Remarks on input-to-state stability and non-coercive Lyapunov functions

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***Abstract*— We consider an abstract class of infinite-dimensional dynamical systems with inputs. For this class the significance of noncoercive Lyapunov functions is analyzed. It is shown that the existence of such Lyapunov functions implies integral input-to-integral state stability. Assuming further regularity it is possible to conclude input-to-state stability. For a particular class of linear systems with unbounded admissible input operators, explicit constructions of noncoercive Lyapunov functions are provided. The theory is applied to a heat equation with Dirichlet boundary conditions.**

I. Introduction

It is well-known that the existence of an ISS Lyapunov function implies ISS. However, the construction of ISS Lyapunov functions for infinite-dimensional systems is a challenging task, especially in the nonlinear case. Already for undisturbed linear systems over Hilbert spaces, "natural" Lyapunov function candidates constructed via solutions of Lyapunov equations are of the form $V(x) := \langle Px, x\rangle$, where $\langle\cdot,\cdot\rangle$ is a scalar product in X and P is a self-adjoint, bounded linear, positive operator the spectrum of which may contain 0. In this case V is not coercive and satisfies only the weaker property that $V(x) > 0$ for $x \neq 0$. Hence the question arises, whether such "non-coercive" Lyapunov functions can be used to conclude that a given system is ISS. A thorough study of a similar question related to characterizations of uniform global asymptotic stability has recently been performed in [1].

In [2, Section III.B] it was shown for a class of semilinear equations in Banach spaces with Lipschitz continuous nonlinearities that the existence of a non-coercive Lyapunov function implies ISS provided the flow of the system has some continuity properties with respect to states and inputs at the origin and the finite-time reachability sets of the system are bounded. However, this class of systems does not include many important systems such as linear control systems with admissible inputs operators, which are crucially important for the study of partial differential equations with boundary inputs.

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In this paper we extend the results from [2, Section III.B] to a broader class of systems, which includes at least some important classes of boundary control systems. The characterizations of ISS developed in [2] will play a central role in these developments.

It is insightful to define another ISS-like property which we call integral-to-integral ISS. Its finite-dimensional counterpart has been studied in [3] and it was shown that integral-to-integral ISS is equivalent to ISS for systems of ordinary differential equations. Further relations of ISS and integral-to-integral ISS have been developed in [4], [5] and other works.

We start by defining a general class of control systems in Section II. This class covers a wide range of infinite-dimensional systems. For this class several stability concepts are defined which relate to the characterization of ISS, in particular to the characterization with the help of noncoercive Lyapunov functions. In Section III we show in Theorem 3.6 that integral-to-integral ISS implies ISS for a broad class of infinite-dimensional systems provided the flow of the system has some continuity properties w.r.t. states and inputs at the origin and the finite-time reachability sets of the system are bounded. The proof of this fact is performed in 3 steps. The first one is to show that integral-to-integral ISS implies a so-called uniform limit property. This result has been already obtained in [2, Section III.B]. The second (technically harder) step, is to show that integral-to-integral ISS implies local stability of a control system provided the flow of the system is continuous w.r.t. state and inputs at the origin. This is done in Proposition 3.5. The third and final step in the proof of Theorem 3.6 is the application of the main result in [2].

In Section IV we derive a constructive converse ISS Lyapunov theorem for certain classes of linear systems with admissible input operators. In particular, our results can be applied for a broad class of subnormal operators, as discussed in Section V-B.

It is well-known that the classic heat equation with Dirichlet boundary inputs is ISS, which has been verified by means of several different methods: [6], [7], [8]. However, no constructions for ISS Lyapunov functions have been proposed. In Section V we show that using the constructions developed in Proposition 4.1 one can construct a non-coercive ISS Lyapunov function for this system. It is still an open question, whether a coercive ISS Lyapunov function for a heat equation with the Dirichlet boundary input exists (note, that for the system with Neumann boundary input a coercive ISS Lyapunov function can be constructed, see [9]).

Notation: We use the following notation. The nonnegative reals are $\mathbb{R}_+ := [0,\infty)$. The open ball of radius r around 0 in X is denoted by $B_r := \{x \in X : \|x\|_X < r\}$. Similarly, $B_{r,\mathscr{U}} := \{u \in \mathscr{U} : \|u\|_{\mathscr{U}} < r\}$. By $\overline{\lim}$ we denote the limit superior. For any normed linear space X, for any $S \subset X$ we denote the closure of S by $\overline{S}$.

For the formulation of stability properties the following classes of comparison functions are useful:

$$\begin{aligned}
\mathscr{K} &:= \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous, strictly increasing and } \gamma(0) = 0\}, \\
\mathscr{K}_\infty &:= \{\gamma \in \mathscr{K} \mid \gamma \text{ is unbounded}\}, \\
\mathscr{L} &:= \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous and strictly decreasing with } \lim_{t\to\infty} \gamma(t) = 0\}, \\
\mathscr{K}\mathscr{L} &:= \{\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \mid \beta \text{ is continuous, } \beta(\cdot,t) \in \mathscr{K},\ \beta(r,\cdot) \in \mathscr{L},\ \forall t \geq 0,\ \forall r > 0\}.
\end{aligned}$$

II. Preliminaries

We begin by defining (time-invariant) forward complete control systems evolving on a Banach space X.

Definition 2.1: Let $(X, \|\cdot\|_X)$, $(U, \|\cdot\|_U)$ be Banach spaces and $\mathscr{U} \subset \{f : \mathbb{R}_+ \to U\}$ be a normed vector space which satisfies the following two axioms:

Axiom of shift invariance: For all $u \in \mathscr{U}$ and all $\tau \geq 0$ we have $u(\cdot + \tau) \in \mathscr{U}$ with $\|u\|_{\mathscr{U}} \geq \|u(\cdot+\tau)\|_{\mathscr{U}}$.

Axiom of concatenation: For all $u_1, u_2 \in \mathscr{U}$ and for all $t > 0$ the concatenation of u_1 and u_2 at time t

$$u(\tau) := \begin{cases} u_1(\tau), & \text{if } \tau \in [0,t], \\ u_2(\tau - t), & \text{otherwise,} \end{cases} \tag{II.1}$$

belongs to $\mathscr{U}$. Assume $\phi : \mathbb{R}_+ \times X \times \mathscr{U} \to X$.

