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Stability of nonlinear infinite dimensional impulsive systems and their interconnections *

Sergey Dashkovskiy and Andrii Mironchenko

Abstract— **We consider input-to-state stability (ISS) of nonlinear infinite dimensional impulsive systems with an emphasis on interconnections of such systems. Stability conditions as a combination of Lyapunov methods and dwell-time inequalities are provided. For stability of interconnections a further condition of a small-gain type comes into play. We illustrate these results on an interconnection of two semilinear parabolic equations.**

Index Terms— **impulsive systems, nonlinear control systems, infinite-dimensional systems, input-to-state stability, Lyapunov methods**

I. Introduction

In many modern applications one has to deal with dynamics of a system that combines both continuous and discontinuous behavior. A general framework for modeling of such phenomena is the hybrid systems theory [9], [8]. An important subclass of hybrid systems are impulsive systems [22], i.e. hybrid systems whose state jumps at predefined time instants, which do not depend on the state of the system. In this work we study stability of impulsive systems, which is crucial for the design and performance of practical systems. To investigate stability of impulsive systems we exploit the notion of input-to-state stability (ISS) [23], [3], which is particularly useful to study interconnections. Input-to-state stability of impulsive systems has been studied in recent papers [11] (finite-dimensional systems) and [2], [17], [24] (time-delay systems). In [4] the interconnected impulsive systems with and without time-delays have been investigated.

In the current literature only exponential ISS Lyapunov functions (or exponential ISS Lyapunov-Razumikhin functions, exponential ISS Lyapunov-Krasovskii functionals) have been exploited for analysis of ISS of impulsive systems (for a short overview see [6]). This restrains the class of systems, which can be investigated by such methods, since it is not proved that an exponential ISS-Lyapunov function for an ISS impulsive system always exists, and even if it does, a nonexponential Lyapunov function may be less restrictive and can be constructed easier. For example, if we apply the small-gain design from [7], [14] to construct a Lyapunov function for an interconnected system, whose subsystems possess exponential Lyapunov functions, then the resulting Lyapunov function will be in general non-exponential, if the gains are nonlinear. Hence restriction of Lyapunov methods to exponential ISS Lyapunov functions only is unsatisfactory. This motivated us to investigate applicability of nonexponential ISS Lyapunov functions.

Another motivation for this work is a rapid development of ISS theory of distributed parameter systems during the last years, see [5], [18], [13], [19] to cite a few. The semigroup approach, exploited in [5], [19], allows for the unified ISS theory of ordinary differential equations (ODEs) and infinite-dimensional systems. Therefore, having in mind possible applications of ISS theory to infinite-dimensional impulsive systems, we consider in this paper ISS of nonlinear impulsive systems on Banach spaces. But we want to stress, that the results presented here are novel already for ODE systems.

We show, that existence of an ISS Lyapunov function (not necessarily exponential) for an impulsive system implies input-to-state stability of the system over impulsive sequences satisfying nonlinear fixed dwell-time (FDT) condition (previously used in [22] to investigate systems without inputs). Under slightly weaker FDT condition we show the uniform global stability of the system over corresponding class of impulse time sequences. To the best of our knowledge, this paper is the first, where nonexponential Lyapunov functions have been exploited to prove ISS of an impulsive system. Next we introduce a generalized average dwell-time (gADT) condition and show that an impulsive system, which possesses an exponential ISS Lyapunov function is uniformly ISS over the class of impulse time sequences, satisfying the gADT condition. This contains a corresponding result from [11] as a special case.

In Section IV we state a small-gain theorem for interconnections of impulsive systems, analogous to corresponding theorem for infinite-dimensional systems with continuous behavior [14], [7], [5]. Next we show, that if all subsystems possess exponential ISS Lyapunov functions, and the gains are power functions, then the exponential ISS Lyapunov function for the whole system can be constructed. This result generalizes Theorem 4.2 from [4], where this statement for linear gains has been proved. We stress, that we consider interconnections, which subsystems may not be ISS. The only requirement is that the instabilities of the systems should be matched, i.e. either all subsystems should possess a stable continuous dynamics, or all subsystems should have a stable discrete dynamics. For such classes of systems our method is more efficient than a small-gain argument from [16] (which, however, is better suited to study systems with non-matched instabilities). For a discussion of these two methods as well for the unified approach, which combines advantages of both methods, see [20].

All theorems are given without proofs, since they can be

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S. Dashkovskiy is with Erfurt University of Applied Sciences, Altonaer Straße 25, 99085 Erfurt, Germany sergey.dashkovskiy@fh-erfurt.de

Andrii Mironchenko is with Department of Systems Design and Informatics, Kyushu Institute of Technology, 680-4 Kawazu, Iizuka, Fukuoka 820-8502, Japan andrii.mironchenko@mathematik.uni-wuerzburg.de

found in [6]. In addition to [6] we illustrate our results on interconnected parabolic systems and provide new discussions and open problems for future investigation.

