Construction of ISS Lyapunov functions for infinite networks of ISS systems

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Abstract— **We show that an infinite network of input-to-state stable (ISS) systems, admitting ISS Lyapunov functions, itself admits an ISS Lyapunov function, provided that the couplings of the subsystems are sufficiently weak. The strength of the couplings is described in terms of the properties of the so-called gain operator, built from the interconnection gains. If this operator satisfies the robust small-gain condition and induces a uniformly globally asymptotically stable discrete-time system, an ISS Lyapunov function for the infinite network can be constructed.**

Keywords: large-scale systems, small-gain theorems, input-to-state stability, nonlinear systems, infinite-dimensional systems.

I. Introduction

Current society is surrounded by networks: social networks, power grids, transportation and manufacturing networks, etc. These networks grow in size from year to year, and emerging technologies, such as the Internet of Things, Cloud Computing, 5G communication, and smart cities, make this trend even more distinct. As the stability properties of the networks may deteriorate with the increase in the number of participating agents [1], it is natural to study infinite-dimensional over-approximations of large-scale networks as a worst-case scenario.

The theory of linear *spatially invariant systems* [2], [3], [4], [5] has a prominent place in these investigations. Here, infinitely many subsystems are coupled via the same pattern. This nice geometric structure together with the *linearity* of subsystems allows to develop powerful stability criteria.

On the other hand, in the stability analysis of *finite networks with nonlinear components*, groundbreaking results have been obtained within the framework of *input-to-state stability (ISS)* [6]. According to the ISS *small-gain approach*, the influence of any subsystem on other subsystems of a network is characterized by so-called gain functions. The *gain operator* constructed from these functions characterizes the interconnection structure of the network. The small-gain theorems for couplings of finitely many input-to-state stable systems of ordinary differential equations (ODEs) [7], [8], [9], [10] state that if the gains are small enough (in an appropriate sense), the network is stable. These results have numerous applications in systems theory [11], [12], and contributed largely to modern nonlinear control theory [13], [14].

Recently, the intensive development of an *infinite-dimensional ISS theory* has been initiated; see [15], [16] for a comprehensive overview of the topic, and [17] for an overview of the linear theory.

This progress motivated the development of the ISS small-gain framework for the stability analysis of infinite interconnections of nonlinear systems without any spatial invariance assumption. This research was initiated in [18], where nonlinear Lyapunov-based small-gain theorems have been obtained under the quite strong assumption that all gains are uniformly less than identity. In [19], tight Lyapunov-based small-gain theorems have been obtained for networks of exponentially ISS systems with linear gains; these results have been applied to distributed observer design for infinite networks in [20].

Nonlinear trajectory-based small-gain theorems for infinite networks have been derived in [21]. Here, it was shown that an infinite network of ISS systems is ISS if the corresponding nonlinear gain operator satisfies the so-called *monotone limit property*, which in turn implies the *uniform small-gain condition* [21], which is equivalent to the monotone bounded invertibility property. The latter played a key role in the derivation of the ISS small-gain theorem for finite networks in [9]; see, e.g., [9, Lem. 13].

This paper is strongly motivated by [22], where the *robust strong small-gain condition* has been introduced and a method to construct paths of strict decay was proposed, based on the concept of the *strong transitive closure* of the gain operator. For finite networks, this method was proposed in [23, Prop. 2.7, Rem. 2.8]; see also [24] for more on the importance of this concept in the small-gain theory. Based on these results, in [22] a small-gain theorem for infinite networks, and the construction of an ISS Lyapunov function for the network were proposed under the assumption that a *linear* path of strict decay exists. In general, this requirement is quite restrictive, and Lyapunov-based small-gain theorems

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for finite networks developed in [10] do not require the linearity assumption for the path of strict decay.

Contribution. In this paper, we consider an infinite network of ISS control systems given by ODEs. We assume that these systems admit ISS Lyapunov functions with corresponding Lyapunov gains, which give rise to the gain operator, characterizing the influence of the subsystems on each other. We show that *the existence of a (possibly nonlinear) path of strict decay for the operator* Γ *(together with some uniformity conditions) implies ISS of the whole network and the existence of an ISS Lyapunov function*. Our result partially extends the nonlinear Lyapunov-based small-gain theorem for finite networks (in maximum formulation) shown in [10] to infinite dimensions, recovers the Lyapunov-based small-gain theorem for infinite networks in [22], and partially recovers the main result in [18].

Additionally, in Theorem V.4, we show that the robust small-gain condition of the (properly scaled) gain operator together with the global attractivity of the corresponding dynamical system guarantees the existence of a path of strict decay, and thus the existence of an ISS Lyapunov function for the interconnection.

We refer the interested readers to [25] for the journal version of this paper, where the full proof of the small-gain theorem and a detailed discussion of the properties of the gain operator are provided. Furthermore, in [25], we state a small-gain theorem for the case of linear gains. We also refer to [26] for a small-gain theorem for networks with a homogeneous, subadditive gain operator, in which case the small-gain condition takes a quite simple form.