The triple $\Sigma = (X, \mathscr{U}, \phi)$ is called a *forward complete control system*, if the following properties hold:

(Σ1) *Identity property*: for every $(x,u) \in X \times \mathscr{U}$ it holds that $\phi(0,x,u) = x$.

(Σ2) *Causality*: for every $(t,x,u) \in \mathbb{R}_+ \times X \times \mathscr{U}$, for every $\tilde{u} \in \mathscr{U}$ with $u(s) = \tilde{u}(s)$ for all $s \in [0,t]$ it holds that $\phi(t,x,u) = \phi(t,x,\tilde{u})$.

(Σ3) *Continuity*: for each $(x,u) \in X \times \mathscr{U}$ the map $t \mapsto \phi(t,x,u)$ is continuous.

(Σ4) *Cocycle property*: for all $t,h \geq 0$, for all $x \in X$, $u \in \mathscr{U}$ we have $\phi(h, \phi(t,x,u), u(t+\cdot)) = \phi(t+h,x,u)$.

The space X ist called the *state space*, U the *input space* and ϕ the *transition map*.
This class of systems encompasses control systems generated by ordinary differential equations (ODEs), switched systems, time-delay systems, evolution partial differential equations (PDEs), abstract differential equations in Banach spaces and many others.

Remark 2.2: Note however, that not all important systems are covered by our definitions. In particular, the input space $C(\mathbb{R}_+, U)$ of continuous U-valued functions does not satisfy the axiom of concatenation. This, however, should not be a big restriction, since already piecewise continuous and L_p inputs, which are used in control theory much more frequently than continuous ones, satisfy the axiom of concatenation.

Some authors consider more general concepts, in which the systems fail to satisfy a cocycle property, see e.g. [10].

We single out two particular cases which will be of interest.

Example 2.3: Let A be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ of bounded linear operators on X and let $f : X \times U \to X$. Consider the system

$$\dot{x}(t) = Ax(t) + f(x(t), u(t)), \quad u(t) \in U, \tag{II.2}$$

where $x(0) \in X$. We study mild solutions of (II.2), i.e. solutions $x : [0,\tau] \to X$ of the integral equation

$$x(t) = T(t)x(0) + \int_0^t T(t-s) f(x(s), u(s)) ds, \tag{II.3}$$

belonging to the space of continuous functions $C([0,\tau], X)$ for some $\tau > 0$.

For system (II.2), we use the following assumption concerning the nonlinearity f:

(i) $f : X \times U \to X$ is Lipschitz continuous on bounded subsets of X, uniformly with respect to the second argument, i.e. for all $C > 0$, there exists a $L_f(C) > 0$, such that for all $x, y \in B_C$ and for all $v \in U$, it holds that

$$\|f(x,v) - f(y,v)\|_X \leq L_f(C) \|x - y\|_X. \tag{II.4}$$

(ii) $f(x,\cdot)$ is continuous for all $x \in X$ and $f(0,0) = 0$.

Let $\mathscr{U} = PC(\mathbb{R}_+, U)$. Then our assumptions on f ensure that mild solutions of initial value problems of the form (II.2) exist and are unique, according to [11, Proposition 4.3.3]. For system (II.2) forward completeness is a further assumption. If these mild solutions exist on $[0,\infty)$ for every $x(0) \in X$ and $u \in PC(\mathbb{R}_+, U)$, then $(X, PC(\mathbb{R}_+, U), \phi)$, defines a forward complete control system, where $\phi(t, x(0), u)$ denotes the mild solution at time t.

Example 2.4: (Linear systems with admissible control operators) We consider linear systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) \in X, t \geq 0, \tag{II.5}$$

where A is the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X and $B \in L(U, X_{-1})$ for some Banach space U. Here X_{-1} is the completion of X with respect to the norm $\|x\|_{X_{-1}} = \|(\beta - A)^{-1} x\|_X$ for some β in the resolvent set $\rho(A)$ of A. The semigroup $(T(t))_{t\geq 0}$ extends uniquely to a C_0-semigroup $(T_{-1}(t))_{t\geq 0}$ on X_{-1} whose generator A_{-1} is an extension of A, see e.g. [12]. Thus we may consider Equation (II.5) on the Banach space X_{-1}. For every $x_0 \in X$ and every $u \in L^1_{\text{loc}}([0,\infty), U)$, the function $x : [0,\infty) \to X_{-1}$,

$$x(t) := T(t)x_0 + \int_0^t T_{-1}(t-s) Bu(s) ds, \quad t \geq 0,$$

is called *mild solution* of Equation (II.5). The operator $B \in L(U, X_{-1})$ is called an *q-admissible control operator* for $(T(t))_{t\geq 0}$, where $1 \leq q \leq \infty$, if

$$\int_0^t T_{-1}(t-s) Bu(s) ds \in X$$

for every $t \geq 0$ and $u \in L^q([0,\infty), U)$.

If B is ∞-admissible and for every initial condition $x_0 \in X$ and every input function $u \in L^\infty([0,\infty),U)$ the mild solution $x : [0,\infty) \to X$ is continuous, then $(X, L^\infty([0,\infty),U), \phi)$, where

$$\phi(t,x_0,u) := T(t)x_0 + \int_0^t T_{-1}(t-s)Bu(s)ds,$$

defines a forward-complete control system as defined in Definition 2.1.

We note that, ∞-admissibility and continuity of all mild solutions $x : [0,\infty) \to X$, where $x_0 \in X$ and $u \in L^\infty([0,\infty),U)$ is implied by each of the following conditions:

- B is q-admissible for some $q \in [1,\infty)$ [6],
- B is ∞-admissible, $\dim U < \infty$, X is a Hilbert space and A generates an analytic semigroup which is similar to a contraction semigroup [13].

Within this article different stability concepts of forward complete control systems are needed.

Definition 2.5: Consider a forward complete control system $\Sigma = (X, \mathscr{U}, \phi)$.

