# Construction of ISS Lyapunov functions for infinite networks of ISS systems

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*Abstract*—We show that an infinite network of input-to-state stable (ISS) systems, admitting ISS Lyapunov functions, itself admits an ISS Lyapunov function, provided that the couplings of the subsystems are sufficiently weak. The strength of the couplings is described in terms of the properties of the socalled gain operator, built from the interconnection gains. If the discrete-time system induced by a slightly scaled gain operator is uniformly globally asymptotically stable, an ISS Lyapunov function for the infinite network can be constructed.

**Keywords**: large-scale systems, small-gain theorems, input-to-state stability, nonlinear systems, infinite-dimensional systems.

# I. INTRODUCTION

Current society is surrounded by networks: social networks, power grids, transportation and manufacturing networks, etc. These networks grow in size from year to year, and emerging technologies, such as the Internet of Things, Cloud Computing, 5G communication, and smart cities, make this trend even more distinct. As the stability properties of the networks may deteriorate with the increase in the number of participating agents [1], it is natural to study infinitedimensional over-approximations of large-scale networks as a worst-case scenario.

The theory of linear *spatially invariant systems* [2], [3], [4], [5] has a prominent place in these investigations. Here, infinitely many subsystems are coupled via the same pattern. This nice geometric structure together with the *linearity* of subsystems allows to develop powerful stability criteria.

On the other hand, in the stability analysis of *finite net-works with nonlinear components*, groundbreaking results have been obtained within the framework of *input-to-state stability (ISS)* [6]. According to the ISS *small-gain approach*, the influence of any subsystem on other subsystems of a network is characterized by so-called gain functions. The *gain operator* constructed from these functions characterizes

the interconnection structure of the network. The smallgain theorems for couplings of finitely many ISS systems of ordinary differential equations (ODEs) [7], [8], [9], [10] state that if the gains are small enough (in an appropriate sense), the network is stable. These results have numerous applications in systems theory [11], [12], [13], [14].

Recently, the intensive development of an *infinitedimensional ISS theory* has been initiated; see [15], [16] for a comprehensive overview of the topic, and [17] for an overview of the linear theory. This progress motivated the development of the ISS small-gain framework for the stability analysis of infinite interconnections of nonlinear systems without any spatial invariance assumption. This research was initiated in [18], where nonlinear Lyapunovbased small-gain theorems have been obtained under the quite strong assumption that all gains are uniformly less than the identity. In [19], tight Lyapunov-based small-gain theorems have been obtained for networks of exponentially ISS systems with linear gains.

Nonlinear trajectory-based small-gain theorems for infinite networks have been derived in [20]. Here, it was shown that an infinite network of ISS systems is ISS if the corresponding nonlinear gain operator satisfies the so-called *monotone limit property*, which in turn implies the *uniform small-gain condition* [20], which is equivalent to the monotone bounded invertibility property. The latter played a key role in the derivation of the ISS small-gain theorem for finite networks in [9]; see, e.g., [9, Lem. 13].

This paper is strongly motivated by [21], where the *ro-bust strong small-gain condition* has been introduced and a method to construct paths of strict decay was proposed, based on the concept of the *strong transitive closure* of the gain operator. For finite networks, this method was proposed in [22, Prop. 2.7, Rem. 2.8]; see also [23] for more details on the importance of this concept in the small-gain theory. Based on these results, in [21] a small-gain theorem for infinite networks, and the construction of an ISS Lyapunov function for the network were proposed under the assumption that a *linear* path of strict decay exists. In general, this requirement is quite restrictive, and Lyapunov-based small-gain theorems for finite networks developed in [10] do not require the linearity assumption for the path of strict decay.

**Contribution.** We consider an infinite network of ISS control systems given by ODEs. We assume that these systems admit ISS Lyapunov functions with corresponding Lyapunov gains,

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giving rise to the gain operator  $\Gamma$  that characterizes the influence of the subsystems on each other. We show that *the* existence of a (nonlinear) path of strict decay for  $\Gamma$  (together with some uniformity conditions) implies ISS of the whole network and the existence of an ISS Lyapunov function. Our result partially extends the nonlinear Lyapunov-based small-gain theorem for finite networks (in maximum formulation) in [10] to infinite dimensions, recovers the Lyapunov-based small-gain theorem for infinite networks in [21], and partially recovers the main result in [18].

In addition, we introduce the *max-robust small-gain condition*, which is less conservative than the robust smallgain condition from [21], but is better suited for max-type gain operators, and can be characterized in terms of the asymptotic properties of the discrete-time system induced by the gain operator. In our second main result (Theorem VI.5), we show that the uniform global asymptotic stability (UGAS) of the system induced by a scaled gain operator guarantees the existence of a path of strict decay. We explicitly construct this path via the concept of the strong transitive closure of the gain operator. Finally, we characterize the UGAS property of the induced system in terms of small-gain conditions and provide sufficient conditions for it. If the gain operator is homogeneous and subadditive, further characterizations for the UGAS property have been shown in [24].