An *important open problem concerns the relationships between the existence of a path of strict decay and other properties of the gain operator, such as robust strong and uniform small-gain conditions, the monotone limit property, uniform global asymptotic stability of the induced discrete-time system*, etc. This would not only provide a better understanding of the applicability of the nonlinear ISS small-gain theorem, but would also provide powerful small-gain type criteria for uniform global asymptotic stability of nonlinear discrete-time systems induced by positive operators (for finite-dimensional criteria of this type, one may consult [27]; for linear infinite-dimensional discrete-time systems, see [28]; max-linear results are discussed in [25]).

Notation. We write $\mathbb{R}$ ($\mathbb{R}_+$) for the set of (nonnegative) real numbers and $\mathbb{Z}$ ($\mathbb{Z}_+$) for the set of (nonnegative) integers. By $C^0(X,Y)$ we denote the set of all continuous mappings from a space X to a space Y. In any metric space, we write $B_\delta(x)$ for the open ball of radius $\delta > 0$ centered at x, and $\mathrm{int}(A)$ for the interior of a subset $A \subset X$. We use the following classes of comparison functions:

$$
\begin{aligned}
\mathcal{P} &:= \left\{\gamma \in C^0(\mathbb{R}_+,\mathbb{R}_+) : \gamma(0) = 0,\ \gamma(r) > 0,\ \forall r > 0\right\},\\
\mathcal{K} &:= \left\{\gamma \in \mathcal{P} :\ \gamma \text{ is strictly increasing}\right\},\\
\mathcal{K}_\infty &:= \left\{\gamma \in \mathcal{K} :\ \gamma \text{ is unbounded}\right\},\\
\mathcal{L} &:= \Big\{\gamma \in C^0(\mathbb{R}_+,\mathbb{R}_+) :\ \gamma \text{ is strictly decreasing with } \lim_{t\to\infty}\gamma(t) = 0\Big\},\\
\mathcal{KL} &:= \{\beta \in C^0(\mathbb{R}_+^2,\mathbb{R}_+) : \beta(\cdot,t) \in \mathcal{K},\ \forall t \geq 0,\ \beta(r,\cdot) \in \mathcal{L},\ \forall r > 0\}.
\end{aligned}
$$

We write ℓ_∞ for the space of bounded real sequences $s = (s_i)_{i\in\mathbb{N}}$, which is a Banach space with the norm $\|s\|_{\ell_\infty} := \sup_{i\in\mathbb{N}}|s_i|$. The *positive cone* in ℓ_∞ is given by $\ell_\infty^+ := \{s \in \ell_\infty : s_i \geq 0,\ \forall i \in \mathbb{N}\}$. We define $\mathbb{1} := (1,1,1,\dots) \in \ell_\infty^+$. By e_i, $i \in \mathbb{N}$, we denote the i-th unit vector in ℓ_∞.

A function $\lambda : \mathbb{R}_+ \to X$ into some space X is called *piecewise right-continuous* if there is a partition of $\mathbb{R}_+$ into disjoint subintervals, $\mathbb{R}_+ = [0,t_1) \cup [t_1,t_2) \cup [t_2,t_3) \cup \dots$, such that λ is continuous on each subinterval.

II. Technical setup

A. Interconnections

Consider a family of control systems of the form

$$\Sigma_i : \quad \dot{x}_i = f_i(x_i,\bar{x}_i,u_i), \quad i \in \mathbb{N}. \tag{1}$$

This family comes with sequences $(n_i)_{i\in\mathbb{N}}$ and $(m_i)_{i\in\mathbb{N}}$ of positive integers as well as *finite* (possibly empty) sets $I_i \subset \mathbb{N}$, $i \notin I_i$, such that the following assumptions are satisfied:

- The *state vector* x_i is an element of $\mathbb{R}^{n_i}$.
- The *internal input vector* $\bar{x}_i$ is composed of the state vectors x_j, $j \in I_i$, and thus is an element of $\mathbb{R}^{N_i}$, where $N_i := \sum_{j\in I_i} n_j$.
- The *external input vector* u_i is an element of $\mathbb{R}^{m_i}$.
- The *right-hand side* f_i is a continuous function $f_i : \mathbb{R}^{n_i} \times \mathbb{R}^{N_i} \times \mathbb{R}^{m_i} \to \mathbb{R}^{n_i}$.
- For every initial state $x_{i0} \in \mathbb{R}^{n_i}$ and all essentially bounded inputs $\bar{x}_i(\cdot)$ and $u_i(\cdot)$, there is a unique solution of Σ_i, which we denote by $\phi_i(t,x_{i0},\bar{x}_i,u_i)$ (it may be defined only on a bounded time interval).