II. Preliminaries

Let X and U denote a state space and a space of input values respectively, and let both of them be Banach. Take the space of admissible inputs as $U_c := PC([t_0,\infty),U)$, i.e. the space of piecewise right-continuous functions from $[t_0,\infty)$ to U equipped with the norm $\|u\|_{U_c} := \sup_{t\geq t_0} \|u(t)\|_U$.

Let $T=\{t_1,\ t_2,\ t_3,\ldots\}$ be a strictly increasing sequence of impulse times without finite accumulation points.

Consider a system of the form

$$\begin{cases} \dot{x}(t) = Ax(t) + f(x(t),u(t)), & t\in[t_0,\infty)\backslash T, \\ x(t) = g(x^-(t),u^-(t)), & t\in T, \end{cases} \tag{1}$$

where $x(t)\in X$, $u(t)\in U$, A is an infinitesimal generator of a C_0-semigroup e^{tA} on X and $f,g : X\times U \to X$ satisfy $f(0,0)=g(0,0)=0$, i.e., $x\equiv 0$ is an equilibrium of the unforced system (1).

Equations (1) together with the sequence of impulse times T define an impulsive system. The first equation of (1) describes the continuous dynamics of the system, and the second describes the jumps of the state at impulse times.

Under solution of the first equation of (1) we understand solutions of an integral equation

$$x(t) = e^{tA}x(0) + \int_0^t e^{(t-s)A} f(x(s),u(s))ds$$

belonging to the class $C([0,\tau],X)$ for all $\tau>0$ (so-called weak solutions).

To ensure that the solution of (1) exists and is unique, we assume that $u\in C(\mathbb{R}_+,U)$ and that $f : X\times U\to X$ is Lipschitz continuous on bounded subsets of X, uniformly w.r.t. the second argument, i.e. $\forall C>0\ \exists L(C)>0$, such that $\forall x,y : \|x\|\leq C,\ \|y\|\leq C\ \forall u\in U$

$$\|f(y,u)-f(x,u)\|_X \leq L(C)\|y-x\|_X.$$

Existence and uniqueness then follows due to a variation of a classical existence and uniqueness theorem [1]. Note that from the continuity assumptions on the inputs u it follows that $x(\cdot)$ is piecewise-continuous, and $x^-(t) = \lim_{s\to t-0} x(s)$ exists for all $t\geq t_0$.

For a given set of impulse times by $\phi(t,t_0,x,u)$ we denote the state of (1) corresponding to the initial value $x\in X$, the initial time t_0 and to the input $u\in U_c$ at time $t\geq t_0$.

We assume that t_0 is fixed and investigate stability properties of the system (1) w.r.t. this initial time.

Definition 1 *Define the following classes of continuous functions:*

$$\begin{array}{lll}
\mathcal{P} & := & \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid\ \gamma(0)=0,\ \gamma(r)>0, r>0\} \\
\mathcal{K} & := & \{\gamma\in\mathcal{P} \mid \gamma \text{ is strictly increasing}\} \\
\mathcal{K}_\infty & := & \{\gamma\in\mathcal{K} \mid \gamma \text{ is unbounded}\} \\
\mathcal{L} & := & \left\{\gamma : \mathbb{R}_+\to\mathbb{R}_+ \mid\ \text{decreasing and } \lim_{t\to\infty}\gamma(t)=0\right\} \\
\mathcal{KL} & := & \left\{\beta : \mathbb{R}_+^2 \to \mathbb{R}_+ \mid \beta(\cdot,t)\in\mathcal{K},\ \beta(r,\cdot)\in\mathcal{L}\ \forall t,r>0\right\}
\end{array}$$

Let us introduce the stability notions for the system (1):

Definition 2 *For a given sequence T of impulse times we call a system* (1) *input-to-state stable (ISS) if there exist $\beta\in\mathcal{KL}$, $\gamma\in\mathcal{K}_\infty$, such that $\forall x\in X$, $\forall u\in U_c$, $\forall t\geq t_0$ it holds*

$$\|\phi(t,t_0,x,u)\|_X \leq \beta(\|x\|_X, t-t_0) + \gamma(\|u\|_{U_c}). \tag{2}$$

System (1) *is called uniformly ISS over a given set $\mathcal{S}$ of admissible sequences of impulse times if it is ISS for every sequence $T\in\mathcal{S}$, with β and γ independent of the choice of the sequence from the class $\mathcal{S}$.*

Definition 3 *For a given sequence T of impulse times we call system* (1) *globally stable (GS) if there exist $\xi,\gamma\in\mathcal{K}_\infty$ such that $\forall x\in X$, $\forall u\in U_c$, $\forall t\geq t_0$ it holds*

$$\|\phi(t,t_0,x,u)\|_X \leq \xi(\|x\|_X) + \gamma(\|u\|_{U_c}). \tag{3}$$

System (1) *is uniformly GS over a given set $\mathcal{S}$ of admissible sequences of impulse times if* (3) *holds for every sequence $T\in\mathcal{S}$, with β and γ independent of the choice of T.*

III. Lyapunov ISS theory for an impulsive system

In this section we derive Lyapunov-type conditions for ISS of an impulsive system of the form (1). The crucial role in this development is played by ISS-Lyapunov functions.