1) We call $0 \in X$ an equilibrium point (of the undisturbed system) if $\phi(t,0,0) = 0$ for all $t \geq 0$.
2) We say Σ has the *CEP property*, if 0 is an equilibrium and for every $\varepsilon > 0$ and for any $h > 0$ there exists a $\delta = \delta(\varepsilon,h) > 0$, so that
$$t \in [0,h],\ \|x\|_X \leq \delta,\ \|u\|_{\mathscr{U}} \leq \delta \ \Rightarrow\ \|\phi(t,x,u)\|_X \leq \varepsilon. \tag{II.6}$$
3) We say that Σ has *bounded reachability sets (BRS)*, if for any $C > 0$ and any $\tau > 0$ it holds that
$$\sup\big\{\|\phi(t,x,u)\|_X : \|x\|_X \leq C,\ \|u\|_{\mathscr{U}} \leq C,\ t \in [0,\tau]\big\} < \infty.$$
4) System Σ is called *uniformly locally stable (ULS)*, if there exist $\sigma \in \mathscr{K}_\infty$, $\gamma \in \mathscr{K}_\infty \cup \{0\}$ and $r > 0$ such that for all $x \in \overline{B_r}$ and all $u \in \overline{B_{r,\mathscr{U}}}$:
$$\|\phi(t,x,u)\|_X \leq \sigma(\|x\|_X) + \gamma(\|u\|_{\mathscr{U}}) \quad \forall t \geq 0. \tag{II.7}$$
5) We say that Σ has the *uniform limit property (ULIM)*, if there exists $\gamma \in \mathscr{K} \cup \{0\}$ so that for every $\varepsilon > 0$ and for every $r > 0$ there exists a $\tau = \tau(\varepsilon,r)$ such that for all x with $\|x\|_X \leq r$ and all $u \in \mathscr{U}$ there is a $t \leq \tau$ such that
$$\|\phi(t,x,u)\|_X \leq \varepsilon + \gamma(\|u\|_{\mathscr{U}}). \tag{II.8}$$
6) System Σ is called *(uniformly) input-to-state stable (ISS)*, if there exist $\beta \in \mathscr{KL}$ and $\gamma \in \mathscr{K}$ such that for all $x \in X$, $u \in \mathscr{U}$ and $t \geq 0$ it holds that
$$\|\phi(t,x,u)\|_X \leq \beta(\|x\|_X,t) + \gamma(\|u\|_{\mathscr{U}}). \tag{II.9}$$
7) We call Σ *integral-to-integral ISS* if there are $\alpha \in \mathscr{K}$ and $\psi \in \mathscr{K}_\infty$, $\sigma \in \mathscr{K}_\infty$ so that for all $x \in X$, $u \in \mathscr{U}$ and $t \geq 0$ it holds that
$$\int_0^t \alpha(\|\phi(s,x,u)\|_X)ds \leq \psi(\|x\|_X) + \int_0^t \sigma(\|u(s+\cdot)\|_{\mathscr{U}})ds. \tag{II.10}$$

Example 2.6: (Linear systems with admissible control operators) We continue with Example 2.4, that is, we consider again Equation (II.5) and assume that A generates a C_0-semigroup, $B \in L(U, X_{-1})$ is ∞-admissible and for every initial condition $x_0 \in X$ and every input function $u \in L^\infty([0,\infty),U)$ the mild solution $x : [0,\infty) \to X$ is continuous. These assumption guarantee that $(X, L^\infty([0,\infty),U), \phi)$, where

$$\phi(t,x_0,u) := T(t)x_0 + \int_0^t T_{-1}(t-s)Bu(s)ds,$$

defines a forward-complete control system. The system has the following properties

1) $0 \in X$ an equilibrium point due to the linearity of the system,
2) $(X, L^\infty([0,\infty),U), \phi)$ has the CEP property, and bounded reachability sets (BRS) [14],
3) If $(T(t))_{t\geq 0}$ is exponentially stable, then $(X, L^\infty([0,\infty),U), \phi)$ has the uniform limit property (ULIM) [14], is uniformly locally stable (ULS) [14] and input-to-state stable (ISS) [6],
4) $(T(t))_{t\geq 0}$ is exponentially stable if and only if $(X, L^\infty([0,\infty),U), \phi)$ is ISS [14].
5) If $(X, L^\infty([0,\infty),U), \phi)$ is integral-to-integral ISS, then $(X, L^\infty([0,\infty),U), \phi)$ is ISS [6].

Remark 2.7: To the best of the knowledge of the authors it is unknown, whether or not the converse statement to item 5) of Example 2.6 holds for every linear system (II.5).

III. Non-coercive Lyapunov theorem

Lyapunov functions are a powerful tool for the investigation of ISS. Let $x \in X$ and V be a real-valued function defined in a neighborhood of x. The (right-hand upper) Dini derivative of V at x corresponding to the input u along the trajectories of Σ is defined by

$$\dot{V}_u(x) = \overline{\lim_{t\to+0}} \frac{1}{t}\big(V(\phi(t,x,u)) - V(x)\big). \tag{III.1}$$

Definition 3.1: A continuous function $V : X \to \mathbb{R}_+$ is called a *non-coercive ISS Lyapunov function* for a system $\Sigma = (X, \mathscr{U}, \phi)$, if there exist $\psi_2, \alpha \in \mathscr{K}_\infty$ and $\sigma \in \mathscr{K}$ such that

$$0 < V(x) \leq \psi_2(\|x\|_X), \quad \forall x \in X \tag{III.2}$$

and the Dini derivative of V along the trajectories of Σ satisfies

$$\dot{V}_u(x) \leq -\alpha(\|x\|_X) + \sigma(\|u\|_{\mathscr{U}}) \tag{III.3}$$

for all $x \in X$ and $u \in \mathscr{U}$.

Moreover, if (III.3) holds just for $u = 0$, we call V a *(non-coercive) Lyapunov function* for the undisturbed system Σ. If additionally there is $\psi_1 \in \mathscr{K}_\infty$ so that the following estimate holds:

$$\psi_1(\|x\|_X) \leq V(x) \leq \psi_2(\|x\|_X), \quad \forall x \in X, \tag{III.4}$$

then V is called a *coercive ISS Lyapunov function* for Σ.

The next proposition shows that integral-to-integral ISS property naturally arises in the theory of ISS Lyapunov functions:

Proposition 3.2: Let $\Sigma = (X, \mathscr{U}, \phi)$ be a forward complete control system. Assume that there exists a (noncoercive)x ISS Lyapunov function for Σ. Then Σ is integral-to-integral ISS.