For each $i \in \mathbb{N}$, we fix (arbitrary) norms on the spaces $\mathbb{R}^{n_i}$ and $\mathbb{R}^{m_i}$, respectively. For brevity in notation, we avoid adding an index to these norms, indicating to which space they belong, and simply write $|\cdot|$ for each of them. The interconnection of the systems Σ_i, $i \in \mathbb{N}$, is defined on the state space $X := \ell_\infty(\mathbb{N},(n_i))$, where

$$\ell_\infty(\mathbb{N},(n_i)) := \{x = (x_i)_{i\in\mathbb{N}} : x_i \in \mathbb{R}^{n_i},\ \sup_{i\in\mathbb{N}}|x_i| < \infty\}.$$

This space is a Banach space with the ℓ_∞-type norm

$$\|x\|_X := \sup_{i\in\mathbb{N}}|x_i|.$$

The space of admissible external input values is likewise defined as the Banach space

$$U := \ell_\infty(\mathbb{N},(m_i)), \quad \|u\|_U := \sup_{i\in\mathbb{N}}|u_i|.$$

The class of admissible external input functions is chosen as

$$\mathcal{U} := \{u \in L_\infty(\mathbb{R}_+,U) : u \text{ is piecewise right-continuous}\},$$

which will be equipped with the L_∞-norm

$$\|u\|_{\mathcal{U}} := \operatorname*{ess\,sup}_{t\in\mathbb{R}_+} |u(t)|_U.$$

The right-hand side of the interconnected system is

$$f : X \times U \to \prod_{i\in\mathbb{N}} \mathbb{R}^{n_i}, \quad f(x,u) := (f_i(x_i, \bar{x}_i, u_i))_{i\in\mathbb{N}}.$$

Hence, the interconnected system can formally be written as the differential equation

$$\Sigma: \quad \dot{x} = f(x,u).$$

For fixed $(u, x^0) \in \mathcal{U} \times X$, a function $\lambda : J \to X$, where $J \subset \mathbb{R}$ is an interval of the form $[0,T)$ with $0 < T \leq \infty$, is called a *solution* of the Cauchy problem

$$\dot{x} = f(x,u), \quad x(0) = x^0,$$

provided that $s \mapsto f(\lambda(s), u(s))$ is a locally integrable X-valued function (in the Bochner integral sense) and

$$\lambda(t) = x^0 + \int_0^t f(\lambda(s), u(s))\,\mathrm{d}s \quad \text{for all } t \in J.$$

Any solution λ is a locally absolutely continuous function, see [29, Prop. 1.2.2].

We say that the system Σ is *well-posed* if for every initial value $x^0 \in X$ and every external input $u \in \mathcal{U}$, a unique maximal solution, which we denote by $\phi(\cdot, x^0, u) : [0, t_{\max}(x^0,u)) \to X$ exists, where $0 < t_{\max}(x^0,u) \leq \infty$.

Sufficient conditions for well-posedness are provided by [19, Cor. III.3]. If Σ is well-posed, it holds that

$$\pi_i(\phi(t, x^0, u)) = \phi_i(t, x_i^0, \bar{x}_i, u_i) \tag{2}$$

for all $t \in [0, t_{\max}(x^0,u))$ and $i \in \mathbb{N}$, where $\pi_i : X \to \mathbb{R}^{n_i}$ denotes the canonical projection onto the i-th component, $\bar{x}_i(\cdot) = (\pi_j(\phi(\cdot, x^0, u)))_{j\in I_i}$, and x_i^0, u_i denote the i-th components of x^0 and u, respectively, see [19, Sec. 3].

In the rest of the paper, we assume the following.

II.1 Assumption: *The system Σ is well-posed, and all of its uniformly bounded maximal solutions $\phi(\cdot, x, u)$ are global, i.e., exist on $\mathbb{R}_+$ (this latter property is also called boundedness-implies-continuation (BIC) property).*

II.2 Remark: A sufficient condition for the BIC property is that the function f is uniformly bounded on bounded balls, and Lipschitz continuous on bounded balls with respect to the first argument (see [30, Thm. 4.3.4] for the related result for systems without inputs).

B. Input-to-state stability

We now recall the definition of input-to-state stability.

II.3 Definition: *A well-posed system Σ is called* (uniformly) input-to-state stable (ISS) *if it is forward complete and there are $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ s.t. for all $(t,x,u) \in \mathbb{R}_+ \times X \times \mathcal{U}$*

$$\|\phi(t,x,u)\|_X \leq \beta(\|x\|_X, t) + \gamma(\|u\|_{\mathcal{U}}).$$

Input-to-state stability is most often verified via the construction of an ISS Lyapunov function which is defined as follows.

II.4 Definition: *A function $V : X \to \mathbb{R}_+$ is called an* ISS Lyapunov function (in an implication form) *for Σ if it satisfies the following properties:*

(i) V is continuous.

(ii) There exist $\psi_1, \psi_2 \in \mathcal{K}_\infty$ such that

$$\psi_1(\|x\|_X) \leq V(x) \leq \psi_2(\|x\|_X) \quad \text{for all } x \in X. \tag{3}$$

(iii) There exist $\gamma \in \mathcal{K}$ and $\alpha \in \mathcal{P}$ such that for all $x \in X$ and $u \in \mathcal{U}$ the following implication holds:

$$V(x) > \gamma(\|u\|_{\mathcal{U}}) \quad \Rightarrow \quad \mathrm{D}^+V_u(x) \leq -\alpha(V(x)), \tag{4}$$

where $\mathrm{D}^+V_u(x)$ denotes the right upper Dini orbital derivative defined as

$$\mathrm{D}^+V_u(x) := \limsup_{t\to 0^+} \frac{V(\phi(t,x,u)) - V(x)}{t}.$$

The importance of ISS Lyapunov functions is due to the following basic fact (cf. [16, Thm. 2.17]).

