Revisiting stability of positive linear discrete-time systems

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Abstract We prove small-gain type criteria of exponential stability for positive linear discrete-time systems in ordered Banach spaces that have a normal and generating positive cone. Such criteria play an important role in the finite-dimensional systems theory but are rather unexplored in the infinite-dimensional setting, yet. Furthermore, we show that our stability criteria can be considerably strengthened if the cone has non-empty interior or if the operator inducing the discrete-time system is quasi-compact.

Keywords: positive systems, discrete-time systems, stability, small-gain condition, linear systems.

1. INTRODUCTION

Positive systems frequently occur in the modeling, analysis, and control of dynamical systems, for instance, in chemical engineering, compartmental systems, and ecological systems (Farina and Rinaldi (2000)). Besides being interesting in their own right, positive systems have been instrumental in establishing stability properties of control systems, which are not positive per se. In particular, within the small-gain approach (Dashkovskiy et al. (2007, 2010)), the stability criteria for large-scale interconnected systems are given in terms of "small-gain" and "no-increase" conditions. For linear systems, they are equivalent to exponential stability of the underlying discrete-time system, see (Rüffer, 2007, Lemma 2.0.1). A decisive tool for the proof of such criteria is the celebrated Perron–Frobenius theorem.

For infinite-dimensional systems set in the realm of ordered Banach spaces, the situation is more complex. Stability of positive and non-positive linear operators and semigroups has already been studied for many decades, as is documented, for instance, in the monographs van Neerven (1996); Emel'yanov (2007); Eisner (2010). Yet, many stability criteria that turned out to be useful guidelines in the finite-dimensional path from the linear to the nonlinear case – in particular, so-called small-gain theorems and non-increasing conditions – have only been partially explored in the infinite-dimensional setting so far (although see (Krasnoselskiy, 1964, Theorem 3.13 on p. 120 and Theorem 4.6 on p. 130), where no-increase conditions have been used for the study of fixed points of monotone operators). This has several reasons. On the one hand, the Krein–Rutman theorem, which is a (partial) infinite-dimensional extension of the Perron–Frobenius theorem, requires the operator under consideration to be quasi-compact, which is a rather strong assumption. On the other hand, many finite-dimensional stability notions exhibit – due to the compactness of the closed unit ball – some intrinsic uniformity, which is essential to characterize properties such as exponential stability. The lack of this uniformity in the infinite-dimensional setting breaks most known finite-dimensional criteria.

Contribution. In this paper, we study discrete-time systems, playing a prominent role in the modeling, analysis, control, and numerics of dynamical systems (Agarwal (2000); LaSalle (2012)). In Section 4, we introduce several novel stability properties, most notably the uniform small-gain condition, and characterize the exponential stability of positive linear discrete-time systems in terms of such properties (Theorem 4). Our assumptions on the ordered Banach space are not particularly restrictive; we merely assume that the cone is normal and generating, which is satisfied for many important classes of spaces. Some, though not all, of the equivalences in Theorem 4 have been shown in Mironchenko et al. (2021a) to hold even in the nonlinear case; these results are related to the input-to-state stability of control systems with inputs and to so-called small-gain theorems.

In Section 5, we devote special attention to ordered Banach spaces whose cone has non-empty interior. Subeigenvectors of operators inducing discrete-time positive exponentially stable systems are a key to the construction of Lyapunov functions for networks of stable systems, see Dashkovskiy et al. (2010) for the case of finite networks. Our results in Section 5 pave the way to extending these results to infinite networks; some results of this type have already been put to use (Kawan et al. (2021)).

In Section 6, we treat systems given by quasi-compact operators. By using the Krein–Rutman theorem for these operators, we show that the assumption of quasi-compactness allows us to extend most of the finite-dimensional results to the infinite-dimensional setting.

Nonlinear versions of small-gain type conditions studied in this paper have been applied for the analysis of nonlinear systems in Mironchenko et al. (2021a). Some of the results in Section 5 have been extended to the case of homogeneous and subadditive operators in Mironchenko et al. (2021b) and applied for the construction of ISS Lyapunov functions for infinite networks of input-to-state stable systems with homogeneous and subadditive gain operators. Nevertheless, in the nonlinear case, many problems remain open.