Complete proofs of the presented results can be found in the journal version [25] of this paper.

**Notation.** We write  $\mathbb{R}$  ( $\mathbb{R}_+$ ) for the set of (nonnegative) real numbers and  $\mathbb{Z}$  ( $\mathbb{Z}_+$ ) for the set of (nonnegative) integers. By  $C^0(X, Y)$ , we denote the set of all continuous mappings from a space X to a space Y. In any metric space, we write  $B_{\delta}(x)$  for the open ball of radius  $\delta > 0$  centered at x, and int(A) for the interior of a subset  $A \subset X$ . A continuous function  $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$  is called *positive definite* if  $\alpha(0) = 0$ and  $\alpha(r) > 0$  for all r > 0. For the sets of comparison functions  $\mathcal{K}, \mathcal{K}_{\infty}, \mathcal{L}$  and  $\mathcal{KL}$ , we refer to [26].

We write  $\ell_{\infty}$  for the space of bounded real sequences  $s = (s_i)_{i \in \mathbb{N}}$ , which is a Banach space with the norm  $||s||_{\ell_{\infty}} := \sup_{i \in \mathbb{N}} |s_i|$ . The *positive cone* in  $\ell_{\infty}$  is given by  $\ell_{\infty}^+ := \{s \in \ell_{\infty} : s_i \geq 0, \forall i \in \mathbb{N}\}$ . We write  $s^1 \geq s^2$  if  $s^1 - s^2 \in \ell_{\infty}^+$  and  $s^1 \not\geq s^2$  if  $s^1 - s^2 \notin \ell_{\infty}^+$ . We define  $\mathbb{1} := (1, 1, 1, \ldots) \in \ell_{\infty}^+$ . By  $e_i, i \in \mathbb{N}$ , we denote the *i*-th unit vector in  $\ell_{\infty}$ . We write  $s^1 \oplus s^2$  for the componentwise maximum of  $s^1, s^2 \in \ell_{\infty}^+$ . By  $\pi_i : \ell_{\infty} \to \mathbb{R}$ , we denote the canonical projection onto the *i*-th component,  $\pi_i(s) = s_i$ .

A function  $\lambda : \mathbb{R}_+ \to X$  into some space X is called *piecewise right-continuous* if there is a partition of  $\mathbb{R}_+$  into disjoint subintervals,  $\mathbb{R}_+ = [0, t_1) \cup [t_1, t_2) \cup [t_2, t_3) \cup \ldots$ , such that  $\lambda$  is continuous on each subinterval.

## **II. INTERCONNECTIONS**

Consider a family of control systems of the form

$$\Sigma_i: \quad \dot{x}_i = f_i(x_i, \bar{x}_i, u_i), \quad i \in \mathbb{N}.$$
(1)

This family comes with sequences  $(n_i)_{i \in \mathbb{N}}$  and  $(m_i)_{i \in \mathbb{N}}$  of positive integers as well as *finite* (possibly empty) sets  $I_i \subset \mathbb{N}$ ,  $i \notin I_i$ , such that the following assumptions are satisfied:

- The state vector  $x_i$  is an element of  $\mathbb{R}^{n_i}$ .
- The *internal input vector* x
  <sub>i</sub> is composed of the state vectors x<sub>j</sub>, j ∈ I<sub>i</sub>, and thus is an element of ℝ<sup>N<sub>i</sub></sup>, where N<sub>i</sub> := ∑<sub>j∈Ii</sub> n<sub>j</sub>.
- The external input vector  $u_i$  is an element of  $\mathbb{R}^{m_i}$ .
- The *right-hand side*  $f_i : \mathbb{R}^{n_i} \times \mathbb{R}^{N_i} \times \mathbb{R}^{m_i} \to \mathbb{R}^{n_i}$  is a continuous function.
- For every initial state  $x_{i0} \in \mathbb{R}^{n_i}$  and all essentially bounded inputs  $\bar{x}_i(\cdot)$  and  $u_i(\cdot)$ , there is a unique solution of  $\Sigma_i$ , which we denote by  $\phi_i(t, x_{i0}, \bar{x}_i, u_i)$  (it may be defined only on a bounded time interval).