II.5 Proposition: *If an ISS Lyapunov function for Σ exists, then Σ is ISS.*

III. Nonlinear small-gain theorem

To find an ISS Lyapunov function V for Σ, we exploit the interconnection structure and construct V from ISS Lyapunov functions of the subsystems Σ_i under an appropriate small-gain condition. We introduce the following assumption:

III.1 Assumption: *For each $i \in \mathbb{N}$, there exists a continuous function $V_i : \mathbb{R}^{n_i} \to \mathbb{R}_+$ which is continuously differentiable outside of $x_i = 0$ and satisfies the following properties:*

(L1) There exist $\psi_{i1}, \psi_{i2} \in \mathcal{K}_\infty$ such that

$$\psi_{i1}(|x_i|) \leq V_i(x_i) \leq \psi_{i2}(|x_i|) \quad \text{for all } x_i \in \mathbb{R}^{n_i}. \tag{5}$$

(L2) There exist $\gamma_{ij} \in \mathcal{K} \cup \{0\}$, where $\gamma_{ij} = 0$ for all $j \in \mathbb{N}\backslash I_i$, and $\gamma_{iu} \in \mathcal{K}$ as well as $\alpha_i \in \mathcal{P}$ such that for all $x = (x_j)_{j\in\mathbb{N}} \in X$ and $u = (u_j)_{j\in\mathbb{N}} \in U$ the following implication holds:

$$\begin{aligned} V_i(x_i) > \max\Big\{\sup_{j\in I_i} \gamma_{ij}(V_j(x_j)), \gamma_{iu}(|u_i|)\Big\} \\ \Rightarrow \nabla V_i(x_i) f_i(x_i, \bar{x}_i, u_i) \leq -\alpha_i(V_i(x_i)). \end{aligned} \tag{6}$$

The function V_i is called an ISS Lyapunov function *for Σ_i. The functions γ_{ij} and γ_{iu} are called* internal gains *and* external gains, *respectively.*

Using the internal gains γ_{ij} from Assumption III.1, we define the gain operator $\Gamma : \ell_\infty^+ \to \ell_\infty^+$ by

$$\Gamma(s) := \Big(\sup_{j\in\mathbb{N}} \gamma_{ij}(s_j)\Big)_{i\in\mathbb{N}}, \quad \Gamma : \ell_\infty^+ \to \ell_\infty^+. \tag{7}$$

The following assumption guarantees that Γ is well-defined and continuous, see [22, Lem. 2.1] and [21, Prop. 2].

III.2 Assumption: *The family $\{\gamma_{ij} : i, j \in \mathbb{N}\}$ is pointwise equicontinuous. That is, for every $r \geq 0$ and every $\varepsilon > 0$ there exists $\delta = \delta(r, \varepsilon) > 0$ such that $|r - \tilde{r}| \leq \delta$, $\tilde{r} \in \mathbb{R}_+$, implies $|\gamma_{ij}(r) - \gamma_{ij}(\tilde{r})| \leq \varepsilon$ for all $i, j \in \mathbb{N}$.*

Additionally, we make the following assumption on the external gains.

III.3 Assumption: *There is $\gamma^u_{\max} \in \mathcal{K}$ such that $\gamma_{iu} \leq \gamma^u_{\max}$ for all $i \in \mathbb{N}$.*

We now introduce the concept of a *path of strict decay* which is of crucial importance in the construction of an ISS Lyapunov function for the interconnected system.

III.4 Definition: *A mapping $\sigma : \mathbb{R}_+ \to \ell^+_\infty$ is called a* path of strict decay (for Γ), *if the following properties hold:*

(i) Each component function σ_i, $i \in \mathbb{N}$, is in $\mathcal{K}_\infty$.

(ii) There exist $\sigma_{\min}, \sigma_{\max} \in \mathcal{K}_\infty$ satisfying

$$\sigma_{\min} \leq \sigma_i \leq \sigma_{\max} \quad \text{for all } i \in \mathbb{N}.$$

(iii) For every compact interval $K \subset (0, \infty)$, there exist $0 < c \leq C < \infty$ such that for all $r_1, r_2 \in K$ and $i \in \mathbb{N}$:

$$c|r_1 - r_2| \leq |\sigma_i^{-1}(r_1) - \sigma_i^{-1}(r_2)| \leq C|r_1 - r_2|.$$

(iv) There exists a function $\rho \in \mathcal{K}_\infty$ such that

$$\Gamma(\sigma(r)) \leq (\mathrm{id} + \rho)^{-1} \circ \sigma(r) \quad \text{for all } r \geq 0,$$

where $(\mathrm{id} + \rho)^{-1}$ is applied componentwise.

In Section V, we provide a method to construct paths of strict decay under suitable assumptions.