This is an abridged conference version of the paper Glück and Mironchenko (2021). Here we state three results characterizing the exponential stability of discrete-time systems. Due to the page limits, we give a proof of only one of these results. For all other proofs and detailed discussions, we refer to Glück and Mironchenko (2021). The journal version also provides an overview of other known characterizations of exponential stability. This supplies our results with some classical context but also makes it possible to treat the paper Glück and Mironchenko (2021) as a survey. The continuous-time counterparts of some of the results of this work are proved in Glück and Mironchenko (2022).

Notation. We use the conventions $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$.

If X is a Banach space, we denote the space of bounded linear operators on X by $\mathcal{L}(X)$, and we denote the dual space – i.e., the space of bounded linear functionals on X – by X'. For $x' \in X'$ and $x \in X$ we use the common notation $\langle x', x\rangle := x'(x)$. The identity operator on a Banach space will be denoted by id (if the space is clear from the context).

If the underlying scalar field of the Banach space X is complex, we denote the *spectrum* an operator $T \in \mathcal{L}(X)$ by $\sigma(T)$. The *spectral radius* of T is denoted by

$$r(T) := \sup\{|\lambda| : \lambda \in \sigma(T)\} \in [0, \infty).$$

If $\lambda \in \mathbb{C}$ is located in the complement of the spectrum (i.e., in the so-called *resolvent set* of T), the operator $(\lambda \operatorname{id} - T)^{-1}$ is called the *resolvent* of T at λ.

For bounded linear operators on real Banach spaces, spectral properties are defined by means of a *complexification.*

2. SETTING THE STAGE: ORDERED BANACH SPACES AND POSITIVE OPERATORS

In this section, we recall some background information on ordered Banach spaces that will be needed throughout the article.

Ordered Banach spaces. By an *ordered Banach space* we mean a pair (X, X^+) where X is a real Banach space, and $X^+ \subseteq X$ is a non-empty closed set such that the set

$$\alpha X^+ + \beta X^+ := \{\alpha x + \beta y : x, y \in X^+\}$$

is a subset of X^+. For all scalars $\alpha, \beta \geq 0$ and such that $X^+ \cap (-X^+) = \{0\}$. The set X^+ is called the *positive cone* in X.

The positive cone of an ordered Banach space (X, X^+) induces a partial order $\leq$ on X which is given by $x \leq y$ if and only if $y - x \in X^+$. In particular, $x \geq 0$ if and only if $x \in X^+$. The partial order $\leq$ is compatible with addition and with multiplication by scalars $\alpha \geq 0$.

Generating and normal cones. Let (X, X^+) be an ordered Banach space. The cone X^+ is called *total* (or *spatial*) if the vector subspace $X^+ - X^+ = \{x - y : x, y \in X^+\}$ of X is dense in X. The cone is called *generating* (or the space X is called *directed*) if we even have $X^+ - X^+ = X$. In other words, the cone is generating if and only if each vector $x \in X$ can be decomposed as $x = y - z$ for two vectors $y, z \in X^+$.

Moreover, if the cone X^+ is generating, then there exists a number $M > 0$ with the following property: for each $x \in X$ there exist $y, z \in X^+$ such that

$$x = y - z \qquad \text{and} \qquad \|y\|, \|z\| \leq M\|x\|; \tag{1}$$

see for instance (Aliprantis and Tourky, 2007, Theorem 2.37(1) and (3)). The cone X^+ is called *normal* if there exists a number $C > 0$ such that we have

$$\|x\| \leq C\|y\| \qquad \text{whenever } 0 \leq x \leq y \tag{2}$$

in X. The cone is normal if and only if there exists an equivalent norm $\|\cdot\|'$ on X which is *monotone* in the sense that $\|x\|' \leq \|y\|'$ whenever $0 \leq x \leq y$; see for instance (Aliprantis and Tourky, 2007, Theorem 2.38(1) and (2)).

Finally, for each set $S \subset X$ denote by $\operatorname{int}(S)$ the *topological interior* of S. If $\operatorname{int}(X^+) \neq \emptyset$, then we say that the cone X^+ has a non-empty interior. Note that a cone with non-empty interior is automatically generating.