For each  $i \in \mathbb{N}$ , we fix (arbitrary) norms on the spaces  $\mathbb{R}^{n_i}$  and  $\mathbb{R}^{m_i}$ , respectively. For brevity in notation, we avoid adding an index to these norms, indicating to which space they belong, and simply write  $|\cdot|$  for each of them. The interconnection of the systems  $\Sigma_i$ ,  $i \in \mathbb{N}$ , is defined on the state space  $X := \ell_{\infty}(\mathbb{N}, (n_i))$ , where

$$\ell_{\infty}(\mathbb{N}, (n_i)) := \{ x = (x_i)_{i \in \mathbb{N}} : x_i \in \mathbb{R}^{n_i}, \sup_{i \in \mathbb{N}} |x_i| < \infty \}.$$

This space is a Banach space with the  $\ell_{\infty}$ -type norm

$$||x||_X := \sup_{i \in \mathbb{N}} |x_i|.$$

The space of admissible external input values is likewise defined as the Banach space

$$U := \ell_{\infty}(\mathbb{N}, (m_i)), \quad \|u\|_U := \sup_{i \in \mathbb{N}} |u_i|.$$

The class of admissible external input functions is chosen as

 $\mathcal{U} := \{ u \in L_{\infty}(\mathbb{R}_+, U) : u \text{ is piecewise right-continuous} \},\$ 

which will be equipped with the  $L_{\infty}$ -norm

$$||u||_{\mathcal{U}} := \operatorname{ess\,sup}_{t \in \mathbb{R}_+} |u(t)|_U.$$

The right-hand side of the interconnected system is

$$f: X \times U \to \prod_{i \in \mathbb{N}} \mathbb{R}^{n_i}, \quad f(x, u) := (f_i(x_i, \bar{x}_i, u_i))_{i \in \mathbb{N}}.$$

Hence, the interconnected system can formally be written as the differential equation

$$\Sigma: \quad \dot{x} = f(x, u).$$

For fixed  $(u, x^0) \in \mathcal{U} \times X$ , a function  $\lambda : J \to X$ , where  $J \subset \mathbb{R}$  is an interval of the form [0,T) with  $0 < T \leq \infty$ , is called a *solution* of the Cauchy problem  $\dot{x} = f(x, u)$ ,  $x(0) = x^0$ , if  $s \mapsto f(\lambda(s), u(s))$  is a locally integrable X-valued function (in the Bochner integral sense) and

$$\lambda(t) = x^0 + \int_0^t f(\lambda(s), u(s)) \,\mathrm{d}s \quad \text{for all } t \in J.$$

We say that the system  $\Sigma$  is *well-posed* if for every initial value  $x^0 \in X$  and every external input  $u \in \mathcal{U}$ , a

unique maximal solution, which we denote by  $\phi(\cdot, x^0, u)$ :  $[0, t_{\max}(x^0, u)) \to X$  exists, where  $0 < t_{\max}(x^0, u) \le \infty$ .

Sufficient conditions for well-posedness are provided by [19, Cor. III.3]. In the rest of the paper, we assume the following.

**II.1 Assumption:** The system  $\Sigma$  is well-posed, and all of its uniformly bounded maximal solutions  $\phi(\cdot, x, u)$  are global, i.e., exist on  $\mathbb{R}_+$  (this latter property is also called boundedness-implies-continuation (BIC) property).

**II.2 Remark:** A sufficient condition for the BIC property is that the function f is uniformly bounded on bounded balls, and Lipschitz continuous on bounded balls with respect to the first argument (see [27, Thm. 4.3.4] for the related result for systems without inputs).

#### III. INPUT-TO-STATE STABILITY

We now recall the definition of input-to-state stability.

**III.1 Definition:** A well-posed system  $\Sigma$  is called input-tostate stable (ISS) if it is forward complete and there are  $\beta \in \mathcal{KL}$  and  $\gamma \in \mathcal{K}_{\infty}$  s.t. for all  $(t, x, u) \in \mathbb{R}_+ \times X \times \mathcal{U}$ 

$$\|\phi(t, x, u)\|_{X} \le \beta(\|x\|_{X}, t) + \gamma(\|u\|_{\mathcal{U}}).$$

Input-to-state stability is most often verified via the construction of an ISS Lyapunov function which is defined as follows.

**III.2 Definition:** A function  $V : X \to \mathbb{R}_+$  is called an ISS Lyapunov function (in an implication form) for  $\Sigma$  if

- (i) V is continuous.
- (ii) There exist  $\psi_1, \psi_2 \in \mathcal{K}_{\infty}$  such that  $\psi_1(||x||_X) \leq V(x) \leq \psi_2(||x||_X)$  for all  $x \in X$ .
- (iii) There exist  $\gamma \in \mathcal{K}$  and  $\alpha \in \mathcal{P}$  such that for all  $x \in X$ and  $u \in \mathcal{U}$  the following implication holds:

$$V(x) > \gamma(||u||_{\mathcal{U}}) \quad \Rightarrow \quad \mathrm{D}^+ V_u(x) \le -\alpha(V(x)),$$

where  $D^+V_u(x)$  denotes the right upper Dini orbital derivative defined as

$$D^+V_u(x) := \limsup_{t \to 0^+} \frac{V(\phi(t, x, u)) - V(x)}{t}.$$

The importance of ISS Lyapunov functions is due to the following basic fact (cf. [16, Thm. 2.17]).