Definition 4 *A continuous function $V : X\to\mathbb{R}_+$ is called an ISS-Lyapunov function for* (1) *if $\exists\ \psi_1,\psi_2,\chi\in\mathcal{K}_\infty$, $\alpha\in\mathcal{P}$ and continuous function $\varphi : \mathbb{R}_+\to\mathbb{R}$ with $\varphi(x)=0 \Leftrightarrow x=0$, such that*

$$\psi_1(\|x\|_X) \leq V(x) \leq \psi_2(\|x\|_X), \quad x\in X \tag{4}$$

and $\forall x\in X$, $\forall\xi\in U$, $\forall u\in U_c$ with $u(0)$=ξ it holds that

$$V(x)\geq\chi(\|\xi\|_U) \Rightarrow \begin{cases} \dot{V}_u(x) \leq -\varphi(V(x)) \\ V(g(x,\xi)) \leq \alpha(V(x)). \end{cases} \tag{5}$$

Here for a given input value $u\in U_c$ the Lie derivative $\dot{V}_u(x)$ is defined by

$$\dot{V}_u(x) = \overline{\lim_{t\to+0}} \frac{1}{t}(V(\phi_c(t,0,x,u)) - V(x)), \tag{6}$$

where ϕ_c is the transition map, corresponding to continuous part of the system (1)*, i.e. $\phi_c(t,0,x,u)$ is the state of system* (1) *at time t, if the state at time $t_0:=0$ was x, input u was applied and $T=\emptyset$. If in addition*

$$\varphi(s) = cs \text{ and } \alpha(s) = e^{-d}s \tag{7}$$

for some $c,d\in\mathbb{R}$, then V is called an exponential ISS-Lyapunov function with rate coefficients c,d.

Note that $\phi(\cdot,0,x,u)$ depends on the sequence T, but if we take t small enough, then $\phi(s,0,x,u)$, $s\in[0,t]$ does not depend on T because the impulse times do not have finite accumulation points. Thus, $\dot{V}_u(x)$ and the Lyapunov function V itself do not depend on the impulse time sequence.

In case $c>0$ and $d>0$ it follows easily that (1) is ISS w.r.t. all impulse time sequences (actually even a stronger result holds, see [11]) and if $c<0$ and $d<0$, then we cannot guarantee ISS of (1) w.r.t. any impulse time sequence. Hence we consider the case of $cd<0$, where stability properties depend on T. In this case input-to-state stability can be guaranteed under certain restrictions on T.

Remark 1 *Recall (see [6]) that ISS-Lyapunov functions can be defined equivalently in the following way that we use for the formulation of the small-gain theorem in Section IV. A continuous function $V : X \to \mathbb{R}_+$ is an ISS-Lyapunov function for* (1) *iff there exist $\psi_1, \psi_2 \in \mathcal{K}_\infty$, so that* (4) *holds and $\exists \gamma \in \mathcal{K}_\infty$, $\alpha \in \mathcal{P}$ and continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}$, $\varphi(0) = 0$ s.t. $\forall \xi \in U$ and $\forall u \in U_c$ with $u(0) = \xi$ it holds*

$$V(x) \geq \gamma(\|\xi\|_U) \quad \Rightarrow \quad \dot{V}_u(x) \leq -\varphi(V(x)) \qquad (8)$$

and $\forall x \in X, \xi \in U$ it holds

$$V(g(x, \xi)) \leq \max\{\alpha(V(x)), \gamma(\|\xi\|_U)\}. \qquad (9)$$

Remark 2 *Similarly, a continuous function $V : X \to \mathbb{R}_+$ is an exponential ISS-Lyapunov function for* (1) *if and only if there exist $\psi_1, \psi_2 \in \mathcal{K}_\infty$, such that* (4) *holds and $\exists \gamma \in \mathcal{K}_\infty$ and $c, d \in \mathbb{R}$ such that for all $\xi \in U$ and all $u \in U_c$ with $u(0) = \xi$ it holds*

$$V(x) \geq \gamma(\|\xi\|_U) \quad \Rightarrow \quad \dot{V}_u(x) \leq -cV(x)$$

and $\forall x \in X, \xi \in U$ it holds

$$V(g(x, \xi)) \leq \max\{e^{-d}V(x), \gamma(\|\xi\|_U)\}.$$

In contrast to continuous systems the existence of an ISS-Lyapunov function for (1) does not automatically imply ISS of the system with respect to all impulse time sequences. In order to find the set of impulse time sequences for which the system is ISS we use the FDT condition (10) from [22], where it was used to guarantee global asymptotic stability of finite-dimensional impulsive systems without inputs.
For $\theta > 0$ let $S_\theta := \{T \subset [t_0, \infty) : \ t_{i+1} - t_i \geq \theta, \ i \in \mathbb{N}\}$ be the set of impulse time sequences with distance between impulse times not less than θ.