The distance to the cone. For a subset S and a vector x in a Banach space X, we denote by

$$\operatorname{dist}(x, S) := \inf\{\|x - y\| : y \in S\}$$

the distance from x to S. If (X, X^+) is an ordered Banach space, we will need the distance of points to the positive cone X^+. Due to the specific properties of cones, the distance function $\operatorname{dist}(\cdot, X^+)$ is quite nicely behaved; in particular, it is not difficult to see that we have for all $x, y \in X$ and all $\alpha \in [0, \infty)$

$$\operatorname{dist}(x + y, X^+) \leq \operatorname{dist}(x, X^+) + \operatorname{dist}(y, X^+),$$
$$\text{and} \qquad \operatorname{dist}(\alpha x, X^+) = \alpha \operatorname{dist}(x, X^+).$$

Examples of ordered Banach spaces. As explained next, classical sequence and function spaces constitute several important classes of ordered Banach spaces.

Example 2.1. Let $X = \ell_p := \{x \in \mathbb{R}^{\mathbb{N}} : \|x\|_{\ell_p} < \infty\}$ for $p \in [1, \infty]$, with the norms $x = (x_n)_{n\in\mathbb{N}} \mapsto \|x\|_{\ell_p} := \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p}$ for $p < \infty$ and $x \mapsto \|x\|_{\ell_\infty} := \sup_{n=1}^{\infty} |x_n|$. We endow each ℓ_p-space with the cone

$$\ell_p^+ := \{(x_n)_{n\in\mathbb{Z}_+} \in \ell_p : x_n \geq 0 \text{ for all } n \in \mathbb{N}\}.$$

Then (ℓ_p, ℓ_p^+) is an ordered Banach space, and the cone ℓ_p^+ is generating and normal. If $p = \infty$, then the cone has non-empty interior, whereas $\operatorname{int}(\ell_p^+) = \emptyset$ for $p \in [1, \infty)$.

Example 2.2. (Spaces of continuous functions). If Ω is a topological space, then the space $C_b(\Omega)$ of all real-valued and bounded continuous functions, endowed with the supremum norm and the cone of all those functions in $C_b(\Omega)$ that are ≥ 0 everywhere on Ω, is an ordered Banach space with generating and normal cone. Moreover, the cone in this space has non-empty interior.

Similarly, if L is a locally compact Hausdorff space and $C_0(L)$ denotes that closed subspace of $C_b(L)$ consisting of functions that vanish at infinity, then $C_0(L)$, with the norm and cone inherited from $C_b(L)$, is an ordered Banach space with normal and generating cone. The interior of the cone in $C_0(L)$ is non-empty if and only if L is compact (in which case we have $C_0(L) = C_b(L) = C(L)$, where $C(L)$ denote the space of all continuous real-valued functions on L).

Positive operators. Let (X, X^+) and (Y, Y^+) be ordered Banach spaces. A linear mapping $A : X \to Y$ is called *positive*, which we denote as $A \geq 0$, if $AX^+ \subseteq Y^+$. A linear mapping A is positive if and only if it respects the order relation (i.e. $Ax_1 \leq Ax_2$ whenever $x_1 \leq x_2$).

We are particularly interested in *bounded* linear operators. Interestingly though, this assumption is often redundant: if the cone X^+ is generating, then every positive linear operator $A : X \to Y$ is automatically bounded (Aliprantis and Tourky, 2007, Theorem 2.32).

Duality of ordered Banach spaces. Let (X, X^+) be an ordered Banach space. The subset

$$(X')^+ := \{x' \in X' : \langle x', x\rangle \geq 0 \text{ for all } x \in X^+\}$$

of the dual space X' is called the *dual wedge* of X^+. The elements of $(X')^+$ are called the *positive functionals* on X.

The dual wedge is also closed (even weak*-closed), convex and invariant with respect to multiplication by non-negative scalars. The dual wedge $(X')^+$ is a cone – i.e., its intersection with $-(X')^+$ is $\{0\}$ – if and only if the cone X^+ is total in X.

3. STABILITY FOR LINEAR OPERATORS

For a Banach space X and an operator $T \in \mathcal{L}(X)$, consider the *discrete-time system* induced by T,

$$x(k+1) = Tx(k) \quad \text{for all } k \in \mathbb{Z}_+. \tag{3}$$

We are interested in whether the solutions to this system converge uniformly to 0 as $k \to \infty$; this is made precise in the following definition.

