad \max_{i,j\ne i}\{\sigma_i^{-1}\circ\gamma_{ij}\circ\sigma_j\circ\sigma_j^{-1}(V_j(x_j))\},\\ &\qquad \max_i\{\sigma_i^{-1}\circ\gamma_i(|\xi|)\}\}.\end{aligned}$$

Define $\tilde\alpha:=\max_i\{\sigma_i^{-1}\circ\alpha_i\circ\sigma_i\}$. Since $\alpha_i\in\mathcal{P}$, then $\tilde\alpha\in\mathcal{P}$. Pick any $\alpha^*\in\mathcal{K}$: $\alpha^*(r)\ge\tilde\alpha(r)$, $r\ge0$. Then the following estimate holds
$$\begin{aligned}\max_i\{\sigma_i^{-1}\circ\alpha_i\circ\sigma_i\circ\sigma_i^{-1}(V_i(x_i))\} &\le \alpha^*(\max_i\{\sigma_i^{-1}(V_i(x_i))\})\\ &= \alpha^*(V(x)).\end{aligned}$$
Define also $\eta:=\max_{i,j\ne i}\{\sigma_i^{-1}\circ\gamma_{ij}\circ\sigma_j\}$ and note that according to (16)
$$\eta=\max_{i,j\ne i}\{\sigma_i^{-1}\circ\gamma_{ij}\circ\sigma_j\}<\max_{i,j\ne i}\{\sigma_i^{-1}\circ\sigma_i\}=id.$$

We continue estimates of $V(g(x,\xi))$:

$$\begin{aligned} V(g(x,\xi)) &\leq \max\{\alpha^*(V(x)),\eta(V(x)),\gamma(|\xi|)\} \\ &= \max\{\alpha(V(x)),\gamma(|\xi|)\}, \end{aligned}$$

where

$$\alpha := \max\{\alpha^*,\eta\}. \qquad (21)$$

According to Proposition 1 the function V is an ISS-Lyapunov function of the interconnection (9). ■

Remark 5: Small-gain theorem has been formulated for Lyapunov functions in the form used in Proposition 1. According to the Proposition 1 this formulation can be transformed to the standard formulation, and from the proof it is clear, that the functions α and φ remain the same after the transformation. In order to find the classes of impulse time-sequences, for which the system (10) is ISS, one should use Theorem 3.

A. Small-gain theorem for exponential ISS-Lyapunov functions

If an exponential ISS-Lyapunov function for a system (1) is given, then the Theorem 4 provides us with the tight estimates of the set of impulse time sequences, w.r.t. which the system (1) is ISS. Unfortunately, even if all subsystems are exponentially ISS, the ISS-Lyapunov function, designed via small-gain approach, is not necessarily exponential.

But we may hope, that if ISS-Lyapunov functions for all subsystems of (9) are *exponential*, then the expression (18) at least for certain type of gains provides the *exponential* ISS-Lyapunov function for the whole system. In this subsection we are going to prove the small-gain theorem of this type.

Define the following class of power functions

$$P := \{f:\mathbb{R}_+ \to \mathbb{R}_+ : \exists a \geq 0,\ b>0:\ f(s)=as^b\ \forall s \in \mathbb{R}_+\}$$

Define also $Q:\mathbb{R}^n_+ \to \mathbb{R}^n_+$ by

$$Q(x) := MAX\{x,\Gamma(x),\Gamma^2(x),\dots,\Gamma^{n-1}(x)\},$$

with $\Gamma^n(x)=\Gamma\circ\Gamma^{n-1}(x)$, for all $n\geq 2$. The function MAX for all $h_i \in \mathbb{R}^n$, $i=1,\dots,m$ is defined by

$$z = MAX\{h_1,\dots,h_m\} \in \mathbb{R}^n, \quad z_i := \max\{h_{1i},\dots,h_{mi}\}.$$

Theorem 6: Let V_i be the eISS Lyapunov function for the i-th subsystem of (9) with corresponding gains γ_{ij}, $i=1,\dots,n$. Let also $\gamma_{ij}\in P$. Then one can construct the exponential ISS-Lyapunov function $V:\mathbb{R}^N \to \mathbb{R}_+$ for the whole system (10) by (18) for certain Ω-path σ. In particular, σcan be chosen by

$$\sigma(t) = Q(at), \forall t\geq 0, \quad \text{for some } a \in int(\mathbb{R}^n_+). \qquad (22)$$

Proof: Take Ω-path σ as in (22). According to the Theorem 5 function V, defined by (18) is an ISS Lyapunov function. We have only to prove, that it is the exponential one.