**III.3 Proposition:** If an ISS Lyapunov function for  $\Sigma$  exists, then  $\Sigma$  is ISS.

#### IV. NONLINEAR SMALL-GAIN THEOREM

To find an ISS Lyapunov function V for  $\Sigma$ , we exploit the interconnection structure and construct V from ISS Lyapunov functions of the subsystems  $\Sigma_i$  under an appropriate smallgain condition. We introduce the following assumption: **IV.1 Assumption:** For each  $i \in \mathbb{N}$ , there exists a continuous function  $V_i : \mathbb{R}^{n_i} \to \mathbb{R}_+$  which is continuously differentiable outside of  $x_i = 0$  and satisfies the following properties:

(L1) There exist  $\psi_{i1}, \psi_{i2} \in \mathcal{K}_{\infty}$  such that

 $\psi_{i1}(|x_i|) \leq V_i(x_i) \leq \psi_{i2}(|x_i|)$  for all  $x_i \in \mathbb{R}^{n_i}$ .

(L2) There exist  $\gamma_{ij} \in \mathcal{K} \cup \{0\}$ , where  $\gamma_{ij} = 0$  for all  $j \in \mathbb{N} \setminus I_i$ , and  $\gamma_{iu} \in \mathcal{K}$  as well as  $\alpha_i \in \mathcal{P}$  such that for all  $x = (x_j)_{j \in \mathbb{N}} \in X$  and  $u = (u_j)_{j \in \mathbb{N}} \in U$  the following implication holds:

$$V_i(x_i) > \max\left\{\sup_{j \in I_i} \gamma_{ij}(V_j(x_j)), \gamma_{iu}(|u_i|)\right\}$$
  
$$\Rightarrow \nabla V_i(x_i) f_i(x_i, \bar{x}_i, u_i) \le -\alpha_i(V_i(x_i)).$$

The function  $V_i$  is called an ISS Lyapunov function for  $\Sigma_i$ . The functions  $\gamma_{ij}$  and  $\gamma_{iu}$  are called internal gains and external gains, respectively.

Using the internal gains  $\gamma_{ij}$  from Assumption IV.1, we define the gain operator  $\Gamma: \ell_{\infty}^+ \to \ell_{\infty}^+$  by

$$\Gamma(s) := \left(\sup_{j \in \mathbb{N}} \gamma_{ij}(s_j)\right)_{i \in \mathbb{N}}, \quad \Gamma : \ell_{\infty}^+ \to \ell_{\infty}^+.$$

The following assumption guarantees that  $\Gamma$  is well-defined and continuous, see [21, Lem. 2.1] and [20, Prop. 2].

**IV.2 Assumption:** The family  $\{\gamma_{ij} : i, j \in \mathbb{N}\}$  is pointwise equicontinuous. That is, for every  $r \ge 0$  and every  $\varepsilon > 0$  there exists  $\delta = \delta(r, \varepsilon) > 0$  such that  $|r - \tilde{r}| \le \delta$ ,  $\tilde{r} \in \mathbb{R}_+$ , implies  $|\gamma_{ij}(r) - \gamma_{ij}(\tilde{r})| \le \varepsilon$  for all  $i, j \in \mathbb{N}$ .

We now introduce the concept of a *path of strict decay* which is of crucial importance in the construction of an ISS Lyapunov function for the interconnected system.

**IV.3 Definition:** A mapping  $\sigma : \mathbb{R}_+ \to \ell_{\infty}^+$  is called a path of strict decay (for  $\Gamma$ ), if the following properties hold:

(i) There exists a function  $\rho \in \mathcal{K}_{\infty}$  such that

$$\Gamma(\sigma(r)) \le (\mathrm{id} + \rho)^{-1} \circ \sigma(r) \quad \text{for all } r \ge 0,$$

where  $(id + \rho)^{-1}$  is applied componentwise.

- (ii) There exist  $\sigma_{\min}, \sigma_{\max} \in \mathcal{K}_{\infty}$  satisfying  $\sigma_{\min} \leq \sigma_i \leq \sigma_{\max}$  for all  $i \in \mathbb{N}$ .
- (iii) Each component function  $\sigma_i$ ,  $i \in \mathbb{N}$ , is in  $\mathcal{K}_{\infty}$ .
- (iv) For every compact interval  $K \subset (0, \infty)$ , there exist  $0 < c \le C < \infty$  such that for all  $r_1, r_2 \in K$  and  $i \in \mathbb{N}$ :

$$c|r_1 - r_2| \le |\sigma_i^{-1}(r_1) - \sigma_i^{-1}(r_2)| \le C|r_1 - r_2|.$$

In Section VI, we provide a method to construct paths of strict decay under suitable assumptions.