Proof: Assume that V is an ISS Lyapunov function for Σ with corresponding ψ_2, α, σ. Integrating (III.3) from 0 to t, we obtain using [1, Lemma 3.4]:

$$V(\phi(t,x,u)) - V(x) \leq -\int_0^t \alpha(\|\phi(s,x,u)\|_X)ds + \int_0^t \sigma(\|u(\cdot+s)\|_{\mathscr{U}})ds.$$

This immediately implies that

$$\begin{aligned}\int_0^t \alpha(\|\phi(s,x,u)\|_X)ds &\leq V(x) - V(\phi(t,x,u)) + \int_0^t \sigma(\|u(\cdot+s)\|_{\mathscr{U}})ds \\ &\leq \psi_2(\|x\|_X) + \int_0^t \sigma(\|u(\cdot+s)\|_{\mathscr{U}})ds. \qquad \text{(III.5)}\end{aligned}$$

This shows integral-to-integral ISS of Σ. ■

In [3, Theorem 1] it was shown that for ODE systems with Lipschitz continuous nonlinearities the notions of ISS and integral-to-integral ISS are equivalent. Next we show that integral-to-integral ISS implies ISS for a class of forward-complete control systems satisfying the CEP and BRS properties. In order to prove this, we are going to use the following characterization of ISS, shown in [2]:

Theorem 3.3: Let $\Sigma = (X, \mathscr{U}, \phi)$ be a forward complete control system. The following statements are equivalent:

(i) Σ is ISS.
(ii) Σ is ULIM, ULS, and BRS.

In [2, Proposition 8] it was shown (with slightly different formulation, but the same proof) that

Proposition 3.4: Let $\Sigma = (X, \mathscr{U}, \phi)$ be a forward complete control system. If Σ is integral-to-integral ISS, then Σ is ULIM.

Next we provide a sufficient condition for the ULS property.

Proposition 3.5: Let $\Sigma = (X, \mathscr{U}, \phi)$ be a forward complete control system satisfying the CEP property. If Σ is integral-to-integral ISS, then Σ is ULS.

Proof: Let Σ be integral-to-integral ISS with the corresponding α, ψ, σ as in Definition 7.

Seeking a contradiction, assume that Σ is not ULS. Then there exist an $\varepsilon > 0$ and sequences $\{x_k\}_{k\in\mathbb{N}}$ in X, $\{u_k\}_{k\in\mathbb{N}}$ in $\mathscr{U}$, and $\{t_k\}_{k\in\mathbb{N}} \subset \mathbb{R}_+$ such that $x_k \to 0$ as $k \to \infty$, $u_k \to 0$ as $k \to \infty$ and

$$\|\phi(t_k, x_k, u_k)\|_X = \varepsilon \quad \forall k \geq 1.$$

Since Σ is CEP, for the above ε there is a $\delta_1 = \delta_1(\varepsilon, 1)$ so that

$$\|x\|_X, \|u\|_{\mathscr{U}} \leq \delta_1,\ t \in [0,1] \quad \Rightarrow \quad \|\phi(t,x,u)\|_X < \varepsilon. \qquad \text{(III.6)}$$

Define for $k \in \mathbb{N}$ the following time sequence:

$$t_k^1 := \sup\{t \in [0, t_k] : \|\phi(t, x_k, u_k)\|_X \leq \delta_1\},$$

if the supremum is taken over a nonempty set, and set $t_k^1 := 0$ otherwise.

Again as Σ is CEP, for the above δ_1 there is a $\delta_2 > 0$ so that

$$\|x\|_X, \|u\|_{\mathscr{U}} \leq \delta_2,\ t \in [0,1] \quad \Rightarrow \quad \|\phi(t,x,u)\|_X < \delta_1. \qquad \text{(III.7)}$$

Without loss of generality we assume that δ_2 is chosen small enough so that

$$\alpha(\delta_1) > \psi(\delta_2). \qquad \text{(III.8)}$$

We now define

$$t_k^2 := \sup\{t \in [0, t_k] : \|\phi(t, x_k, u_k)\|_X \leq \delta_2\},$$

if the supremum is taken over a nonempty set, and set $t_k^2 := 0$ otherwise.

Since $u_k \to 0$ and $x_k \to 0$ as $k \to \infty$, there is $K > 0$ so that $\|u_k\|_{\mathscr{U}} \leq \delta_2$ and $\|x_k\|_X \leq \delta_2$ for $k \geq K$.

From now on, we always assume that $k \geq K$.

Using (III.6), (III.7) and the cocycle property, it is not hard to show that for $k \geq K$ it must hold that $t_k \geq 2$, as otherwise we arrive at a contradiction to $\|\phi(t_k, x_k, u_k)\|_X = \varepsilon$.

Assume that $t_k - t_k^1 < 1$. This implies (since $t_k \geq 2$), that $t_k^1 > 0$. By the cocycle property we have

$$\|\phi(t_k, x_k, u_k)\|_X = \|\phi(t_k - t_k^1, \phi(t_k^1, x_k, u_k), u_k(\cdot + t_k^1))\|_X.$$

The axiom of shift invariance justifies the inequalities

$$\|u_k(\cdot + t_k^1)\|_{\mathscr{U}} \leq \|u_k\|_{\mathscr{U}} \leq \delta_2 \leq \delta_1.$$

Since $\|\phi(t_k^1, x_k, u_k)\|_X = \delta_1$, and $t_k - t_k^1 < 1$, we have by (III.6) that $\|\phi(t_k, x_k, u_k)\|_X < \varepsilon$, a contradiction. Hence $t_k - t_k^1 \geq 1$ for all $k \geq K$.

Analogously, we obtain that $t_k^1 - t_k^2 \geq 1$ and $t_k - t_k^2 \geq 2$.

Define

$$x_k^2 := \phi(t_k^2, x_k, u_k), \quad u_k^2 := u_k(\cdot + t_k^2)$$

and

$$x_k^1 := \phi(t_k^1, x_k, u_k), \quad u_k^1 := u_k(\cdot + t_k^1).$$

Due to the axiom of shift invariance $u_k^1, u_k^2 \in \mathscr{U}$ and

$$\|u_k^1\|_{\mathscr{U}} \leq \|u_k^2\|_{\mathscr{U}} \leq \|u_k\|_{\mathscr{U}} \leq \delta_2.$$

Also by the definition of t_k^2 we have $\|x_k^2\|_X = \delta_2$.