Definition 3.1. For a Banach space X and an operator $T \in \mathcal{L}(X)$, the system (3) is called

(a) *uniformly asymptotically stable*, if there is a sequence of real numbers $0 \leq a_k \to 0$ such that
$$\left\|T^k x\right\| \leq a_k \|x\| \quad \text{for all } x \in X,\ k \in \mathbb{Z}_+;$$
(b) *uniformly exponentially stable* (or *uniformly power stable*), if there exist real numbers $a \in [0,1)$ and $M > 0$ such that
$$\left\|T^k x\right\| \leq Ma^k \|x\| \quad \text{for all } x \in X \text{ and all } k \in \mathbb{Z}_+;$$
(c) *uniformly weakly attractive* if, for each $r > 0$ and each $\varepsilon > 0$, there is a time τ with the following property: for each $x \in X$ of norm $\|x\| \leq r$ there is $k \leq \tau$ such that $\left\|T^k x\right\| \leq \varepsilon$.

A well-known result in the stability theory of discrete-time systems (3) is the following (see, e.g., (Przyluski, 1988, Lemma 2.1 and Theorem 2.2), (Mironchenko, 2017, Proposition 5.1)):

Proposition 3.1. Let X be a Banach space and $T \in \mathcal{L}(X)$. The following assertions are equivalent:

(i) We have $r(T) < 1$.
(ii) The system (3) is uniformly asymptotically stable.
(iii) The system (3) is uniformly exponentially stable.
(iv) The system (3) is uniformly weakly attractive.

4. GENERAL STABILITY CRITERIA FOR POSITIVE SYSTEMS

From now on we assume that T is a positive linear operator on an *ordered Banach space*. Positivity of the operator T does not simplify the criteria for uniform exponential stability considered in Proposition 3.1. On the other hand, positivity allows to show diverse characterizations of *small-gain* type, as shown in our main result stated next:

Theorem 4.1. Let (X, X^+) be an ordered Banach space with generating and normal cone and let $T \in \mathcal{L}(X)$ be positive. Then the following assertions are equivalent:

(i) *Uniform exponential stability:* The system (3) satisfies the equivalent criteria of Proposition 3.1, i.e., we have $r(T) < 1$.
(ii) *Positivity of the resolvent at* 1*:* The operator $\operatorname{id} - T : X \to X$ is bijective and its inverse $(\operatorname{id} - T)^{-1}$ is positive.[1]
(iii) *Monotone bounded invertibility property:* There exists a number $c \geq 0$ such that
$$(\operatorname{id} - T)x \leq y \qquad \Rightarrow \qquad \|x\| \leq c\,\|y\|$$
holds for all $x, y \in X^+$.
(iv) *Uniform small-gain condition:* There is a number $\eta > 0$ such that
$$\operatorname{dist}\big((T - \operatorname{id})x, X^+\big) \geq \eta\,\|x\|$$
for each $x \in X^+$.
(v) *Robust small-gain condition:* There exists a number $\varepsilon > 0$ such that
$$(T + P)x \not\geq x \tag{4}$$
for every $0 \neq x \in X^+$ and for every positive operator $P \in \mathcal{L}(X)$ of norm $\|P\| \leq \varepsilon$.
(vi) *Rank-*1 *robust small-gain condition:* There exists a number $\varepsilon > 0$ such that
$$(T + P)x \not\geq x$$
for every $0 \neq x \in X^+$ and for every positive operator $P \in \mathcal{L}(X)$ of rank 1 and of norm $\|P\| \leq \varepsilon$.

Proof. "(i) ⇒ (ii)" If the $r(T) < 1$, then $\operatorname{id} - T$ is clearly invertible, and it follows from the Neumann series representation of the resolvent that

$$(\operatorname{id} - T)^{-1} = \sum_{k=0}^{\infty} T^k \geq 0;$$

the inequality at the end follows by applying the operator series to vectors $x \in X^+$ and using that X^+ is closed.