For all $f,g\in P$ it follows $f\circ g \in P$, thus for all i it holds that $\sigma_i(t)=\max\{f^i_1(t),\dots,f^i_{r_i}(t)\}$, where all $f^i_k \in P$ and r_i is finite.

Thus, for all i there exists a partition of $\mathbb{R}_+$ into sets S^i_j, $j=1,\dots,k_i$ (i.e. $\cup_{j=1}^{k_i} S^i_j = \mathbb{R}_+$ and $S^i_j \cap S^i_s = \emptyset$, if $j\neq s$), such that $\sigma_i^{-1}(t)=a_{ij}t^{p_{ij}}$ for $t\in S^i_j$. This partition is always finite, because all $f^i_j \in P$, and such two functions intersect in no more than one point, distinct from zero.

Note, that σ is indeed an Ω-path, since the first two properties of Definition 3 are satisfied and $\Gamma(\sigma(t)) \leq \sigma(t)$ for all $t\geq 0$ according to [12, Proposition 2.7 and Remark 2.8].

Let $x\in M_i$ and $V_i(x_i)\in S^i_j$. Then the condition (13) implies

$$\begin{aligned} \frac{d}{dt}V(x) &= \frac{d}{dt}(\sigma_i^{-1}(V_i(x_i))) = \frac{d}{ds}(a_{ij}s^{p_{ij}})|_{s=V_i(x_i)}\frac{d}{dt}(V_i(x_i)) \\ &\leq -c_i a_{ij} p_{ij}(V_i(x_i))^{p_{ij}} \leq -cV(x) \end{aligned}$$

where $c=\min_{i,j}\{c_i p_{ij}\}$.

We have to prove, that the function α from (21) can be estimated from above by linear function. We choose $\alpha^* := \tilde{\alpha} = \max_i\{\sigma_i^{-1}\circ\alpha_i\circ\sigma_i\}$.

For any $t\geq 0$ it holds that $\sigma_i^{-1}\circ\alpha_i\circ\sigma_i(t) = c_i = const$ since α_i are linear and σ_i^{-1} are piecewise power functions. This implies that for some constant k it holds that $\alpha^*(t)\leq kt$ for all $t\geq 0$.

Since $\eta < id$, it is clear that one can take $\alpha := \max\{k,1\}\,\mathrm{Id}$. Recalling Proposition 2 we obtain the claim of the theorem. ■

Remark 6: The obtained exponential ISS-Lyapunov function can be transformed to the implication form with the help of Proposition 2. Then Theorem 4 can be used in order to characterize the set of impulse time sequences for which the system (10) is ISS.

Let us demonstrate how one can analyze interconnected impulsive systems on a simple example. Let $T=\{t_k\}$ be a sequence of impulse times. Consider two interconnected nonlinear impulsive systems

$$\begin{aligned} \dot{x}_1(t) &= -x_1(t)+x_2^2(t),\ t\notin T, \\ x_1(t) &= e^{-1}x_1^-(t),\ t\in T \end{aligned}$$

and

$$\begin{aligned} \dot{x}_2(t) &= -x_2(t)+3\sqrt{|x_1(t)|},\ t\notin T, \\ x_2(t) &= e^{-1}x_2^-(t),\ t\in T. \end{aligned}$$

Both subsystems are uniformly ISS (even strongly uniformly ISS, see [10]) for all impulse time sequences, since continuous and discrete dynamics stabilize the subsystems and one can easily construct exponential ISS Lyapunov functions with positive rate coefficients for both subsystems and the corresponding gains χ_{12}, χ_{21}. But for arbitrary gains, which allow to prove the ISS of the subsystems w.r.t. all time-sequences the small-gain condition will fail, since the continuous dynamics of the interconnected system is not stable. Therefore in order to apply the small-gain theorem we have to choose other gains, which make it possible to prove ISS of the interconnected system for certain classes of impulse time sequences.