Our small-gain result now reads as follows.

**IV.4 Theorem:** Consider the interconnected system  $\Sigma$ , composed of the subsystems  $\Sigma_i$ ,  $i \in \mathbb{N}$  and let the following assumptions be satisfied.

- (i)  $\Sigma$  is well-posed and satisfies the BIC property (Ass. II.1).
- (ii) There exist ISS Lyapunov functions V<sub>i</sub> for the subsystems Σ<sub>i</sub> with associated internal gains γ<sub>ij</sub> and external gains γ<sub>iu</sub> (Ass. IV.1). Moreover, there exist ψ<sub>1</sub>, ψ<sub>2</sub> ∈ K<sub>∞</sub> such that ψ<sub>1</sub> ≤ ψ<sub>i1</sub> and ψ<sub>i2</sub> ≤ ψ<sub>2</sub> for all i ∈ N.
- (iii) The family of internal gains  $\gamma_{ij}$  is pointwise equicontinuous (Ass. IV.2) and there exists  $\gamma_{\max}^u \in \mathcal{K}$  such that  $\gamma_{iu} \leq \gamma_{\max}^u$  for all  $i \in \mathbb{N}$ .
- (iv) There exists a path  $\sigma : \mathbb{R}_+ \to \ell_{\infty}^+$  of strict decay for the gain operator  $\Gamma$ , defined from the internal gains  $\gamma_{ij}$ .
- (v) For each R > 0, there is L(R) > 0 such that

$$|V_i(x_i) - V_i(y_i)| \le L(R)|x_i - y_i|$$

for all  $i \in \mathbb{N}$  and  $x_i, y_i \in B_R(0) \subset \mathbb{R}^{n_i}$ .

(vi) There exists  $\tilde{\alpha} \in \mathcal{P}$  such that  $\alpha_i \geq \tilde{\alpha}$  for all  $i \in \mathbb{N}$ .

Then the following is an ISS Lyapunov function for  $\Sigma$ :

$$V(x) := \sup_{i \in \mathbb{N}} \sigma_i^{-1}(V_i(x_i)) \quad \text{for all } x \in X.$$

Moreover, V is locally Lipschitz continuous on  $X \setminus \{0\}$ . In particular,  $\Sigma$  is ISS.

#### V. GAIN OPERATORS AND THEIR PROPERTIES

A crucial assumption in Theorem IV.4 is the existence of a path of strict decay for the operator  $\Gamma$ . Our next aim is to understand under which conditions such a path exists, and to provide an explicit expression for it. We base our analysis on the properties of the gain operator, presented in this section.

From now on, we always assume that the family  $\{\gamma_{ij}\}$  is pointwise equicontinuous (Ass. IV.2), implying that  $\Gamma$  is well-defined and continuous. Observe that  $\Gamma(0) = 0$  and that  $\Gamma$  is monotone:  $\Gamma(s^1) \leq \Gamma(s^2)$  whenever  $s^1 \leq s^2$ .

Now we recall the important *robust and robust strong smallgain conditions*, introduced in [21]. We modify these properties to improve compatibility with max-type gain operators.

**V.1 Definition:** We say that the operator  $\Gamma$  satisfies

(i) the small-gain condition (SGC) if

$$\Gamma(s) \not\geq s \quad \text{for all } s \in \ell_{\infty}^+ \setminus \{0\}.$$

(ii) the strong small-gain condition if there is  $\rho \in \mathcal{K}_{\infty}$  with

$$D_{\rho} \circ \Gamma(s) \not\geq s$$
 for all  $s \in \ell_{\infty}^+ \setminus \{0\}$ 

for the operator  $D_{\rho}: \ell_{\infty}^+ \to \ell_{\infty}^+$ , defined by

$$D_{\rho}(s) := \left( (\mathrm{id} + \rho)(s_i) \right)_{i \in \mathbb{N}}$$

(iii) the max-robust small-gain condition if there is  $\omega \in \mathcal{K}_{\infty}$ with  $\omega < \text{id}$  such that for all  $i, j \in \mathbb{N}$  the operator

$$\Gamma_{ij}(s) := \Gamma(s) \oplus \omega(s_j)e_i \quad \text{for all } s \in \ell_{\infty}^+$$
 (2)

satisfies the small-gain condition.

(iv) the max-robust strong small-gain condition if there are  $\omega \in \mathcal{K}_{\infty}$  with  $\omega < \text{id}$  and  $\rho \in \mathcal{K}_{\infty}$  such that for all  $i, j \in \mathbb{N}$  the operator  $\Gamma_{ij}$ , as defined in (2), satisfies the strong small-gain condition with the same  $\rho$  for all i, j.