Applying (II.10), and estimating the integral on the right hand side of (II.10), we obtain for $t := t_k - t_k^2$ that

$$\begin{aligned}\int_0^{t_k - t_k^2} \alpha(\|\phi(s, x_k^2, u_k^2)\|_X)ds &\leq \psi(\|x_k^2\|_X) + (t_k - t_k^2)\sigma(\|u_k^2\|_{\mathscr{U}}) \\ &\leq \psi(\delta_2) + (t_k - t_k^2)\sigma(\|u_k\|_{\mathscr{U}}).\end{aligned} \qquad \text{(III.9)}$$

On the other hand, changing the integration variable and using the cocycle property we obtain that

$$\begin{aligned}
&\int_0^{t_k-t_k^2} \alpha(\|\phi(s,x_k^2,u_k^2)\|_X)ds \\
&= \int_0^{t_k^1-t_k^2} \alpha(\|\phi(s,x_k^2,u_k^2)\|_X)ds + \int_{t_k^1-t_k^2}^{t_k-t_k^2} \alpha(\|\phi(s,x_k^2,u_k^2)\|_X)ds \\
&= \int_0^{t_k^1-t_k^2} \alpha(\|\phi(s,x_k^2,u_k^2)\|_X)ds \\
&\quad + \int_0^{t_k-t_k^1} \alpha(\|\phi(s+t_k^1-t_k^2,x_k^2,u_k^2)\|_X)ds \\
&= \int_0^{t_k^1-t_k^2} \alpha(\|\phi(s,x_k^2,u_k^2)\|_X)ds + \int_0^{t_k-t_k^1} \alpha(\|\phi(s,x_k^1,u_k^1)\|_X)ds.
\end{aligned}$$

By definition of t_k^2 and t_k^1 we have that

$$\|\phi(s,x_k^2,u_k^2)\|_X \geq \delta_2, \quad s \in [t_k^2,t_k]$$

and

$$\|\phi(s,x_k^1,u_k^1)\|_X \geq \delta_1, \quad s \in [t_k^1,t_k].$$

Continuing the above estimates and using that $t_k - t_k^1 \geq 1$ and $\alpha(\delta_1) > \alpha(\delta_2)$, we arrive at

$$\begin{aligned}
\int_0^{t_k-t_k^2} \alpha(\|\phi(s,x_k^2,u_k^2)\|_X)ds &\geq (t_k^1-t_k^2)\alpha(\delta_2) + (t_k-t_k^1)\alpha(\delta_1) \\
&\geq (t_k-t_k^2-1)\alpha(\delta_2) + \alpha(\delta_1).
\end{aligned}$$

Since $t_k - t_k^2 \geq 2$ and in view of (III.8), we derive

$$\int_0^{t_k-t_k^2} \alpha(\|\phi(s,x_k^2,u_k^2)\|_X)ds > \frac{t_k-t_k^2}{2}\alpha(\delta_2) + \psi(\delta_2) \tag{III.10}$$

Combining inequalities (III.9) and (III.10), we obtain

$$\frac{t_k-t_k^2}{2}\alpha(\delta_2) < (t_k-t_k^2)\sigma(\|u_k\|_{\mathscr{U}}).$$

This leads to

$$\frac{1}{2}\alpha(\delta_2) < \sigma(\|u_k\|_{\mathscr{U}}), \quad k \geq K.$$

Finally, since $\lim_{k\to\infty}\|u_k\|_{\mathscr{U}} = 0$, letting $k \to \infty$ we come to a contradiction. This shows that Σ is ULS. ■

Now we combine the derived results to state a relationship between ISS and integral-to-integral ISS.

Theorem 3.6: Let Σ be a forward complete control system, which is CEP and BRS. If Σ is integral-to-integral ISS, then Σ is ISS.

Proof: Propositions 3.4 and 3.5 imply that Σ is ULIM and ULS. Since Σ is assumed to be BRS, Theorem 3.3 shows that Σ is ISS. ■

We may now state our main result on noncoercive ISS Lyapunov functions.

Theorem 3.7: Let Σ be a forward complete control system, which is CEP and BRS. If there exists a (noncoercive) ISS Lyapunov function for Σ, then Σ is ISS.

Proof: Follows from a combination of Proposition 3.2 and Theorem 3.6. ■

Remark 3.8: Note that forward complete ODE systems with locally Lipschitz nonlinearities are BRS (see [15, Proposition 5.1]) and CEP. Hence for this special class of systems we recover the result of Sontag that integral-to-integral ISS implies ISS (which is a part of [3, Theorem 1]).

One of the requirements in Theorem 3.6 is that the CEP property holds. If this property is not available, we can still infer input-to-state practical stability of Σ, using the main result in [16]. The notion of input-to-state practical stability, a relaxation of the ISS concept has been proposed in [17]. This concept is very useful for control under quantization errors [18], [19], sample-data control [20] to name a few examples.

Definition 3.9: A control system $\Sigma = (X, \mathscr{U}, \phi)$ is called *(uniformly) input-to-state practically stable (ISpS)*, if there exist $\beta \in \mathscr{KL}$, $\gamma \in \mathscr{K}_\infty$ and $c > 0$ such that for all $x \in X$, $u \in \mathscr{U}$ and $t \geq 0$ the following holds:

$$\|\phi(t,x,u)\|_X \leq \beta(\|x\|_X,t) + \gamma(\|u\|_{\mathscr{U}}) + c. \tag{III.11}$$

Theorem 3.10: Let Σ be a forward complete control system, which is BRS. If Σ is integral-to-integral ISS, then Σ is ISpS.

Proof: Proposition 3.4 implies that Σ is ULIM. Since Σ is also BRS, [16, Theorem III.1] shows that Σ is ISpS. ■

IV. Lyapunov functions for linear systems with admissible operators

In this section we return to systems of the form (II.5), which we call $\Sigma(A,B)$ for short. We show how non-coercive ISS Lyapunov functions can be constructed for systems $\Sigma(A,B)$ with an admissible input operator B provided the operator A has some additional properties.

Here we generally assume that X is a Hilbert space and that the input space is given by $\mathscr{U} := L^\infty([0,\infty),U)$.

Our main result in this section is a constructive converse ISS Lyapunov theorem for certain classes of linear systems with admissible input operators.

Proposition 4.1: Let A be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on a Hilbert space X and let $B \in L(U, X_{-1})$ and assume that the system $\Sigma(A,B)$ is ISS.

Further, assume that $D(A) \subseteq D(A^*)$ and the inequality

$$\mathrm{Re}\,\langle A^*A^{-1}x,x\rangle_X + \delta\|x\|_X^2 \geq 0 \tag{IV.1}$$

holds for some $\delta < 1$ and every $x \in X$, and $\mathrm{Re}\,\langle Ax,x\rangle_X < 0$ for every $x \in D(A)\backslash\{0\}$.