Theorem 1 *Let V be an ISS-Lyapunov function for* (1) *and φ, α be as in Definition 4 and $\varphi \in \mathcal{P}$. If for some $\theta, \delta > 0$*

$$\int_a^{\alpha(a)} \frac{ds}{\varphi(s)} \leq \theta - \delta \quad \forall\, a > 0, \qquad (10)$$

then (1) *is ISS for all impulse time sequences $T \in S_\theta$.*

Remark 3 *If the discrete dynamics does not destabilize the system, i.e. $\alpha(a) \leq a$ for all $a \neq 0$, then the integral on the right hand side of* (10) *is non-positive for all $a \neq 0$, and the dwell-time condition* (10) *is satisfied for arbitrary small $\theta > 0$, that is the system is ISS for all impulse time sequences without finite accumulation points.*

A counterpart of Theorem 1 for the GS property is

Theorem 2 *Let all the assumptions of the Theorem 1 hold with $\delta = 0$. Then* (1) *is globally stable uniformly over S_θ.*

Now consider the case, when continuous dynamics destabilizes the system and the discrete one stabilizes it. To this end we define $\tilde{S}_\theta := \{\{t_i\}_1^\infty \subset [t_0, \infty) \ : \ t_{i+1} - t_i \leq \theta, \ i \in \mathbb{N}\}$.

Theorem 3 *Let V be an ISS-Lyapunov function for* (1) *and φ, α be as in Definition 4 with $-\varphi \in \mathcal{P}$. If for some $\theta, \delta > 0$*

$$\int_{\alpha(a)}^a \frac{ds}{-\varphi(s)} \geq \theta + \delta \quad \forall a > 0, \qquad (11)$$

then (1) *is ISS w.r.t. every sequence from $\tilde{S}_\theta$.*

Theorem 4 *Let the assumptions of the Theorem 3 hold with $\delta = 0$. Then the system* (1) *is GS uniformly over $\tilde{S}_\theta$.*

If system (1) possesses an exponential ISS-Lyapunov function a stronger result can be proved. For a given sequence of impulse times denote by $N(t, s)$ the number of jumps within the interval $(s, t]$.

Theorem 5 *Let V be an exponential ISS-Lyapunov function for* (1) *with corresponding coefficients $c \in \mathbb{R}$, $d \neq 0$. For arbitrary function $h : \mathbb{R}_+ \to (0, \infty)$, for which there exists $p \in \mathcal{L}$: $h(x) \leq p(x)$ for all $x \in \mathbb{R}_+$ consider the class $\mathcal{S}[h]$ of impulse time-sequences, satisfying the generalized average dwell-time (gADT) condition:*

$$-dN(t, s) - c(t - s) \leq \ln h(t - s) \quad \forall t > s \geq t_0. \quad (12)$$

Then the system (1) *is uniformly ISS over $\mathcal{S}[h]$.*

The flexibility in the choice of h in Theorem 5 should be used to ensure that the set $\mathcal{S}[h]$ is not empty but becomes possibly large. Theorem 5 generalizes Theorem 1 from [11], where this result for the function $h : x \mapsto e^{\mu - \lambda x}$ has been proved. For this h gADT condition resolves to the ADT condition from [12], [11]:

$$-dN(t, s) - (c - \lambda)(t - s) \leq \mu \quad \forall t > s \geq t_0. \qquad (13)$$

Condition (12) is tight, i.e., if for some sequence T the function $N(\cdot, \cdot)$ does not satisfy the condition (12) for every function h from the statement of the Theorem 5, then one can construct (see [6]) a certain system (1) which is not ISS w.r.t. this impulse time sequence T.

For the system (1) which possesses an exponential ISS-Lyapunov function we have introduced two different types of dwell-time conditions: generalized ADT condtion (12) and fixed dwell-time condition (10). Relations between ADT, gADT and FDT were discussed in [6] in detail.

IV. ISS of interconnected impulsive systems

Here we show how an ISS-Lyapunov function can be constructed for interconnected systems based on the knowledge of ISS-Lyapunov functions for subsystems.

Let a Banach space X_i be the state space of the i-th subsystem, $i = 1, \ldots, n$, and U and $U_c = PC(\mathbb{R}_+, U)$ be the space of input values and of input functions respectively.