Our small-gain result now reads as follows.

III.5 Theorem: *Consider the interconnected system Σ, composed of the subsystems Σ_i, $i \in \mathbb{N}$ and let the following assumptions be satisfied.*

(i) Assumption II.1 is satisfied, i.e., Σ is well-posed and satisfies the BIC property.

(ii) Assumption III.1 is satisfied, i.e., there exist ISS Lyapunov functions V_i for the subsystems Σ_i with associated internal gains γ_{ij} and external gains γ_{iu}. Moreover, there exist $\psi_1, \psi_2 \in \mathcal{K}_\infty$ such that

$$\psi_1 \leq \psi_{i1} \quad \text{and} \quad \psi_{i2} \leq \psi_2 \quad \text{for all } i \in \mathbb{N}. \tag{8}$$

(iii) Assumptions III.2 and III.3 are satisfied, i.e., the family of internal gains γ_{ij} is pointwise equicontinuous and the external gains γ_{iu} are upper bounded by a $\mathcal{K}$-function.

(iv) There exists a path $\sigma : \mathbb{R}_+ \to \ell^+_\infty$ of strict decay for the gain operator Γ, defined from the internal gains γ_{ij}.

(v) For each $R > 0$, there is $L(R) > 0$ such that

$$|V_i(x_i) - V_i(y_i)| \leq L(R)|x_i - y_i| \tag{9}$$

for all $i \in \mathbb{N}$ and $x_i, y_i \in B_R(0) \subset \mathbb{R}^{n_i}$.

(vi) There exists $\tilde{\alpha} \in \mathcal{P}$ such that $\alpha_i \geq \tilde{\alpha}$ for all $i \in \mathbb{N}$.

Then the following is an ISS Lyapunov function for Σ:

$$V(x) := \sup_{i \in \mathbb{N}} \sigma_i^{-1}(V_i(x_i)) \quad \text{for all } x \in X. \tag{10}$$

Moreover, V is locally Lipschitz continuous on $X \backslash \{0\}$. In particular, Σ is ISS.

An example, on how the small-gain theorem can be applied to the stability analysis of infinite networks with nonlinear gains, can be found in [22, Sec. 6]. We do not include one more example here due to the space constraints.

IV. Gain operators and their properties

A crucial assumption in our small-gain theorem is the existence of a path of strict decay for the operator Γ. Our next aim is to understand under which conditions such a path exists, and to provide an explicit expression for it. We base our analysis on the properties of the gain operator, presented in this section.

From now on, we always assume that the family $\{\gamma_{ij}\}$ is pointwise equicontinuous (Assumption III.2), implying that Γ is well-defined and continuous. Observe that $\Gamma(0) = 0$ and that Γ is monotone: for all $s^1, s^2 \in \ell^+_\infty$:

$$s^1 \leq s^2 \quad \Rightarrow \quad \Gamma(s^1) \leq \Gamma(s^2).$$

Now we recall the important *robust and robust strong small-gain conditions*, introduced in [22].

IV.1 Definition: *We say that the operator Γ satisfies*

(i) the small-gain condition *if*

$$\Gamma(s) \not\geq s \quad \text{for all } s \in \ell^+_\infty \backslash \{0\}. \tag{11}$$

(ii) the strong small-gain condition *if there is $\rho \in \mathcal{K}_\infty$ with*

$$D_\rho \circ \Gamma(s) \not\geq s \quad \text{for all } s \in \ell^+_\infty \backslash \{0\} \tag{12}$$

for the operator $D_\rho : \ell^+_\infty \to \ell^+_\infty$, defined by

$$D_\rho(s) := \big((\mathrm{id} + \rho)(s_i)\big)_{i \in \mathbb{N}}.$$

(iii) the robust small-gain condition *if there is $\omega \in \mathcal{K}_\infty$ with $\omega < \mathrm{id}$ such that for all $i, j \in \mathbb{N}$ the operator*

$$\Gamma_{ij}(s) := \Gamma(s) + \omega(s_j)e_i \quad \text{for all } s \in \ell^+_\infty \tag{13}$$

satisfies the small-gain condition.

(iv) the robust strong small-gain condition *if there are $\omega \in \mathcal{K}_\infty$ with $\omega < \mathrm{id}$ and $\rho \in \mathcal{K}_\infty$ such that for all $i, j \in \mathbb{N}$ the operator Γ_{ij}, as defined in (13), satisfies the strong small-gain condition with the same ρ for all i, j.*

In the next lemma, we introduce the *strong transitive closure* (or Kleene star operator) Q of the gain operator Γ, which provides the crucial tool for the construction of a path of strict decay. This result was shown in [22, Lem. 4.3], and slightly strengthened in [21, Lem. 14]. For finite networks, the idea to use the operator Q for the construction of the paths of decay was proposed in [23, Prop. 2.7, Rem. 2.8].