In the next lemma, we introduce the *strong transitive closure* (or Kleene star operator) Q of  $\Gamma$ , which provides the crucial tool for the construction of a path of strict decay. This result was shown in [21, Lem. 4.3], slightly strengthened in [20, Lem. 14] and is now even more strengthened, since the robust SGC is replaced by the weaker max-robust SGC.

**V.2 Lemma:** Assume that  $\Gamma$  satisfies the max-robust SGC. Then the operator

$$Q(s) := \bigoplus_{k \in \mathbb{Z}_+} \Gamma^k(s) \quad \text{for all } s \in \ell_\infty^+ \tag{3}$$

is well-defined and has the following properties:

$$s \le Q(s) \le \omega^{-1}(\|s\|_{\ell_{\infty}}) \mathbb{1} \quad \text{for all } s \in \ell_{\infty}^+,$$
  
 
$$\Gamma(Q(s)) \le Q(s) \quad \text{for all } s \in \ell_{\infty}^+.$$

Some further simple properties of the operator Q are summarized in the following proposition.

**V.3 Proposition:** Assume that  $\Gamma : \ell_{\infty}^{+} \to \ell_{\infty}^{+}$  is welldefined, continuous and satisfies the max-robust SGC. Then the operator Q in (3) is monotone, satisfies Q(0) = 0, and its image is given by  $\operatorname{im} Q = \{s \in \ell_{\infty}^{+} : \Gamma(s) \leq s\}$ . This set is closed, contains s = 0, is cofinal (i.e., for any  $x \in \ell_{\infty}^{+}$  there is  $s \in \operatorname{im} Q$  with  $x \leq s$ ) and satisfies  $\Gamma(\operatorname{im} Q) \subset \operatorname{im} Q$ .

For the gain operator  $\Gamma$  and any  $\theta \in \mathcal{K}_{\infty}$ , we define the operator  $\Gamma_{\theta} : \ell_{\infty}^+ \to \ell_{\infty}^+$  by  $\Gamma_{\theta}(s) := (\mathrm{id} + \theta) \circ \Gamma(s)$  for all  $s \in \ell_{\infty}^+$ . Here we apply the function  $\mathrm{id} + \theta$  componentwise:

$$\Gamma_{\theta}(s) = \left(\sup_{j \in \mathbb{N}} (\mathrm{id} + \theta) \circ \gamma_{ij}(s_j)\right)_{i \in \mathbb{N}}$$

Hence, the operator  $\Gamma_{\theta}$  is structurally the same as  $\Gamma$ , but with scaled gain functions.

We close the section with basic properties of gain operators satisfying the *max-robust strong SGC*:

**V.4 Lemma:** Assume that  $\Gamma$  satisfies the max-robust strong SGC with given  $\rho, \omega \in \mathcal{K}_{\infty}$ . Then  $\Gamma_{\rho}$  satisfies the max-robust SGC with the same  $\omega \in \mathcal{K}_{\infty}$ . Furthermore, there is  $\theta \in \mathcal{K}_{\infty}$  such that  $\Gamma_{\theta}$  also satisfies the max-robust strong SGC.

# VI. CONSTRUCTION OF PATHS OF STRICT DECAY

For the construction of paths of strict decay, the stability properties of the following discrete-time dynamical system induced by the gain operator  $\Gamma$  are of great importance:

$$s(k+1) = \Gamma(s(k)), \quad k \in \mathbb{Z}_+.$$
(4)

**VI.1 Definition:** System (4) is called

uniformly globally stable (UGS) if there is φ ∈ K<sub>∞</sub>, s.t. for any initial state s ∈ ℓ<sup>+</sup><sub>∞</sub>, the solution of (4) satisfies

$$\|\Gamma^k(s)\|_{\ell_{\infty}} \le \varphi(\|s\|_{\ell_{\infty}}), \quad \forall k \in \mathbb{Z}_+$$

 uniformly globally asymptotically stable (UGAS) if there is β ∈ KL, so that for any s ∈ ℓ<sup>+</sup><sub>∞</sub>

$$\|\Gamma^k(s)\|_{\ell_{\infty}} \le \beta(\|s\|_{\ell_{\infty}}, k), \quad \forall k \in \mathbb{Z}_+$$

• globally attractive if  $\lim_{k\to\infty} \|\Gamma^k(s)\|_{\ell_\infty} = 0 \quad \forall s \in \ell_\infty^+.$ 

The next proposition characterizes the max-robust SGC in terms of the stability properties of the system (4). It shows that the max-robust SGC is not quite equivalent to UGAS, but to a weaker property.