"(ii) ⇒ (iii)" Let $C \in [0, \infty)$ be the normality constant from inequality (2). If $x, y \in X^+$ and $(\operatorname{id} - T)x \leq y$ we obtain from the positivity of the resolvent $(\operatorname{id} - T)^{-1}$ that $x \leq (\operatorname{id} - T)^{-1}y$, and hence

$$\|x\| \leq C\left\|(\operatorname{id} - T)^{-1}\right\|\|y\|.$$

[1] Note that as $\operatorname{id} - T$ is invertible and bounded, then $(\operatorname{id} - T)^{-1}$ is closed, and since $\operatorname{id} - T$ is surjective, $(\operatorname{id} - T)^{-1}$ is bounded by a closed graph theorem. Thus, $1 \in \rho(T)$ and $(\operatorname{id} - T)^{-1}$ is indeed a resolvent.

This proves the monotone bounded invertibility property with $c = C \left\|(\mathrm{id} - T)^{-1}\right\|$.

"(iii) $\Rightarrow$ (iv)" Let $c > 0$ be as in (iii). Since the cone X^+ is assumed to be generating, we can find a constant $M > 0$ as in the decomposition property (1).

Now fix $x \in X^+$ and let $\varepsilon > 0$ be arbitrary; we are going to show that

$$\operatorname{dist}\big((T - \mathrm{id})x, X^+\big) \geq \frac{1}{cM} \|x\| - \varepsilon. \tag{5}$$

For convenience, denote $a := (T - \mathrm{id})x$. By the definition of the distance, we can find a vector $z \in X^+$ such that $\|a - z\| \leq \operatorname{dist}(a, X^+) + \varepsilon$, and we set $y := a - z$. Now we decompose the vector y as

$$y = u - v,$$

where u, v are in X^+ and satisfy the norm estimate

$$\|u\|, \|v\| \leq M \|y\| \leq M \operatorname{dist}(a, X^+) + M\varepsilon.$$

For the vector $(\mathrm{id} - T)x$ we have the estimate

$$(\mathrm{id} - T)x = -a = -y - z = v - u - z \leq v,$$

so the monotone bounded invertibility property from (iii) implies that

$$\|x\| \leq c \|v\| \leq cM\big(\operatorname{dist}(a, X^+) + \varepsilon\big).$$

So we have indeed shown the claimed inequality (5). Since ε was arbitrary, this gives us the uniform small-gain condition with $\eta = \frac{1}{cM}$.

"(iv) $\Rightarrow$ (v)" Choose $\varepsilon = \eta/2$, where η is the number from (iv). Now, let x be a non-zero element of X^+ and let $P \in \mathcal{L}(X)$ be a positive linear operator of norm at most ε. We have to show that $Tx + Px \not\geq x$, and to this end we may – and will, in order to simplify the notation – assume that x has norm 1.

For each vector $z \in X^+$, we know from the uniform small-gain condition that $Tx - x$ has distance at least η from z. Since $\|Px\| \leq \eta/2$, it follows that $Tx + Px - x$ still has distance at least $\eta/2$ from z, so

$$\operatorname{dist}(Tx + Px - x, X^+) \geq \frac{\eta}{2} > 0.$$

In particular, $Tx + Px - x$ is not in the cone, so $Tx + Px \not\geq x$.

"(v) $\Rightarrow$ (vi)" This implication is obvious.

"(vi) $\Rightarrow$ (i)" Let $\varepsilon > 0$ be as in (vi). We argue by contraposition: assume that $r(T) \geq 1$.

By (Glück and Mironchenko, 2021, Lemma 3.5), $r(T)$ is an approximate eigenvalue of T and there exists a corresponding approximate eigenvector $(x_n)_{n\in\mathbb{N}}$ in X^+; more precisely, this means that each vector x_n has norm 1 and that

$$(T - r(T)\,\mathrm{id})x_n \to 0.$$

Since the cone in our space is generating, we can choose a number $M \in [0, \infty)$ as in the decomposition property (1). Since the dual cone $(X')^+$ in X' is generating, too (due to the normality of X^+, see (Krasnosel'skii et al., 1989, Theorem 4.5)), there also exists a constant $M' \in [0, \infty)$ with the same property for the dual cone $(X')^+$.

For each index n we can decompose the vector $(T - r(T)\,\mathrm{id})x_n$ as

$$(T - r(T)\,\mathrm{id})x_n = y_n - z_n,$$

where y_n, z_n are vectors in X^+ of norm at most $M \|(T - r(T)\,\mathrm{id})x_n\|$. If we choose a sufficiently large number $n_0 \in \mathbb{N}$, we thus have $M' \|z_{n_0}\| \leq \varepsilon$.