Define $X = X_1 \times \ldots \times X_n$, which is a Banach space, endowed with the norm $\| \cdot \|_X := \| \cdot \|_{X_1} + \ldots + \| \cdot \|_{X_n}$. The input space for the i-th subsystem is $\tilde{X}_i := X_1 \times \ldots \times X_{i-1} \times X_{i+1} \times \ldots \times X_n \times U$ with the norm given by

$$\|\cdot\|_{\tilde{X}_i} := \|\cdot\|_{X_1} + \ldots + \|\cdot\|_{X_{i-1}} + \|\cdot\|_{X_{i+1}} + \ldots + \|\cdot\|_{X_n} + \|\cdot\|_U$$

and elements $\tilde{x}_i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, \xi)$.

Consider n interconnected impulsive subsystems for $i = 1, \dots, n$

$$\begin{aligned} \dot{x}_i(t) =& A_i x_i(t) + f_i(x_1(t), \dots, x_n(t), u(t)),\ t \notin T, \\ x_i(t) =& g_i(x_1^-(t), \dots, x_n^-(t), u^-(t)),\ t \in T. \end{aligned} \tag{14}$$

Here A_i denotes the generator of a C_0-semigroup on X_i, $f_i, g_i : X \times U \to X_i$, and we assume that for each subsystem there exists a unique and forward-complete solution. Note that $T = \{t_1, \dots, t_k, \dots\}$ is assumed to be the same for all subsystems. This allows to consider the interconnection (14) as one large system by setting $x = (x_1, \dots, x_n)^T$, $f(x,u) = (f_1(x,u), \dots, f_n(x,u))^T$, $g(x,u) = (g_1(x,u), \dots, g_n(x,u))^T$ and $A := \operatorname{diag}(A_1, \dots, A_n)$ with the domain $D(A) = D(A_1) \times \ldots \times D(A_n)$. Clearly, A is the generator of a C_0-semigroup on X. Now the interconnection (14) can be written as one system

$$\left\{ \begin{array}{l} \dot{x}(t) = Ax(t) + f(x(t), u(t)), \quad t \notin T \\ x(t) = g(x^-(t), u^-(t)), \quad t \in T. \end{array} \right. \tag{15}$$

According to Remark 1 for the i-th subsystem in (14) the definition of an ISS-LF can be written as follows. A continuous function $V_i : X_i \to \mathbb{R}_+$ is an ISS-Lyapunov function for the i-th subsystem of (14), if three properties hold:

1) There exist functions $\psi_{i1}, \psi_{i2} \in \mathcal{K}_\infty$, such that:

$$\psi_{i1}(\|x_i\|_{X_i}) \leq V_i(x_i) \leq \psi_{i2}(\|x_i\|_{X_i}) \quad \forall x_i \in X_i.$$

2) There exist $\chi_{ij}, \chi_i \in \mathcal{K}$, $j = 1, \dots, n$, $\chi_{ii} := 0$ and $\varphi_i \in \mathcal{P}$, so that for all $x_i \in X_i$, for all $\tilde{x}_i \in \tilde{X}_i$ and for all $v \in PC(\mathbb{R}_+, \tilde{X}_i)$ with $v(0) = \tilde{x}_i$ from

$$V_i(x_i) \geq \max\{\max_{j=1}^{n} \chi_{ij}(V_j(x_j)), \chi_i(\|\xi\|_U)\} \tag{16}$$

it follows

$$\dot{V}_i(x_i(t)) \leq -\varphi_i\left(V_i(x_i(t))\right), \tag{17}$$

where

$$\dot{V}_i(x_i) = \overline{\lim_{t \to +0}} \frac{1}{t}(V_i(\phi_{i,c}(t, 0, x_i, v))) - V_i(x_i))$$

and $\phi_{i,c} : \mathbb{R}_+ \times \mathbb{R}_+ \times X_i \times PC(\mathbb{R}_+, \tilde{X}_i) \to X_i$ is the solution (transition map) of the i-th subsystem of (14) for the case $T = \emptyset$.

3) There exists $\alpha_i \in \mathcal{P}$, such that for gains defined above and for all $x \in X$ and for all $\xi \in U$ it holds

$$V_i(g_i(x, \xi)) \leq \max\{\alpha_i(V_i(x_i)), \max_{j=1}^{n} \chi_{ij}(V_j(x_j)), \chi_i(\|\xi\|_U)\}. \tag{18}$$

If $\varphi_i(y) = c_i y$ and $\alpha_i(y) = e^{-d_i} y$ for all $y \in \mathbb{R}_+$, then V_i is called an exponential ISS-Lyapunov function for the i-th subsystem of (14) with rate coefficients $c_i, d_i \in \mathbb{R}$.

Lyapunov gains χ_{ij} characterize the interconnection structure of subsystems. Let us introduce the gain operator $\Gamma : \mathbb{R}_+^n \to \mathbb{R}_+^n$ defined by

$$\Gamma(s) := \left(\max_{j=1}^{n} \chi_{1j}(s_j), \dots, \max_{j=1}^{n} \chi_{nj}(s_j) \right),\ s \in \mathbb{R}_+^n. \tag{19}$$

We recall the notion of Ω-path (see [7], [21]), useful for investigation of stability of interconnected systems and for a construction of a Lyapunov function of the whole system.