IV.2 Lemma: *([22, Lem. 4.3], [21]) Assume that Γ satisfies the robust small-gain condition. Then the operator*

$$Q(s) := \sup_{k\in\mathbb{Z}_+} \Gamma^k(s) \quad \text{for all } s \in \ell_\infty^+, \tag{14}$$

where the supremum is defined componentwise, is well-defined and satisfies

$$s \leq Q(s) \leq \omega^{-1}(\|s\|_{\ell_\infty})\mathbb{1} \quad \text{for all } s \in \ell_\infty^+, \tag{15}$$

where $\omega \in \mathcal{K}_\infty$, $\omega < \mathrm{id}$ stems from Definition IV.1(iii) and $\mathbb{1} := (1,1,1,\ldots) \in \ell_\infty^+$. Furthermore,

$$\Gamma(Q(s)) \leq Q(s) \quad \text{for all } s \in \ell_\infty^+. \tag{16}$$

Some further simple properties of the operator Q are summarized in the following proposition.

IV.3 Proposition: *Assume that $\Gamma : \ell_\infty^+ \to \ell_\infty^+$ is well-defined, continuous and satisfies the robust small-gain condition. Then the operator Q in (14) is monotone, satisfies $Q(0) = 0$, and its image is given by*

$$\mathrm{im}Q = \{s \in \ell_\infty^+ : \Gamma(s) \leq s\}.$$

This set is closed, contains $s = 0$, is cofinal (i.e., for any $x \in \ell_\infty^+$ there is $s \in \mathrm{im}Q$ with $x \leq s$) and forward-invariant with respect to Γ, i.e., $\Gamma(\mathrm{im}Q) \subset \mathrm{im}Q$.

Proof: Monotonicity and $Q(0) = 0$ are easy to see. Lemma IV.2 implies that $\mathrm{im}Q \subset \{s \in \ell_\infty^+ : \Gamma(s) \leq s\}$. Conversely, $\Gamma(s) \leq s$ implies $\Gamma^k(s) \leq s$ for all $k \geq 0$, and hence $Q(s) = s$ implying $s \in \mathrm{im}Q$. Since Γ is continuous, it then follows that $\mathrm{im}Q$ is closed. As for each $s \in \ell_\infty^+$ we have $s \leq Q(s) \in \mathrm{im}Q$, the set $\mathrm{im}Q$ is cofinal. As $\Gamma(s) \leq s$ for any $s \in \mathrm{im}Q$, by monotonicity of Γ it holds that $\Gamma(\Gamma(s)) \leq \Gamma(s)$, which shows the invariance of $\mathrm{im}Q$. ∎

For the gain operator Γ and any $\theta \in \mathcal{K}_\infty$, we define the operator $\Gamma_\theta : \ell_\infty^+ \to \ell_\infty^+$ by

$$\Gamma_\theta(s) := (\mathrm{id} + \theta) \circ \Gamma(s) \quad \text{for all } s \in \ell_\infty^+. \tag{17}$$

Here we apply the function $\mathrm{id} + \theta$ componentwise to all components of $\Gamma(s)$. For each $s \in \ell_\infty^+$, we have

$$\begin{aligned}\Gamma_\theta(s) &= (\mathrm{id} + \theta) \circ \big(\sup_{j\in\mathbb{N}} \gamma_{ij}(s_j)\big)_{i\in\mathbb{N}} \\ &= \big(\sup_{j\in\mathbb{N}} (\mathrm{id} + \theta) \circ \gamma_{ij}(s_j)\big)_{i\in\mathbb{N}}.\end{aligned}$$

Hence, the operator Γ_θ is structurally the same as Γ, but with scaled gain functions.

We close the section with basic properties of gain operators satisfying the *robust strong small-gain condition*:

IV.4 Lemma: *Assume that Γ satisfies the robust strong small-gain condition with given $\rho, \omega \in \mathcal{K}_\infty$. Then Γ_ρ satisfies the robust small-gain condition with the same $\omega \in \mathcal{K}_\infty$. Furthermore, there is $\theta \in \mathcal{K}_\infty$ such that Γ_θ also satisfies the robust strong small-gain condition.*

V. Construction of paths of strict decay

We consider the following discrete-time dynamical system induced by the gain operator Γ:

$$s(k+1) = \Gamma(s(k)), \quad k \in \mathbb{Z}_+. \tag{18}$$

V.1 Definition: (18) *is called* uniformly globally asymptotically stable (UGAS), *if there is $\beta \in \mathcal{KL}$, such that for any initial condition $s \in \ell_\infty^+$ the solution of* (18) *satisfies*

$$\|\Gamma^k(s)\|_{\ell_\infty} \leq \beta(\|s\|_{\ell_\infty}, k), \quad \forall k \in \mathbb{Z}_+. \tag{19}$$

The following proposition is crucial for the construction of paths of strict decay. For its proof, we refer to [25].

V.2 Proposition: *Assume that $\Gamma : \ell_\infty^+ \to \ell_\infty^+$ is well-defined, continuous and satisfies the robust small-gain condition. If* (18) *is UGAS, then the following statements hold:*

(i) For all $s_1, s_2 \in \mathrm{int}(\ell_\infty^+)$ with $s_1 \leq s_2$, there is a finite $m \in \mathbb{N}$ such that

$$Q(s) = \sup_{0\leq k\leq m} \Gamma^k(s), \quad \forall s : s_1 \leq s \leq s_2. \tag{20}$$

(ii) Q is continuous on ℓ_∞^+.