We now choose a functional $z' \in (X')^+$ of norm at most M' such that $\langle z', x_{n_0}\rangle \geq 1$; such a functional z' exists in view of (Glück and Mironchenko, 2021, Lemma 3.6). The rank-1 operator $P \in \mathcal{L}(X)$ that is defined by the formula

$$Pv = \langle z', v\rangle z_{n_0} \quad \text{for each } v \in X,$$

is positive and has norm

$$\begin{aligned}\|P\| = \sup_{\|v\|=1} \|\langle z', v\rangle z_{n_0}\| &= \sup_{\|v\|=1} |\langle z', v\rangle| \, \|z_{n_0}\| \\ &= \|z'\| \, \|z_{n_0}\| \leq M' \|z_{n_0}\| \leq \varepsilon.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}Tx_{n_0} + Px_{n_0} - x_{n_0} &\geq (T - r(T)\,\mathrm{id})x_{n_0} + Px_{n_0} \\ &= y_{n_0} - z_{n_0} + \langle z', x_{n_0}\rangle z_{n_0} \geq 0\end{aligned}$$

since $\langle z', x_{n_0}\rangle \geq 1$. Hence, we have $Tx + Px \geq x$ for $x := x_{n_0}$. □

Remarks 4.1. (a) The terminology "small-gain condition" stems from the study of interconnected systems in systems and control theory. In this context, the gain describes the response of the system on the applied input. As an example, consider two systems Σ_1 and Σ_2. If $\gamma_{12} > 0$ is the gain describing the influence of the system Σ_2 on the system Σ_1, and $\gamma_{21} > 0$ is the gain describing the influence of the system Σ_1 on the Σ_2, then the condition $\gamma_{12} \cdot \gamma_{21} < 1$ guarantees in a proper context the stability of the feedback interconnection of Σ_1 and Σ_2 and is referred to as a "small-gain condition". At the same time, the condition $\gamma_{12} \cdot \gamma_{21} < 1$ is equivalent to

$$\begin{pmatrix} 0 & \gamma_{12} \\ \gamma_{21} & 0 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \not\geq \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \quad \text{for all } (s_1, s_2) \in \mathbb{R}^2_+ \setminus \{0\}$$

(for the implication "$\Leftarrow$", just take $s_1 := 1$, $s_2 := \gamma_{21}$). This explains the use of the term "small-gain condition" for conditions like (4).

(b) In assertion (v) of the theorem, it does not suffice to consider only a single non-zero operator P as a perturbation. As a counterexample, let $p \in [1, \infty]$ and let $T : \ell_p \to \ell_p$ denote the right shift operator given by

$$T : (x_1, x_2, \dots) \mapsto (0, x_1, x_2, \dots).$$

Moreover, let $P = \frac{1}{2}\,\mathrm{id} : \ell_p \to \ell_p$ denote half the identity operator. Then it is easy to check that $(T + P)x \not\geq x$ for each non-zero vector $x \geq 0$. Yet, T has spectral radius 1.

A related observation is made in Example 6.2.

(c) We note that all the equivalent conditions in Theorem 4.1 can also be formulated in terms of the dual operator T', since the dual operator is positive, too, and since $r(T') = r(T)$.

5. CONES WITH NON-EMPTY INTERIOR

If (X, X^+) is an ordered Banach space and the cone X^+ has non-empty interior, we will write $x \ll y$ (or, synonymously, $y \gg x$) if $y - x \in \operatorname{int}(X^+)$.

If the cone X^+ in an ordered Banach space contains an interior point, further powerful characterizations for stability of positive systems can be obtained:

Theorem 5.1. Let (X, X^+) be an ordered Banach space with normal cone and suppose that the cone has non-empty interior. For every positive operator $T \in \mathcal{L}(X)$, the following assertions are equivalent:

(i) *Uniform exponential stability:* The system (3) satisfies the equivalent criteria of Proposition 3.1, i.e., we have $r(T) < 1$.
(ii) *Dual small-gain condition:* For each $0 \neq x' \in (X')^+$ we have
$$T'x' \not\geq x'.$$
(iii) *Interior point small-gain condition, first version:* For every interior point z of X^+ there is a number $\eta > 0$ such that
$$Tx \not\geq x - \eta \|x\| z \qquad \text{for all } x \in X^+ \setminus \{0\}.$$
(iv) *Interior point small-gain condition, second version:* There exists an interior point z of X^+ and a number $\eta > 0$ such that
$$Tx \not\geq x - \eta \|x\| z \qquad \text{for all } x \in X^+ \setminus \{0\}.$$
(v) *Strong decreasing property, first version:* There exists an interior point z of X^+ such that $Tz \ll z$.
(vi) *Strong decreasing property, second version:* There exists an interior point z of X^+ and $\lambda \in (0,1)$ such that
$$Tz \leq \lambda z. \tag{6}$$
(vii) *Strong stability:* $T^k x \to 0$ as $k \to \infty$ for all $x \in X$.
(viii) *Weak attractivity in the cone:* For each $x \in X^+$ we have $\inf_{k \geq 0} \|T^k x\| = 0$.

It is particularly worthwhile to point out that condition (ii) in Theorem 5.1 is formulated in a non-uniform way (as opposed to the uniform small-gain condition in Theorem 4.1(iv)). This is because, on spaces whose cone is normal and has non-empty interior, the spectral radius of a positive operator is always an eigenvalue of the dual operator with a positive eigenvector.

Remark 5.1. In some applications, in particular in Lyapunov-based small-gain theorems Mironchenko et al. (2021b), it is of interest to explicitly compute the number λ and the corresponding vector z as in (6), also called a *point of strict decay*. We refer to Mironchenko et al. (2021b) for explicit formulas for the computations of such points.

6. THE QUASI-COMPACT CASE

The characterisation of stability becomes considerably easier when the operator T under consideration is compact or, more generally *quasi-compact*. Here we call a bounded linear operator T on a Banach space X *quasi-compact* if there exists an integer $n_0 \in \mathbb{N}$ and a compact operator K on X such that $\|T^{n_0} - K\| < 1$ (alternatively, if there exists n_1 and a compact operator K such that $r(T^{n_1} - K) < 1$). This is equivalent to saying that the equivalence class of T in the *Calkin algebra* $\mathcal{L}(X)/\mathcal{K}(X)$ – where $\mathcal{K}(X)$ denotes the ideal of compact operators on X – has spectral radius strictly less than 1. The latter spectral radius is known to coincide with the so-called *essential spectral radius* $r_{\text{ess}}(T)$ of T. Hence, T is quasi-compact if and only if $r_{\text{ess}}(T) < 1$, and the latter condition means that on the unit circle, and outside of it, all spectral values of T (if there exist any at all) are poles of the resolvent $(\,\cdot - T)^{-1}$ with a finite-dimensional spectral space. Clearly, every compact operator and every power compact operator is quasi-compact.

Our next theorem gives additional stability criteria for positive linear operators in case they are quasi-compact. In contrast to Theorem 4.1 we do not need the cone to be normal now, and moreover, it suffices if the cone is total rather than generating. In case the cone is normal and generating, though, the following criteria are complemented by those in Theorem 4.1, of course.

Theorem 6.1. Let (X, X^+) be an ordered Banach space with total cone and let $T \in \mathcal{L}(X)$ be positive. If T is quasi-compact, then the following assertions are equivalent:

(i) *Uniform exponential stability:* The system (3) satisfies the equivalent criteria of Proposition 3.1, i.e., we have $r(T) < 1$.
(ii) *Positivity of the resolvent at* 1*:* The operator $\mathrm{id} - T : X \to X$ is bijective and $(\mathrm{id} - T)^{-1}$ is positive.
(iii) *All sub-fixed vectors of T are positive:* If $x \in X$ satisfies
$$Tx \leq x,$$
then $x \geq 0$.
(iv) *Small-gain condition:* For each $0 \neq x \in X^+$ we have
$$Tx \not\geq x.$$
(v) *Attractivity on the cone:* For each $x \in X^+$ we have $T^k x \to 0$ as $k \to \infty$.
(vi) *Weak attractivity on the cone:* For each $x \in X^+$ we have $\inf_{k \geq 0} \|T^k x\| = 0$.