Definition 5 *A function $\sigma = (\sigma_1, \dots, \sigma_n)^T : \mathbb{R}_+^n \to \mathbb{R}_+^n$, where $\sigma_i \in \mathcal{K}_\infty$, $i = 1, \dots, n$ is called an Ω-path, if it possesses the following properties:*

1) *σ_i^{-1} is locally Lipschitz continuous on $(0, \infty)$;*
2) *for every compact set $P \subset (0, \infty)$ there are finite constants $0 < K_1 < K_2$ such that for all points of differentiability of σ_i^{-1} we have*

$$0 < K_1 \leq (\sigma_i^{-1})'(r) \leq K_2, \quad \forall r \in P;$$

3)

$$\Gamma(\sigma(r)) < \sigma(r)\ \forall r > 0. \tag{20}$$

If operator Γ satisfies the small-gain condition

$$\Gamma(s) \not\geq s\ \forall\ s \in \mathbb{R}_+^n \setminus \{0\}, \tag{21}$$

then Ω-path exists [7] and provides a scaling of Lyapunov functions of subsystems to derive a Lyapunov function for the whole interconnection:

Theorem 6 *Let V_i be the ISS-Lyapunov function for i-th subsystem of* (14) *with corresponding gains χ_{ij}. If the corresponding operator Γ defined by* (19) *satisfies the small-gain condition* (21)*, then an ISS-Lyapunov function V for the whole system is given by*

$$V(x) := \max_i \{\sigma_i^{-1}(V_i(x_i))\}, \tag{22}$$

where $\sigma = (\sigma_1, \dots, \sigma_n)^T$ is an Ω-path. The Lyapunov gain of the whole system can be chosen as

$$\chi(r) := \max_i \sigma_i^{-1}(\chi_i(r)). \tag{23}$$

Remark 4 *Our small-gain theorem has been formulated for Lyapunov functions in the form used in Remark 1. According to Remark 1 this formulation can be transformed to the standard formulation, and from the proof it is clear, that the functions α and φ remain the same after the transformation.*

Now consider the case of exponential ISS-Lyapunov functions. If an exponential ISS-Lyapunov function for a system (1) is given, then Theorem 5 provides us with the tight estimates of the set of impulse time sequences, w.r.t. which the system (1) is ISS and hence exponential ISS-Lyapunov functions are "more valuable", than the general ones.

We may hope, that if ISS-Lyapunov functions for all subsystems of (14) are *exponential*, then the expression (22) at least for certain type of gains provides an *exponential* ISS-Lyapunov function for the whole system. Here we provide a small-gain theorem of this type.

Firstly note the following fact due to [15]

Lemma 5 *Let operator Γ satisfy small-gain condition. Then for arbitrary $a \in \mathbb{R}_+^n$: $a_j > 0$, for $j = 1, \dots, n$ the function*

$$\sigma(t) = Q(at), \forall t \geq 0 \tag{24}$$

satisfies

$$\Gamma(\sigma(r)) \leq \sigma(r),\ \forall r > 0. \tag{25}$$

Here $Q : \mathbb{R}_+^n \to \mathbb{R}_+^n$ is defined by

$$Q(x) := MAX\{x, \Gamma(x), \Gamma^2(x), \dots, \Gamma^{n-1}(x)\},$$

with $\Gamma^n(x) = \Gamma \circ \Gamma^{n-1}(x)$, for all $n \geq 2$. The function MAX for all $h_i \in \mathbb{R}^n$, $i = 1, \ldots, m$ is defined by

$$z = MAX\{h_1, \ldots, h_m\} \in \mathbb{R}^n, \quad z_i := \max\{h_{1i}, \ldots, h_{mi}\}.$$

Define the following class of functions

$$P := \{f : \mathbb{R}_+ \to \mathbb{R}_+ : \exists a \geq 0,\ b > 0 :\ f(s) = as^b\ \forall s \in \mathbb{R}_+\}.$$

Theorem 7 *Let V_i be an eISS Lyapunov function for the i-th subsystem of* (14) *with corresponding gains χ_{ij}, $i = 1, \ldots, n$. Let also $\chi_{ij} \in P$ and let the small-gain condition* (21) *hold. Then $V : X \to \mathbb{R}_+$, defined by* (22)*, with σ given by* (24)*, is an eISS Lyapunov function for* (15).

Remark 6 *The obtained exponential ISS-Lyapunov function can be transformed to the implication form with the help of Remark 2. Then Theorem 5 can be used in order to verify ISS of the system* (15).