Additionally, we will need a technical lemma:

V.3 Lemma: *Let $f_1, \ldots, f_n$ be strictly increasing functions $f_i : [a,b] \to \mathbb{R}_+$ with $a, b \in \mathbb{R}$, $a < b$. Further assume that $|f_i(r_1) - f_i(r_2)| \geq l_i|r_1 - r_2|$ for all $r_1, r_2 \in [a,b]$ and $i \in \mathbb{N}$, where $l_i > 0$. Put $f(r) := \max_{i=1,\ldots,n} f_i(r)$, $f : [a,b] \to \mathbb{R}_+$. Then, with $l := \min\{l_1, \ldots, l_n\}$ we have*

$$|f(r_1) - f(r_2)| \geq l|r_1 - r_2| \quad \text{for all } r_1, r_2 \in [a,b].$$

We can finally present our main result on the existence and construction of paths of strict decay. It extends the first result of this kind in [22, Lem. 4.5], where items (i)–(iii) of the next result have been shown under similar assumptions.

V.4 Theorem: *Let the following assumptions be satisfied:*

(a) $\Gamma_\theta = (\mathrm{id}+\theta)\circ\Gamma$ satisfies the robust small-gain condition for some $\theta \in \mathcal{K}_\infty$.

(b) The system induced by Γ_θ is globally attractive.

(c) For each $i \in \mathbb{N}$ there are only finitely many nonzero γ_{ij}.

Then there exists a curve $\sigma : \mathbb{R}_+ \to \ell_\infty^+$ with the following properties:

(i) With $\lambda := (\mathrm{id} + \theta)^{-1} < \mathrm{id}$, the inequality $\Gamma(\sigma(r)) \le \lambda \circ \sigma(r)$ holds for all $r \in \mathbb{R}_+$.

(ii) There exist $\sigma_{\min}, \sigma_{\max} \in \mathcal{K}_\infty$ such that $\sigma_{\min} \le \sigma_i \le \sigma_{\max}$ for all $i \in \mathbb{N}$.

(iii) Each component function σ_i is a $\mathcal{K}_\infty$-function.

(iv) Additionally assume that for each compact interval $K \subset (0,\infty)$ there are $0 < l \le L < \infty$ with $l(r_2 - r_1) \le \gamma_{ij}(r_2) - \gamma_{ij}(r_1) \le L(r_2 - r_1)$ for all nonzero γ_{ij} and $r_1 < r_2$ in K. Then for each compact interval $K \subset (0,\infty)$, there exist $0 < c \le C < \infty$ such that

$$c|r_1 - r_2| \le |\sigma_i^{-1}(r_1) - \sigma_i^{-1}(r_2)| \le C|r_1 - r_2|$$

for all $r_1, r_2 \in K$ and $i \in \mathbb{N}$.

In particular, under the additional assumption of (iv), σ is a path of strict decay.

Proof: First, we fix $\theta \in \mathcal{K}_\infty$ such that Γ_θ satisfies the robust small-gain condition and $s = 0$ is a globally attractive fixed point for Γ_θ. We also put $\gamma_{ij}^\theta := (\mathrm{id} + \theta) \circ \gamma_{ij}$ for all $i, j \in \mathbb{N}$, and define $\sigma(r) := Q_\theta(r\mathbb{1})$ for all $r \in \mathbb{R}_+$, where $Q_\theta(s) = \sup_{k \in \mathbb{Z}_+} \Gamma_\theta^k(s)$. Then we can verify all assertions:

(i) Since $\Gamma_\theta(Q_\theta(r\mathbb{1})) \le Q_\theta(r\mathbb{1})$ by Lemma IV.2, with $\lambda := (\mathrm{id}+\theta)^{-1} < \mathrm{id}$ we obtain $\Gamma(\sigma(r)) \le \lambda \circ \sigma(r)$ for all $r \in \mathbb{R}_+$.

(ii) By definition, $\sigma(r) \ge r\mathbb{1}$, i.e., $\sigma_i(r) \ge r$ for all $i \in \mathbb{N}$ and $r \in \mathbb{R}_+$. Hence, the lower bound is satisfied with $\sigma_{\min} = \mathrm{id}$. By Lemma IV.2, the upper bound holds with $\sigma_{\max} := \omega^{-1}$.