Then

$$V(x) := -\mathrm{Re}\,\langle A^{-1}x,x\rangle_X \tag{IV.2}$$

is an ISS Lyapunov function satisfying

$$\begin{aligned}
\dot{V}_u(x) &\leq -(1-\delta-\varepsilon)\|x_0\|_X^2 \\
&\quad + \left(\frac{(\|A^*A^{-1}\|_{L(X)}+1)^2\|A_{-1}^{-1}B\|^2}{4\varepsilon} + \|A_{-1}^{-1}B\|\kappa(0)\right)\|u\|_\infty^2
\end{aligned}$$

for $\varepsilon \in (0, 1-\delta)$, $x_0 \in X$ and $u \in L^\infty([0,\infty),U)$. Here $\kappa(0) = \lim_{t\searrow 0}\kappa(t)$, where $\kappa(t) > 0$ satisfies

$$\left\|\int_0^t T_{-1}(t-s)Bu(s)\,ds\right\|_X \leq \kappa(t)\|u\|_\infty, \qquad u \in L^\infty([0,t),U).$$

Proof:

Since Σ is ISS, A generates an exponentially stable semigroup, which implies (see e.g. [21, Proposition 5.2.4]) that

$0 \in \rho(A)$ and so $A^{-1} \in L(X)$. Thus, $V(x) \le \|A^{-1}\|_{L(X)} \|x\|_X^2$ for any $x \in X$. Moreover, for any $x \in X \backslash \{0\}$ there is $y \in D(A) \backslash \{0\}$ so that $x = Ay$. Then by the assumptions of the proposition it holds that

$$V(x) = -\mathrm{Re}\,\langle Ay, y\rangle > 0.$$

Thus, property (III.2) is satisfied. It remains to show the dissipation inequality (III.3) for V.

For $x_0 \in X$ and $u \in L^\infty([0,\infty), U)$ we have

$$\begin{aligned}
&V(\phi(t,x_0,u)) - V(x_0) \qquad\qquad\qquad\qquad \text{(IV.3)}\\
&= \mathrm{Re}\,\langle A^{-1}x_0, x_0\rangle_X \\
&\quad - \mathrm{Re}\left\langle A^{-1}\left(T(t)x_0 + \int_0^t T_{-1}(t-s)Bu(s)ds\right),\right. \\
&\qquad\qquad \left. T(t)x_0 + \int_0^t T_{-1}(t-s)Bu(s)ds \right\rangle_X \\
&= \mathrm{Re}\,\langle A^{-1}x_0, x_0\rangle_X - \mathrm{Re}\,\langle A^{-1}T(t)x_0, x_0\rangle_X \\
&\quad + \mathrm{Re}\,\langle A^{-1}T(t)x_0, x_0\rangle_X - \mathrm{Re}\,\langle A^{-1}T(t)x_0, T(t)x_0\rangle_X \quad \text{(IV.4)}\\
&\quad - \mathrm{Re}\,\langle A^{-1}T(t)x_0, \int_0^t T_{-1}(t-s)Bu(s)ds\rangle_X \qquad \text{(IV.5)}\\
&\qquad - \mathrm{Re}\left\langle A_{-1}^{-1}\int_0^t T_{-1}(t-s)Bu(s)ds, T(t)x_0\right\rangle_X \\
&\quad - \mathrm{Re}\left\langle A_{-1}^{-1}\int_0^t T_{-1}(t-s)Bu(s)ds, \int_0^t T_{-1}(t-s)Bu(s)ds\right\rangle_X.
\end{aligned}$$

The terms in the second line (IV.4) of the previous expression can be transformed into:

$$\begin{aligned}
&\mathrm{Re}\,\langle A^{-1}T(t)x_0, x_0\rangle_X - \mathrm{Re}\,\langle A^{-1}T(t)x_0, T(t)x_0\rangle_X \\
&\quad = \mathrm{Re}\,\langle A^{-1}T(t)x_0, x_0 - T(t)x_0\rangle_X \\
&\quad = \mathrm{Re}\,\langle A^{-1}T(t)x_0, AA^{-1}(x_0 - T(t)x_0)\rangle_X \\
&\quad = \mathrm{Re}\,\langle A^*A^{-1}T(t)x_0, A^{-1}(x_0 - T(t)x_0)\rangle_X. \qquad \text{(IV.6)}
\end{aligned}$$

Next we estimate the expression in (IV.5) by means of the Cauchy-Schwarz inequality:

$$\begin{aligned}
&-\mathrm{Re}\,\langle A^{-1}T(t)x_0, \int_0^t T_{-1}(t-s)Bu(s)ds\rangle_X \\
&\quad = -\mathrm{Re}\,\langle A^*A^{-1}T(t)x_0, A_{-1}^{-1}\int_0^t T_{-1}(t-s)Bu(s)ds\rangle_X \\
&\quad \le \|A^*A^{-1}T(t)x_0\|_X \cdot \|A_{-1}^{-1}\int_0^t T_{-1}(t-s)Bu(s)ds\|_X \\
&\quad = \|A^*A^{-1}T(t)x_0\|_X \cdot \|\int_0^t A_{-1}^{-1}T_{-1}(t-s)Bu(s)ds\|_X.
\end{aligned}$$

Due to the fact, that A_{-1}^{-1} and $T_{-1}(t-s)$ commute, we proceed to

$$\begin{aligned}
&-\mathrm{Re}\,\langle A^{-1}T(t)x_0, \int_0^t T_{-1}(t-s)Bu(s)ds\rangle_X \\
&\quad \le \|A^*A^{-1}T(t)x_0\|_X \cdot \left\|\int_0^t T_{-1}(t-s)A_{-1}^{-1}Bu(s)ds\right\|_X.
\end{aligned}$$

Now, since $A_{-1} : X \to D(A)$ is a bounded operator, then also $A_{-1}^{-1} : X_{-1} \to X$ is a bounded operator, and since $B \in L(U, X_{-1})$, we have that $A_{-1}^{-1}B \in L(U,X)$. In particular, $T_{-1}(t-s)A_{-1}^{-1}Bu(s) \in X$ for all $s \ge 0$, and hence we can continue the estimates as

$$\begin{aligned}
&-\mathrm{Re}\,\langle A^{-1}T(t)x_0, \int_0^t T_{-1}(t-s)Bu(s)ds\rangle_X \\
&\quad \le \|A^*A^{-1}T(t)x_0\|_X \cdot \int_0^t \left\|T_{-1}(t-s)A_{-1}^{-1}Bu(s)\right\|_X ds \\
&\quad \le \|A^*A^{-1}T(t)x_0\|_X \cdot \int_0^t M ds \|A_{-1}^{-1}B\|_{L(U,X)}\|u\|_\infty \\
&\quad \le \|A^*A^{-1}T(t)x_0\|_X \cdot Mt\|A_{-1}^{-1}B\|_{L(U,X)}\|u\|_\infty. \qquad \text{(IV.7)}
\end{aligned}$$