V. An example

To illustrate the application of the above results we consider two parabolic equations with impulses:

$$\begin{cases} \frac{\partial x_1}{\partial t} = q_1 \frac{\partial^2 x_1}{\partial z^2} + x_2^2, \quad z \in (0,\pi),\ t \neq T, \\ x_1(z,t) = g_1(x_1^-(z,t)) := e^{-\tilde{d}_1} \cdot x_1^-(z,t),\ t \in T \\ x_1(0,t) = x_1(\pi,t) = 0; \\ \frac{\partial x_2}{\partial t} = q_2 \frac{\partial^2 x_2}{\partial z^2} + \sqrt{|x_1|}, \quad z \in (0,\pi),\ t \neq T, \\ x_2(z,t) = g_2(x_2^-(z,t)) := e^{-\tilde{d}_2} \cdot x_2^-(z,t),\ t \in T \\ x_2(0,t) = x_2(\pi,t) = 0. \end{cases} \tag{26}$$

Here $q_1, q_2 > 0$ are diffusion coefficients and $\tilde{d}_1, \tilde{d}_2 < 0$ describe how much the state increases at the jumps.

This system can model a chemical reaction network with two reagents, whose densities at a point z and at time t are given by $x_i(z,t)$, $i = 1, 2$. The impulse means an instantaneous increase of a concentration of substances within the whole domain of a reactor.

We assume that $x_1 \in L_2(0,\pi) =: X_1$ and $x_2 \in L_4(0,\pi) := X_2$. The state of the whole system (26) is $X = X_1 \times X_2$.

We choose the following Lyapunov functions for subsystems 1 and 2 respectively:

$$V_1(x_1) = \int_0^\pi x_1^2(z)dz = \|x_1\|^2_{L_2(0,\pi)},$$

$$V_2(x_2) = \int_0^\pi x_2^4(z)dz = \|x_2\|^4_{L_4(0,\pi)}.$$

Assume for a while that x_i are twice continuously differentiable functions. Consider the Lie derivative of V_1:

$$\begin{aligned} \dot{V}_1(x_1) &= 2\int_0^\pi x_1(z,t)\left(q_1 \frac{\partial^2 x_1}{\partial z^2}(z,t) + x_2^2(z,t)\right)dz \\ &\leq -2q_1 \left\|\frac{dx_1}{dz}\right\|^2_{L_2(0,\pi)} + 2\|x_1\|_{L_2(0,\pi)}\|x_2\|^2_{L_4(0,\pi)}. \end{aligned}$$

In the last estimate we have used the Cauchy-Schwarz inequality.

According to the Friedrichs' inequality [10, p.85]

$$\int_0^\pi \left(\frac{\partial^2 x}{\partial z^2}\right)^2 dz \geq \int_0^\pi \left(\frac{\partial x}{\partial z}\right)^2 dz$$

we obtain the estimate

$$\begin{aligned} \dot{V}_1(x_1) &\leq -2q_1\|x_1\|^2_{L_2(0,\pi)} + 2\|x_1\|_{L_2(0,\pi)}\|x_2\|^2_{L_4(0,\pi)} \\ &= -2q_1 V_1(x_1) + 2\sqrt{V_1(x_1)}\sqrt{V_2(x_2)}. \end{aligned}$$

Take

$$\chi_{12}(r) = \frac{1}{a} r,\ \forall r > 0,$$

with arbitrary $a > 0$. We obtain

$$V_1(x_1) \geq \chi_{12}(V_2(x_2)) \ \Rightarrow\ \frac{d}{dt}V_1(x_1) \leq -2(q_1 - a^{\frac{1}{2}})V_1(x_1).$$

The derivation was made under assumption that x_1, x_2 are twice continuously differentiable functions. For general $x_1 \in L_2(0,\pi)$ the above estimate holds due to the density argument.

Now consider the Lie derivative of V_2:

$$\begin{aligned} \dot{V}_2(x_2) =& 4\int_0^\pi x_2^3(z,t)\Big(q_2 \frac{\partial^2 x_2}{\partial z^2}(z,t) + \sqrt{|x_1(z,t)|}\Big)dz \\ \leq& -3q_2 \int_0^\pi \Big(\frac{\partial}{\partial z}(x_2^2)\Big)^2 dz + 4\int_0^\pi x_2^3(z,t)|x_1(z,t)|^{\frac{1}{2}}dz \end{aligned}$$

Applying Friedrichs' inequality for the first term and the Hölder's inequality for the last one we obtain

$$\dot{V}_2(x_2) \leq -3q_2 V_2(x_2) + 4(V_2(x_2))^{3/4}(V_1(x_1))^{1/4}.$$

Let

$$\chi_{21}(r) = \frac{1}{b} r,\ \forall r > 0,$$

where $b > 0$ is an arbitrary constant. It holds the implication

$$V_2(x_2) \geq \chi_{21}(V_1(x_1)) \quad \Rightarrow \quad \dot{V}_2(x_2) \leq -(3q_2 - 4b^{\frac{1}{4}})V_2(x_2).$$

On the jumps the following inequalities hold

$$V_1(g_1(x_1)) = V(e^{-\tilde{d}_1}x_1) \leq e^{-2\tilde{d}_1}V_1(x_1),$$

$$V_2(g_2(x_2)) = V(e^{-\tilde{d}_2}x_2) \leq e^{-4\tilde{d}_2}V_2(x_2).$$

Hence V_1 and V_2 are exponential ISS Lyapunov functions for the subsystems with rate coefficients $c_1 = 2(q_1 - a^{\frac{1}{2}})$, $d_1 = 2\tilde{d}_1$ and $c_2 = 3q_2 - 4b^{\frac{1}{4}}$, $d_2 = 4\tilde{d}_2$ respectively.