**VI.2 Proposition:** Assume that  $\Gamma$  is well-defined and continuous. Then the following statements are equivalent:

- (i) The system (4) is UGS and each of its trajectories converges to zero componentwise, i.e. π<sub>i</sub> ∘ Γ<sup>k</sup>(s) → 0 as k → ∞ for every s ∈ l<sup>+</sup><sub>∞</sub> and i ∈ N.
- (ii)  $\Gamma$  satisfies the max-robust SGC.

We provide a brief sketch of the proof of Proposition VI.2: For the implication "(i)  $\Rightarrow$  (ii)", we first show that (i) implies the existence of  $\varphi \in \mathcal{K}_{\infty}$  such that for any  $s, b \in \ell_{\infty}^+$ 

$$s \leq \Gamma(s) \oplus b \quad \Rightarrow \quad \|s\|_{\ell_{\infty}} \leq \varphi(\|b\|_{\ell_{\infty}}).$$
 (5)

To show this, we use in particular that  $\Gamma$  is a max-preserving operator, i.e.  $\Gamma(s^1 \oplus s^2) = \Gamma(s^1) \oplus \Gamma(s^2)$  for any  $s^1, s^2 \in \ell_{\infty}^+$ . From (5), it easily follows that the max-robust SGC holds for any  $\omega < \varphi^{-1}$ . For the implication "(ii)  $\Rightarrow$  (i)", we show that for any  $s \in \operatorname{im}(Q)$ , the trajectory  $(\Gamma^n(s))_{n \in \mathbb{N}_0}$  converges monotonically and componentwise to a fixed point  $s^*$  of  $\Gamma$ . Since  $\Gamma$  satisfies the SGC, this implies  $s^* = 0$ . The proof is completed by noting that  $s \leq Q(s)$  for any s.

By an example (see [25, Ex. V.3]), which is not provided here due to space constraints, one can show that the two statements in Proposition VI.2 are not equivalent to UGAS. Instead, we have the following relations:

It is well-known that for finite networks the max-preserving gain operator  $\Gamma$  induces a UGAS system if and only if all cycles composed of gains are contractions, see, e.g., [28, Thm. 6.4]. In the case of infinite networks, UGAS of the induced system can be characterized in terms of sufficiently long chains of gains, as shown in the next proposition.

**VI.3 Proposition:** Assume that the gain operator  $\Gamma$  is well-defined, continuous and satisfies the max-robust small-gain condition. Then the following statements are equivalent:

- (i) The induced system (4) is UGAS.
- (ii) There exist η ∈ K with η < id and i<sub>0</sub> ∈ N such that for every r ≥ 0 there is n ∈ N with

$$\sup_{\substack{j_0, j_1, \dots, j_n \in \mathbb{N} \\ j_0 \ge i_0}} \gamma_{j_0 j_1} \circ \dots \circ \gamma_{j_{n-1} j_n}(r) \le \eta(r).$$

The following proposition is crucial for the construction of paths of strict decay.

**VI.4 Proposition:** Let  $\Gamma : \ell_{\infty}^+ \to \ell_{\infty}^+$  be well-defined and continuous. If (4) is UGAS, then Q is continuous and for all  $s_1, s_2 \in \operatorname{int}(\ell_{\infty}^+)$  with  $s_1 \leq s_2$ , there is  $m \in \mathbb{N}$  such that

$$Q(s) = \sup_{0 \le k \le m} \Gamma^k(s), \quad \forall s : s_1 \le s \le s_2.$$

We can finally present our main result on the existence of paths of strict decay. It extends the first result of this kind in [21, Lem. 4.5], where properties (i)–(iii) of a path of strict decay have been shown under similar assumptions.

VI.5 Theorem: Let the following assumptions hold:

- (a) There exists  $\theta \in \mathcal{K}_{\infty}$  such that the system induced by  $\Gamma_{\theta} = (\mathrm{id} + \theta) \circ \Gamma$  is UGAS.
- (b) For each compact interval  $K \subset (0, \infty)$ , there are  $0 < l \leq L < \infty$  with  $l(r_2 r_1) \leq \gamma_{ij}(r_2) \gamma_{ij}(r_1) \leq L(r_2 r_1)$  for all nonzero  $\gamma_{ij}$  and  $r_1 < r_2$  in K.

Then there exists a path of strict decay  $\sigma : \mathbb{R}_+ \to \ell_{\infty}^+$  for  $\Gamma$ .

The proof of Theorem VI.5, given in [25], provides the following explicit expression for a path of strict decay:

$$\sigma(r) = Q_{\theta}(r\mathbb{1}) = \bigoplus_{k=0}^{\infty} \Gamma_{\theta}^{k}(r\mathbb{1}).$$

Proposition VI.4 is used to prove that the component functions  $\sigma_i$  are actually elements of  $\mathcal{K}_{\infty}$ . Lemma V.2 is used to prove that property (ii) of paths of strict decay is satisfied with  $\sigma_{\min} = \text{id}$  and  $\sigma_{\max} = \omega^{-1}$ . Finally, Assumption (b) in Proposition VI.5 is used to prove the local uniform Lipschitz property (iv) required in Def. IV.3. It is also interesting to note that the max-robust SGC for  $\Gamma_{\theta}$  already implies that  $\sigma$ , as defined above, satisfies properties (i)–(iii) in Def. IV.3.