Substituting (IV.6) and (IV.7) into (IV.3), and using that $T(t)$ commutes with the resolvent of A, so in particular with A^{-1}, we obtain:

$$\begin{aligned}
V(\phi(t,x_0,u)) - V(x_0) &\le -\mathrm{Re}\,\langle T(t)A^{-1}x_0 - A^{-1}x_0, x_0\rangle_X \\
&\quad - \mathrm{Re}\,\langle A^*A^{-1}T(t)x_0, T(t)A^{-1}x_0 - A^{-1}x_0\rangle_X \\
&\quad + \|A^*A^{-1}T(t)x_0\|_X \cdot Mt\|A_{-1}^{-1}B\|_{L(U,X)}\|u\|_\infty \\
&\quad + Mt\|A_{-1}^{-1}B\|_{L(U,X)}\|u\|_\infty \cdot \|T(t)x_0\|_X \\
&\quad + Mt\|A_{-1}^{-1}B\|_{L(U,X)}\kappa(t)\|u\|_\infty^2,
\end{aligned}$$

which implies for every $\varepsilon > 0$

$$\begin{aligned}
\dot{V}_u(x_0) &= \overline{\lim_{t \searrow 0}} \frac{1}{t}(V(\phi(t,x_0,u)) - V(x_0)) \\
&\le -\|x_0\|_X^2 - \mathrm{Re}\,\langle A^*A^{-1}x_0, x_0\rangle_X \\
&\quad + \|A^*A^{-1}x_0\|_X \|A_{-1}^{-1}B\|_{L(U,X)} M\|u\|_\infty \\
&\quad + \|A_{-1}^{-1}B\|_{L(U,X)} M\|u\|_\infty \|x_0\|_X \\
&\quad + \|A_{-1}^{-1}B\|_{L(U,X)} M\kappa(0)\|u\|_\infty^2 \\
&\le -\|x_0\|_X^2 - \mathrm{Re}\,\langle A^*A^{-1}x_0, x_0\rangle_X \\
&\quad + \|x_0\|_X(\|A^*A^{-1}\|_{L(X)} + 1)\|A_{-1}^{-1}B\|M\|u\|_\infty \\
&\quad + \|A_{-1}^{-1}B\|_{L(U,X)} M\kappa(0)\|u\|_\infty^2
\end{aligned}$$

Using Young's inequality and the estimate (IV.1) we proceed to

$$\begin{aligned}
\dot{V}_u(x_0) &\le -\|x_0\|_X^2 - \mathrm{Re}\,\langle A^*A^{-1}x_0, x_0\rangle_X + \varepsilon\|x_0\|_X^2 \\
&\quad + \frac{(\|A^*A^{-1}\|_{L(X)} + 1)^2\|A_{-1}^{-1}B\|_{L(U,X)}^2 M^2}{4\varepsilon}\|u\|_\infty^2 \\
&\quad + \|A_{-1}^{-1}B\|_{L(U,X)} M\kappa(0)\|u\|_\infty^2 \\
&\le (\varepsilon - 1 + \delta)\|x_0\|_X^2 \\
&\quad + \Big(\frac{1}{4\varepsilon}(\|A^*A^{-1}\|_{L(X)} + 1)^2\|A_{-1}^{-1}B\|_{L(U,X)}^2 M^2 \\
&\qquad\qquad + \|A_{-1}^{-1}B\|_{L(U,X)}\kappa(0)\Big)\|u\|_\infty^2
\end{aligned}$$

Here we have used in particular that A^*A^{-1} is a bounded operator, which follows from $\mathrm{ran}(A^{-1}) = D(A) \subset D(A^*)$ and where the last inclusion holds by assumption. This shows the dissipation inequality (III.3). ■

Remark 4.2: Inequality (IV.1) is equivalent to the existence of a constant $\delta' < 1$ satisfying

$$\|(A + A^*)x\|_X^2 + \delta'\|Ax\|_X^2 \ge \|A^*x\|_X^2, \qquad x \in D(A).$$

If A generates a strongly continuous contraction semigroup, then (IV.1) implies that the semigroup $(T(t))_{t\geq 0}$ is 2-hypercontractive [22]. In particular, subnormal and normal operators whose spectrum lie in a sector satisfy (IV.1), see Proposition 5.2.

Corollary 4.3: Let A generate an exponentially stable analytic semigroup on a Hilbert space X and assume that A is a normal operator. Further, let $B \in L(\mathbb{C}^n, X_{-1})$ be ∞-admissible. Then

$$V(x) := -\mathrm{Re}\,\langle A^{-1}x, x\rangle_X \tag{IV.8}$$

is an ISS Lyapunov function satisfying

$$\dot{V}_u(x) \leq -c_1\|x_0\|_X^2 + c_2\|u\|_\infty^2$$

for some constants $c_1, c_2 > 0$ and all $x_0 \in X$ and $u \in L^\infty([0,\infty), U)$.

Proof: Section V-B shows that the assumption of Proposition 4.1 are satisfied. ■

Remark 4.4: By Section V-B normality in Corollary 4.3 can be replaced by subnormality.

V. Applications of Proposition 4.1

A. ISS Lyapunov functions for a heat equation with Dirichlet boundary input

It is well-known that a classical heat equation with Dirichlet boundary inputs is ISS, which has been verified by means of several different methods: [6], [7], [8]. However, no constructions for ISS Lyapunov functions have been proposed. In the next example we show that using Proposition 4.1 one can construct a non-coercive ISS Lyapunov function for this system.

Example 5.1: Let us consider the following boundary control system given by the one-dimensional heat equation on the spatial domain $[0,1]$ with Dirichlet boundary control at the point 1,

$$\begin{aligned}
&x_t(\xi,t) = a x_{\xi\xi}(\xi,t), \quad \xi \in (0,1),\ t > 0,\\
&x(0,t) = 0, \quad x(1,t) = u(t), \quad t > 0,\\
&x(\xi,0) = x_0(\xi),
\end{aligned}$$

where $a > 0$.