(iii) From (ii), we can conclude that $\sigma_i(0) = 0$, $\sigma_i(r) > 0$ for all $r > 0$, and $\sigma_i(r) \to \infty$ as $r \to \infty$. As 0 is a globally attractive fixed point for Γ_θ, Q_θ is continuous on ℓ_∞^+ by Proposition V.2. Thus, all σ_i are continuous either. Furthermore, for $r_1, r_2 \in (0,\infty)$ with $r_1 < r_2$ we obtain by Proposition V.2, that

$$\sigma_i(r) = \max_{0 \le k < k_0} \pi_i \circ \Gamma_\theta^k(r\mathbb{1}) \quad \text{for all } r \in [r_1, r_2].$$

By our assumption that for each i only finitely many γ_{ij} are nonzero, the supremum in

$$\sigma_i(r) = \max_{0 \le k < k_0} \sup_{j_1,\ldots,j_k} \gamma_{ij_1}^\theta \circ \cdots \circ \gamma_{j_{k-1}j_k}^\theta(r)$$

is in fact a supremum over finitely many strictly increasing functions (since we can ignore all chains which contain a zero function). This implies that σ_i is also strictly increasing on $[r_1, r_2]$, and hence everywhere.

(iv) It suffices to prove the statement for σ_i in place of σ_i^{-1}. Indeed, assume that for every compact interval $L \subset (0,\infty)$ we have constants $\tilde{c}, \tilde{C} > 0$ satisfying

$$\tilde{c}|r_1 - r_2| \le |\sigma_i(r_1) - \sigma_i(r_2)| \le \tilde{C}|r_1 - r_2| \ \forall r_1, r_2 \in L.$$

If $K = [a,b] \subset (0,\infty)$, then $\sigma_i^{-1}(K) = [\sigma_i^{-1}(a), \sigma_i^{-1}(b)]$ which is a subset of $[\sigma_{\max}^{-1}(a), \sigma_{\min}^{-1}(b)] =: L \subset (0,\infty)$. Hence, the above estimates imply

$$\frac{1}{\tilde{C}}|r_1 - r_2| \le |\sigma_i^{-1}(r_1) - \sigma_i^{-1}(r_2)| \le \frac{1}{\tilde{c}}|r_1 - r_2| \ \forall r_1, r_2 \in K.$$

To verify the statement for σ_i, we first prove the following claim: If $K = [a,b] \subset (0,\infty)$ is a compact interval, then there exists another compact interval $[c,d] \subset (0,\infty)$ such that $\gamma_{ij}^\theta(K) \subset [c,d]$ for all nonzero γ_{ij}. By uniform equicontinuity of the functions $\{\gamma_{ij}^\theta\}$ on compact intervals, we know that $\gamma_{ij}^\theta(b)$ is uniformly bounded from above by some $d > 0$. Now pick $\rho > 0$ such that $a - \rho > 0$. Then by assumption there is $l > 0$ with $\gamma_{ij}^\theta(a) - \gamma_{ij}^\theta(a - \rho) \ge l\rho$ for all $i, j \in \mathbb{N}$ such that $\gamma_{ij} \ne 0$. Hence, $\gamma_{ij}^\theta(a) \ge (1+\theta)l\rho$ whenever $\gamma_{ij} \ne 0$. This implies $\gamma_{ij}^\theta(K) \subset [c,d]$ with $c := (1+\theta)l\rho$.

Now fix a compact interval $K \subset (0,\infty)$, $k \in \mathbb{N}$ and consider all chains of the form

$$c_{j_1 \ldots j_k} := \gamma_{j_1 j_2}^\theta \circ \gamma_{j_2 j_3}^\theta \circ \cdots \circ \gamma_{j_{k-1} j_k}^\theta$$

which are built from nonzero gains. From the claim and our assumptions, it then follows that

$$\begin{aligned} l_1 l_2 \cdots l_k |r_1 - r_2| &\le |c_{j_1 \ldots j_k}(r_1) - c_{j_1 \ldots j_k}(r_2)| \\ &\le L_1 L_2 \cdots L_k |r_1 - r_2| \end{aligned}$$

for certain positive numbers $l_i, L_i > 0$, $i = 1, \ldots, k$ and all $r_1, r_2 \in K$. The same Lipschitz bounds then also hold for the functions $r \mapsto \pi_i \circ \Gamma_\theta^k(r\mathbb{1}) = \sup_{j_2 \ldots j_k} c_{ij_2 \ldots j_k}(r)$, where for the lower bound we need to require that at least one nonzero chain $c_{ij_2 \ldots j_k}$ exists. By (iii), on every compact interval $K \subset (0,\infty)$, σ_i can be written as the maximum over finitely many of such functions:

$$\sigma_i(r) = \max_{0 \le k < k_0} \max_{j_2 \ldots j_k} c_{ij_2 \ldots j_k}(r) \quad \text{for all } r \in K. \qquad (21)$$

With $C := \max\{1, L_1, L_1 L_2, \ldots, L_1 \cdots L_{k_0}\}$ (keeping in mind that $\pi_i \circ \Gamma_\theta^0(r\mathbb{1}) = r$), it then holds that

$$|\sigma_i(r_1) - \sigma_i(r_2)| \le C|r_1 - r_2| \quad \text{for all } r_1, r_2 \in K.$$

For the lower bound, we put $c := \min\{1, l_1, l_1 l_2, \ldots, l_1 l_2 \cdots l_{k_0}\}$ and apply Lemma V.3. Observe that in taking the supremum we can ignore all functions which are identically zero. Considering $k = 0$, we see that at least one nonzero function is involved in taking the supremum, namely the identity. ∎

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