For finite-dimensional discrete-time systems the equivalence "(i) $\Leftrightarrow$ (iv)" of Theorem 6.1 and the equivalence "(i) $\Leftrightarrow$ (vi)" of Theorem 5.1 can be found in (Rüffer, 2007, Lemma 2.0.1) and (Rüffer, 2010, Lemma 1.1). In the finite-dimensional continuous-time case, an analogue of "(i) $\Leftrightarrow$(iv)" can be found in (Stern, 1982, Theorem 1.4).

Without the assumption $r_{\text{ess}}(T) < 1$, none of the conditions (iii) or (iv) in Theorem 6.1 is sufficient to ensure that $r(T) < 1$. Here are simple counterexamples:

Example 6.1. Let X denote the ordered Banach space $C_b([0, +\infty))$ of bounded continuous functions with pointwise order (compare Example 2.2), and let $T : X \to X$ be given for each $f \in X$ by
$$(Tf)(\omega) = (1 - e^{-\omega}) f(\omega) \quad \text{for all } \omega \in [0, +\infty).$$
Then T is positive, has spectrum $[0, 1]$ and hence spectral radius 1; but T satisfies condition (iii), as well as the property $(T(x))(s) < x(s)$ for all $x \in X^+$ and $s \geq 0$, which is a stronger property than the small-gain condition (iv) in Theorem 6.1.

The theorem is not applicable since the essential spectral radius of T is equal to 1. Also note that $Tx \not\ll x$ for all $x \in X^+$.

Example 6.2. Consider the ordered Banach space $X := \ell_\infty$, ordered by its usual cone (see Example 2.1). The cone X^+ is normal and has non-empty interior. We consider the system
$$x(k+1) = 2Rx(k), \quad k \in \mathbb{Z}_+,$$

where R is the right shift on X, i.e., R acts on $x = (x_0, x_1, x_2, \dots) \in \ell_\infty$ as $Rx := (0, x_0, x_1, x_2, \dots)$. Clearly, R is a positive operator in $\mathcal{L}(X)$.

Consider an arbitrary strictly positive diagonal operator $D \in \mathcal{L}(X)$, defined for $x := (x_i)_{i\in\mathbb{Z}_+} \in X$ by $Dx := (d_i x_i)_{i\in\mathbb{Z}_+}$, where $(d_i)_{i\in\mathbb{Z}_+}$ is a sequence of a real numbers that satisfy $0 < d_i \le M$ for a fixed constant $M > 0$ and all indices $i \in \mathbb{Z}_+$.

Let $x = (x_1, x_2, \dots) \in X^+ \setminus \{0\}$ and let i be the index of the first non-zero component of x (which is well-defined and finite as $x \in X^+$ and $x \neq 0$). Then the components of $2R(I + D)x$ with indices $j = 0, \dots, i$ are equal to 0. This shows that

$$2R(I + D)x \not\geq x \qquad \text{for all } x \in X^+ \setminus \{0\},$$

which implies so-called *strong small-gain condition* for the operator $2R$ used in, e.g., (Dashkovskiy et al., 2010, p. 11), Dashkovskiy et al. (2019), Mironchenko et al. (2021a); in particular, this implies the small-gain condition for $T := 2R$ in Theorem 6.1(iv).

The strong small-gain condition says that there are positive perturbations of the operator, under which the operator still satisfies the small-gain condition in Theorem 6.1(iv). In this way, it resembles the *robust small-gain condition* in Theorem 4.1(v). However, note that while in the robust small-gain condition, the operator is being disturbed by arbitrary additive small enough perturbations, in the strong small-gain condition above, the operator is disturbed by multiplicative perturbations of a specific form.

However, R is not quasicompact, and thus Theorem 6.1 is not applicable. In fact, R is an *isometry* as $\|Rx\| = \|x\|$ for each $x \in X$. Thus, also $\|(2R)^k x\| = 2^k \|x\| \to \infty$ as $k \to \infty$ provided that $x \neq 0$. This also shows that the robust small-gain condition is much stronger than the strong small-gain condition.

Finally, note (see, e.g., (Hundertmark et al., 2013, Example B.7)) that $\sigma(2R) = \overline{B(0, 2)}$, where $B(0, 2)$ is the open ball of radius 2 with the center at 0 in the complex plane; at the same time the point spectrum of $2R$ is empty (which already implies that the claim of Krein–Rutman theorem does not hold for $2R$).

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