We choose $X = L^2(0,1)$, $U = \mathbb{C}$,

$$\begin{aligned}
Af &= f'', \quad f \in D(A),\\
D(A) &= \left\{f \in H^2(0,1) \mid f(0) = f(1) = 0\right\}.
\end{aligned}$$

and $B = a\delta_1'$. Clearly, A is a self-adjoint operator on X generating an exponentially stable analytic C_0-semigroup on X. Moreover, $B \in X_{-1} = L(U, X_{-1})$ is ∞-admissible, for every $x_0 \in X$ and $u \in L^\infty(0,\infty)$ the corresponding mild solution is continuous and $\kappa(0) = 0$ [6]. Further, in [6] the following ISS-estimates has been shown:

$$\|x(t)\|_{L^2(0,1)} \leq \mathrm{e}^{-a\pi^2 t}\|x_0\|_{L^2(0,1)} + \frac{1}{\sqrt{3}}\|u\|_{L^\infty(0,t)},$$

$$\|x(t)\|_{L^2(0,1)} \leq \mathrm{e}^{-a\pi^2 t}\|x_0\|_{L^2(0,1)} + c\left(\int_0^t |u(s)|^p ds\right)^{1/p},$$

for every $x_0 \in X$, $u \in L^\infty(0,\infty)$, $p > 2$ and some constant $c = c(p) > 0$. Due to the self-adjointness of A, Equation (IV.1) holds for every $\delta \geq -1$. Then we may compute that

$$\begin{aligned}
V(x) &= -\langle A^{-1}x, x\rangle_X\\
&= \int_0^1 \left(\int_\xi^1 (\xi - \tau)x(\tau)d\tau\right)\overline{x(\xi)}d\xi
\end{aligned}$$

is a non-coercive ISS Lyapunov function for the one-dimensional heat equation on the spatial domain $[0,1]$ with Dirichlet boundary control at the point 1.

B. An inequality for subnormal A

In this section we would like to argue that the inequality (IV.1), which is one of the central assumptions in Proposition 4.1, holds for a broad class of subnormal operators over Hilbert spaces.

Let A be closed, densely-defined and subnormal operator on a Hilbert space X. Here A is called *subnormal*, if $A = N_{|X}$ where N is a normal operator on a Hilbert space Z and X is an invariant subspace for N, that is, $N(D(N) \cap X) \subseteq X$. We write P for the orthogonal projection from Z onto X. That is, up to unitary equivalence $N = M_\phi$, a multiplication operator on some $L^2(\mu)$ space, and $Af = \phi f$, $A^*f = P(\overline{\phi} f)$. See, for example [23, Th. X.4.19]. Moreover, a closed, densely-defined and subnormal operator A satisfies $D(A) \subset D(A^*)$, since $D(N) = D(N^*)$ [23, Prop. X.4.3].

For $\theta \in [0, \pi/2)$ we define

$$\Sigma_\theta := \{s \in \mathbb{C} \mid |\arg(-s)| \leq \theta\}.$$

Proposition 5.2: Let A be closed, densely-defined and subnormal operator on a Hilbert space X satisfying $\sigma(A) \subseteq \Sigma_\theta$, for some $\theta \in [0, \pi/2)$. Then for $\delta \geq 1 - 2\cos^2\theta$ we have

$$\mathrm{Re}\,\langle x, A^2x\rangle_X + \delta\|Ax\|_X^2 \geq 0, \qquad f \in D(A). \tag{V.1}$$

Proof: Expanding (V.1) we obtain the equivalent assertion

$$\mathrm{Re}\,\langle \phi f, P\overline{\phi} f\rangle + \delta\|\phi f\|^2 \geq 0, \tag{V.2}$$

and we note that $\langle \phi f, P\overline{\phi} f\rangle = \langle \phi f, \overline{\phi} f\rangle = \langle \phi^2 f, f\rangle$. The left hand side of (V.2) is

$$\langle(\mathrm{Re}\,\phi^2 + \delta|\phi|^2)f, f\rangle = \langle(2(\mathrm{Re}\,\phi)^2 + (\delta - 1)|\phi|^2)f, f\rangle.$$

Now, since the essential range of ϕ lies in $\sigma(A)$, we have by sectoriality

$$2(\mathrm{Re}\,\phi)^2 \geq 2\cos^2\theta|\phi|^2$$

and hence

$$\langle(2(\mathrm{Re}\,\phi)^2 + (\delta - 1)|\phi|^2)f, f\rangle \geq 0,$$

for $\delta \geq 1 - 2\cos^2\theta$. ■

Proposition 5.3: Let A be closed, densely-defined and subnormal operator on a Hilbert space X satisfying $\sigma(A) \subseteq \Sigma_\theta$, for some $\theta \in [0, \pi/2)$. Then A generates a analytic C_0-semigroup of contractions.

Example 5.4:
1) Clearly, every normal operator on a Hilbert space is subnormal.
2) Symmetric operators on Hilbert spaces are subnormal.

3) Isometries are subnormal, and hence a right shift operator on $L^2(0,\infty)$ is subnormal.
4) Multiplication operators (analytic Toeplitz operators T_g) on the Hardy space $H^2(\mathrm{D})$ are subnormal, and T_g is sectorial if $g(\mathrm{D}) \subseteq \Sigma_\theta$ for some $\theta \in [0, \pi/2)$.

VI. Conclusion

In this paper we have investigated the question to what extent the existence of a noncoercive ISS Lyapunov function implies that a forward complete system is ISS. It was shown that the property of integral-to-integral ISS follows from the existence of such Lyapunov functions for a large class of systems. In order to arrive at ISS in its own right further assumptions were necessary. These further assumptions, the CEP property and the BRS property relate to questions of the richness of the possible dynamics both close to the origin and in the large.

The construction of noncoercive Lyapunov functions is to some extent natural in infinite dimesions. Already for Datko's construction of quadratic Lyapunov functions for exponentially stable linear systems on Hilbert space it sometimes cannot be avoided to use a noncoercive version. Also we have seen in this paper for some classes of linear systems with unbounded input operators the construction of Lyapunov functions using the resolvent at 0 is a natural choice and one that leads to noncoercive Lyapunov functions.

In future work we plan to extend the class of systems for which explicit constructions are possible and to deepen our understanding of noncoercive ISS Lyapunov functions.

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