Since the discrete dynamics is destabilizing, the continuous dynamics has to be stabilizing in order to assure ISS for each of the impulsive systems for some classes of impulse time sequences. This leads us to the conditions $a^{\frac{1}{2}} < q_1$ and $b < (\frac{3q_2}{4})^4$. The classes of impulse time sequences for which the system is ISS, can be found from gADT condition (12). In particular, the average density of jumps for i-th subsystem has to be less than $\frac{c_i}{-d_i}$.

To assure stability of the interconnection we apply the small-gain condition

$$\chi_{12} \circ \chi_{21} < \mathrm{Id} \quad \Leftrightarrow \quad ab > 1. \tag{27}$$

Assuming that this condition is satisfied we construct an ISS Lyapunov function for the whole interconnection. To this end take an arbitrary constant k such that $\frac{1}{b} < \frac{1}{k} < a$. Then Ω-path, corresponding to the above gains can be chosen as

$$\sigma_1(r) = r, \quad \sigma_2(r) = \frac{1}{k} r,\ \forall r \geq 0$$

and an ISS-Lyapunov function for the interconnection, constructed by small-gain design, is given by

$$V(x) = \max\{V_1(x_1), kV_2(x_2)\},$$

where $x = (x_1, x_2)$. The estimate for discrete dynamics for the whole system can be taken as follows

$$V(g(x)) = \max\{V_1(e^{-\tilde{d}_1}x_1), kV_2(e^{-\tilde{d}_2}x_2)\} \leq e^{-d}V(x),$$

with $\quad d := \min\{d_1, d_2\} < 0.$

The estimate of the continuous dynamics for V is

$$\frac{d}{dt}V(x) \leq -\min\{c_1, c_2\}V(x). \tag{28}$$

This shows that we have constructed an exponential ISS Lyapunov function for the interconnection.

The classes of impulse time sequences for which the system (26) is ISS, can be found from gADT condition (12). In particular, the average density of jumps for (26) has to be less than $\frac{c}{-d}$. We are going to maximize this value. Since we are not able to influence coefficient d, we will seek a and b which maximize c.

It is easy to see that in order to maximize $c := \min\{c_1, c_2\}$, we have to choose $c_1 = c_2$. Then $a = \frac{1}{4}(4b^{\frac{1}{4}} + 2q_1 - 3q_2)^2$ and b we can find using (27) we obtain the inequality $(4b^{\frac{1}{4}} + 2q_1 - 3q_2)^2 b > 4$.

E.g. for $q_1 = 2$ and $q_2 = 3$ a good choice for b is $b \approx 4.79$ and V is an exponential ISS-Lyapunov function for (26) with rate coefficients $d = \min\{d_1, d_2\}$ and $c \approx 3.082$.

The ISS-Lyapunov function for an interconnection is constructed, and one can apply Theorem 5 in order to obtain the classes of impulse time sequences for which the interconnection is ISS.

VI. Discussion of results and future work

Our example demonstrates how stability properties of an interconnection can be studied and in particular, how a Lyapunov function can be constructed, provided that the small-gain condition is satisfied. Moreover we see the interplay between the small-gain and dwell-time conditions. We have demonstrated that there is a certain flexibility for the choice of gains (a and b) of the subsystems. Certainly this choice depends on the systems properties and on the choice of the corresponding Lyapunov functions for subsystems, moreover the function spaces taken for the input signals play an important role for this choice. Further we have required that these gains satisfy the small-gain condition, which restricts the choice of a, b and hence the diffusion rates q_1, q_2. To assure the ISS property we then need to require a dwell-time condition that restricts the set of possible impulse time sequences. To obtain the least restriction for this set we have in general to solve an optimization problem over all a, b such that the small-gain condition is satisfied and the average density of jumps is maximized.

In this example we were able to derive and to use linear gains. The situation becomes more challenging if the gains are nonlinear. The relation between the small-gain and the dwell-time conditions becomes more involved, especially if we consider interconnections of a large number of subsystems. In the future we are going to investigate this dependence in detail.

In this paper we have assumed that the impulse time sequences are the same for each subsystem. Hence another important direction of research is to investigate the case of different impulse time sequences for subsystems. This case is very relevant for applications.

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