Take the following exponential ISS-Lyapunov functions and Lyapunov gains for subsystems

$$V_1(x_1) = |x_1|, \quad \gamma_{12}(r) = \tfrac{1}{a}r^2,$$
$$V_2(x_2) = |x_2|, \quad \gamma_{21}(r) = \tfrac{1}{b}\sqrt{r},$$

where $a, b > 0$. We have the estimates

$$|x_1| \geq \gamma_{12}(|x_2|) \Rightarrow \dot{V}_1(x_1) \leq (a-1)V_1(x_1),$$
$$|x_2| \geq \gamma_{21}(|x_1|) \Rightarrow \dot{V}_2(x_2) \leq (3b-1)V_2(x_2).$$

The small-gain condition

$$\gamma_{12} \circ \gamma_{21}(r) = \frac{1}{ab^2} r < r, \quad \forall r > 0 \tag{23}$$

is satisfied, if it holds

$$h(a,b) := ab^2 > 1. \tag{24}$$

Take arbitrary constant c such that $\frac{1}{b} < \frac{1}{c} < \sqrt{a}$. Then Ω-path can be chosen as

$$\sigma_1(r) = r, \quad \sigma_2(r) = \frac{1}{c}\sqrt{r}, \ \forall r \geq 0.$$

Then

$$\sigma_2^{-1}(r) = c^2 r^2, \ \forall r \geq 0.$$

In this case an ISS-Lyapunov function for the interconnection, constructed by small-gain design, is given by

$$V(x) = \max\{|x_1|, c^2|x_2|^2\}, \quad \text{where } \frac{1}{b} < \frac{1}{c} < \sqrt{a}$$

and we have the estimate

$$V(g(x)) = V(e^{-1} \cdot x) \leq e^{-1} V(x). \tag{25}$$

Thus, $d = -1$ for the interconnection. The estimates of the continuous dynamics for V are as follows: For $|x_1| \geq c^2 x_2^2 > \frac{1}{a} x_2^2 = \gamma_{12}(|x_2|)$ it holds

$$\frac{d}{dt}V(x) = \frac{d}{dt}|x_1| \leq (a-1)|x_1| = (a-1)V(x),$$

and $|x_1| \leq c^2 x_2^2 < \gamma_{21}^{-1}(|x_2|)$ implies

$$\frac{d}{dt}V(x) = \frac{d}{dt}\left(c^2 x_2^2\right) = \frac{d}{dt}\left(c^2 V_2(x_2)^2\right) \leq 2(3b-1)c^2|x_2|^2 = 2(3b-1)V(x).$$

Overall, for all x we have:

$$\frac{d}{dt}V(x) \leq \max\{(a-1), 2(3b-1)\}V(x). \tag{26}$$

Function h, defined by (24), is increasing w.r.t. both arguments since $a, b > 0$, hence in order to minimize $\varepsilon := \max\{(a-1), 2(3b-1)\}$, we have to choose $(a-1) = 2(3b-1)$. Then, from (23) we obtain the inequality

$$(1 + 2(3b-1))b^2 > 1.$$

Thus, the best choice for b is $b \approx 0.612$ and V is an exponential ISS-Lyapunov function for the interconnection with coefficients with $d = -1$ and $c = 2 \cdot (3 \cdot 0.612 - 1) = 1.672$.

The ISS-Lyapunov function for the interconnection is constructed, and one can apply Theorem 4 in order to obtain the classes of impulse time sequences for which the interconnection is GAS.

References

[1] Wu-Hua Chen and Wei Xing Zheng. Input-to-state stability and integral input-to-state stability of nonlinear impulsive systems with delays. *Automatica (Journal of IFAC)*, 45(6):1481–1488, 2009.

[2] S. Dashkovskiy, D. Efimov, and E. Sontag. Input to state stability and allied system properties. *Automation and Remote Control*, 72:1579–1614, 2011.

[3] Sergey Dashkovskiy and Michael Kosmykov. Stability of networks of hybrid ISS systems. In *Proceedings of the 48th IEEE Conference on Decision and Control, Shanghai, China, Dec. 16-18, 2009*, pages 3870–3875, 2009.

[4] Sergey Dashkovskiy, Michael Kosmykov, Andrii Mironchenko, and Lars Naujok. Stability of interconnected impulsive systems with and without time-delays using Lyapunov methods. *Nonlinear Analysis: Hybrid Systems.*, 6(3):899–915, 2012.