Since UGAS of the discrete-time system, induced by the scaled gain operator  $\Gamma_{\theta}$ , is a key requirement for the existence of a path of strict decay, we provide some sufficient conditions for UGAS of the system induced by a gain operator. The next proposition describes a way of reducing the proof of UGAS of (4) to finitely many computations.

**VI.6 Proposition:** Assume that there exists a positive integer N and a map  $p : \mathbb{N} \to \{1, \dots, N\}$  as well as a family  $\{\bar{\gamma}_{ij} : i, j = 1, \dots, N\} \subset \mathcal{K} \cup \{0\}$  of virtual gains such that

$$\gamma_{ij} \leq \bar{\gamma}_{p(i)p(j)}$$
 for all  $i, j \in \mathbb{N}$ .

Let  $\overline{\Gamma} : \mathbb{R}^N_+ \to \mathbb{R}^N_+$ ,  $s \mapsto (\sup_{1 \le j \le N} \overline{\gamma}_{ij}(s_j))_{1 \le i \le N}$  be the associated virtual gain operator. If  $\Gamma$  satisfies the SGC,  $\overline{\Gamma}(s) \ge s$  for all  $s \in \mathbb{R}^N_+ \setminus \{0\}$ , the system (4) induced by  $\Gamma$  is UGAS.

The proof of the proposition relies on majorization of the trajectories of  $\Gamma$  by those of  $\overline{\Gamma}$  and uses that in finite dimensions, the SGC suffices to obtain UGAS in the case of max-type gain operators.

Another method of checking UGAS of (4) via the introduction of virtual gains, based on a compactification of the index set  $\mathbb{N}$ , is described in the next proposition.

**VI.7 Proposition:** Let  $\mathbb{N}^* := \mathbb{N} \cup \{\infty\}$  and assume that there exist virtual gains  $\bar{\gamma}_{ij} \in \mathcal{K} \cup \{0\}, i, j \in \mathbb{N}^*$  (where  $\bar{\gamma}_{\infty\infty} \neq 0$  is allowed), satisfying the following assumptions:

- (i)  $\bar{\gamma}_{ij} = \gamma_{ij}$  whenever  $(i, j) \in \mathbb{N} \times \mathbb{N}$ .
- (ii) The virtual gain operator

$$\bar{\Gamma}: \ell^+_{\infty}(\mathbb{N}^*) \to \ell^+_{\infty}(\mathbb{N}^*), \quad s \mapsto (\sup_{j \in \mathbb{N}^*} \bar{\gamma}_{ij}(s_j))_{i \in \mathbb{N}^*},$$

is well-defined, continuous and satisfies the max-robust SGC with some  $\omega \in \mathcal{K}_{\infty}$ .

- (iii) For each  $i \in \mathbb{N}^*$ ,  $\bar{\gamma}_{ij} \neq 0$  only for finitely many  $j \in \mathbb{N}^*$ .
- (iv) There exists  $k_0 \in \mathbb{N}$  such that for all r > 0

$$\limsup_{i \to \infty} \sup_{j_1, \dots, j_{k_0} \in \mathbb{N}^*} \bar{\gamma}_{ij_1} \circ \dots \circ \bar{\gamma}_{j_{k_0-1}j_{k_0}} \circ \omega^{-1}(r)$$
  
$$\leq \sup_{j_1, \dots, j_{k_0} \in \mathbb{N}^*} \bar{\gamma}_{\infty j_1} \circ \dots \circ \bar{\gamma}_{j_{k_0-1}j_{k_0}}(r).$$

Then, the system (4) induced by  $\Gamma$  is UGAS.

The proof of Proposition VI.7 relies on a general result provided in [29, Fact A.1.24] about interchanging the order of infimum and supremum in certain expressions. We know from Proposition VI.2 that the max-robust SGC implies

$$\sup_{i\in\mathbb{N}}\inf_{k\in\mathbb{Z}_+}\Gamma^k_i(s)=0\quad\text{for all }s\in\ell_\infty^+.$$

The assumptions in Proposition VI.7 guarantee that sup and inf in this identity can be interchanged, which is equivalent to global attractivity and in combination with UGS, to UGAS of system (4). The special choice of  $\bar{\gamma}_{ij} = 0$  whenever  $i = \infty$ or  $j = \infty$  in Proposition VI.7 characterizes the compactness of the operator  $\Gamma^{k_0}$ , see [25, Prop. VII.5].

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