[5] Sergey Dashkovskiy and Andrii Mironchenko. Dwell-time conditions for robust stability of impulsive systems. In *Proceedings of 20th International Symposium on Mathematical Theory of Systems and Networks, Melbourne, Australia, Jul. 09-13, 2012*, 2012. Paper 72.

[6] Sergey Dashkovskiy, Björn S. Rüffer, and Fabian R. Wirth. An ISS Lyapunov function for networks of ISS systems. In *Proceedings of the 17th International Symposium on Mathematical Theory of Networks and Systems (MTNS), Kyoto, Japan, July 24-28, 2006*, pages 77–82, 2006.

[7] Sergey Dashkovskiy, Björn S. Rüffer, and Fabian R. Wirth. An ISS small gain theorem for general networks. *Math. Control Signals Systems*, 19(2):93–122, 2007.

[8] Sergey N. Dashkovskiy, Björn S. Rüffer, and Fabian R. Wirth. Small Gain Theorems for Large Scale Systems and Construction of ISS Lyapunov Functions. *SIAM Journal on Control and Optimization*, 48(6):4089–4118, 2010.

[9] Wassim M. Haddad, Vijay Sekhar Chellaboina, and Sergey G. Nersesov. *Impulsive and hybrid dynamical systems*. Princeton Series in Applied Mathematics. Princeton University Press, Princeton, NJ, 2006.

[10] João P. Hespanha, Daniel Liberzon, and Andrew R. Teel. Lyapunov conditions for input-to-state stability of impulsive systems. *Automatica J. IFAC*, 44(11):2735–2744, 2008.

[11] Iasson Karafyllis and Zhong-Ping Jiang. A small-gain theorem for a wide class of feedback systems with control applications. *SIAM J. Control Optim.*, 46(4):1483–1517 (electronic), 2007.

[12] Iasson Karafyllis and Zhong-Ping Jiang. A vector small-gain theorem for general non-linear control systems. *IMA Journal of Mathematical Control and Information*, 28(3):309–344, June 2011.

[13] M. Kosmykov. *Hybrid dynamics in large-scale logistics networks*. PhD thesis, Department of Mathematics and Computer Science, University of Bremen, 2011.

[14] V. Lakshmikantham, D. D. Baĭnov, and P. S. Simeonov. *Theory of impulsive differential equations*, volume 6 of *Series in Modern Applied Mathematics*. World Scientific Publishing Co. Inc., Teaneck, NJ, 1989.

[15] Jun Liu, Xinzhi Liu, and Wei-Chau Xie. Input-to-state stability of impulsive and switching hybrid systems with time-delay. *Automatica*, 47(5):899–908, May 2011.

[16] Andrii Mironchenko. *Input-to-state stability of infinite-dimensional control systems*. PhD thesis, University of Bremen, 2012.

[17] D. Nesic and A.R. Teel. A Lyapunov-based small-gain theorem for hybrid ISS systems. In *Proceedings of the 47th IEEE Conference on Decision and Control, Cancun, Mexico, Dec. 9-11, 2008*, pages 3380–3385, 2008.

[18] B. S. Rüffer. Monotone inequalities, dynamical systems, and paths in the positive orthant of Euclidean *n*-space. *Positivity.*, 14(2):257–283, 2010.

[19] A. M. Samoilenko and N. A. Perestyuk. *Impulsive differential equations*, volume 14 of *World Scientific Series on Nonlinear Science. Series A: Monographs and Treatises*. World Scientific Publishing Co. Inc., River Edge, NJ, 1995.

[20] E. D. Sontag. Input to state stability: basic concepts and results. In *Nonlinear and optimal control theory*, volume 1932 of *Lecture Notes in Math.*, pages 163–220, Berlin, 2008. Springer.

[21] Eduardo D. Sontag. Smooth stabilization implies coprime factorization. *IEEE Trans. Automat. Control*, 34(4):435–443, 1989.

[22] Shujie Yang, Bao Shi, and Shuyan Hao. Input-to-state stability for discrete-time nonlinear impulsive systems with delays. *Int. J. Robust. Nonlinear Control*, 2011. published online.

[23] Tao Yang. *Impulsive control theory*, volume 272 of *Lecture Notes in Control and Information Sciences*. Springer-Verlag, Berlin, 2001.