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A small-gain approach to ISS of infinite networks with homogeneous gain operators

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Abstract— **We present a Lyapunov-based small-gain theorem for input-to-state stability (ISS) of networks composed of infinitely many finite-dimensional systems. A key assumption in our results is that the internal Lyapunov gains, modeling the influence of the subsystems on each other, are linear functions. Moreover, the gain operator constructed from the internal gains is assumed to be subadditive and homogeneous. This covers both max-type and sum-type formulations for the ISS Lyapunov functions of the subsystems. We formulate the small-gain condition in terms of a generalized spectral radius of the gain operator. The effectiveness of our results are illustrated through an example. Particularly, we show that the small-gain condition can easily be checked if the interconnection topology of the network has some kind of symmetry.**

I. INTRODUCTION

Rapid technological advances in computation and communication have made network systems ever more ubiquitous in many socioeconomic and engineering domains. Concepts such as smart cities, autonomous connected vehicles and multi-carrier energy networks are no longer out of our reach. In these applications, the system size can be very large, time-varying and possibly uncertain. To guarantee safety and functionality of such emerging network systems, one needs to rigorously study effects of the system size on the stability and performance indices. Standard tools in analysis and design of control systems, however, scale poorly with the system size. A promising approach to address the scalability issue is to over-approximate a large-but-finite network with an infinite network. The latter refers to a network of infinitely many (finite-dimensional) subsystems and can be viewed as the limit case of the former in terms of the number of participating subsystems. Having studied an infinite network, it can be shown that the performance/stability indices achieved for the infinite network are transferable to any finite truncation of the infinite network [1]–[3].

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Small-gain theory is widely used for analysis and design of finite-dimensional networks. In particular, the integration of Lyapunov functions with small-gain theory leads to Lyapunov-based small-gain theorems, where each subsystem satisfies a so-called input-to-state stability (ISS) Lyapunov condition [4], [5]. If the gain functions associated to the ISS Lyapunov conditions fulfill a small-gain condition, ISS of the overall network can be concluded.

Advances in infinite-dimensional ISS theory, e.g. [6]–[8], created a firm basis for the development of Lyapunov-based small-gain theorems for infinite networks, recently studied in [9]–[12]. Particularly, it has been observed that existing small-gain conditions for finite networks may fail to ensure ISS of an infinite network, even if all subsystems are linear [12].

To address such issues, we recently developed a Lyapunov-based small-gain theorem in a summation form (called sum-type small-gain theorems) in [9], i.e., the overall ISS Lyapunov function is a linear combination of ISS Lyapunov functions of the subsystems. The small-gain condition, proposed in [9], is tight in the sense that it cannot be relaxed under the assumptions imposed on the network; see [9, Sec. VI.A] for more details. The overall state space (as well as the overall input space) is, however, modeled in [9] as an ℓ_p-type space with $1 \leq p < \infty$. This condition on the state space requires that each state vector has a vanishing tail, which is reasonable for some applications, but inappropriate for other ones. To remove this condition, one needs to model the overall state space as an ℓ_∞-type space. This motivates the introduction of small-gain theorems in a maximum formulation (called max-type small-gain theorems). In this type of small-gain theorems, each component of the gain operator, encoding the influence of neighboring subsystems on a fixed subsystem, is expressed as a maximum over gain functions associated to the neighboring systems. In [11], [1], max-type small-gain conditions are provided for continuous-time and discrete-time infinite networks, respectively. Nevertheless, in both works, the gain functions are assumed to be uniformly less than the identity function. To get rid of this overly restrictive assumption, the so-called robust strong small-gain condition has been introduced in [12] and a method to construct a *path of strict decay* was proposed. Motivated by [5], this path of strict decay is then used in [12] to construct an ISS Lyapunov function for the overall network. However, in [12], the existence of a linear path of strict decay is assumed, which is very restrictive. This assumption is removed in [13], where ISS of an infinite network is shown provided that there is a nonlinear path of strict decay with respect to the gain operator. Furthermore, in [13], sufficient conditions for the existence of a nonlinear path of strict decay are provided. In particular, it is shown that a (nonlinear) path of strict decay exists if the discrete-time system induced by a slightly scaled gain operator is uniformly globally asymptotically stable and, additionally, the interconnection gains satisfy a local uniform Lipschitz condition.

All of the above small-gain results developed in an ℓ_∞-type space are given in a pure maximum formulation. Clearly, a pure max-type formulation is not necessarily the best choice, in general, as one might need to add conservatism in the calculation of ISS Lyapunov bounds. For instance, one might unnecessarily upper-bound terms expressed in summation by those in maximum. Inspired by [5], our work introduces a new, more general small-gain theorem (cf. Theorem IV.1 below), where the gain operator is monotone, subadditive and homogeneous of degree one.

Therefore, our setting treats sum and maximum formulations in a unified, generalized way, making our results very flexible in different contexts. To obtain such a formulation, we assume that the internal gain functions are linear. Our small-gain condition is expressed in terms of a spectral radius condition for the gain operator. Under this assumption, we show that an infinite network of ISS systems is ISS provided that the spectral radius of the gain operator is less than one. We show several equivalent criteria for the spectral small-gain condition, which give a rich machinery to check this condition in practice.

Our small-gain theorem is built on characterizations of the spectral radius condition, cf. Proposition III.1 below. In particular, we show that the *small-gain condition is equivalent to uniform global exponential stability of the monotone discrete-time system induced by the gain operator*, which is of great significance on its own, and extends several known criteria for linear discrete-time systems summarized in [14]. Equivalently, *there exists a linear path of strict decay through which we construct an ISS Lyapunov function for the overall network.* For maximum and sum formulations, the spectral radius condition (i.e., the small-gain condition) admits explicit formulas for which a graph-theoretic description is provided. A linear version of the robust strong small-gain condition is also presented as another reformulation of the spectral radius condition in the maximum formulation. Via an example of a spatially invariant network, the effectiveness of these conditions is illustrated. Proofs are omitted due to space constraints and can be found in [15].

II. TECHNICAL SETUP

A. Notation

We write $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}_+$ for the sets of positive integers, real numbers and nonnegative real numbers, respectively. We also write $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Elements of $\mathbb{R}^n$ are by default regarded as column vectors. If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $x \in \mathbb{R}^n$, $\nabla f(x)$ denotes its gradient which is regarded as a row vector. By ℓ_∞, we denote the Banach space of all bounded real sequences $x = (x_i)_{i\in\mathbb{N}}$ with the norm $\|x\|_{\ell_\infty} := \sup_{i\in\mathbb{N}} |x_i|$. The subset $\ell_\infty^+ := \{(x_i)_{i\in\mathbb{N}} \in \ell_\infty : x_i \geq 0,\ \forall i \in \mathbb{N}\}$ is a closed cone with nonempty interior, and is called the *positive cone* in ℓ_∞. It induces a partial order on ℓ_∞ by "$x \leq y$ if and only if $y - x \in \ell_\infty^+$" (which simply reduces to $x_i \leq y_i$ for all $i \in \mathbb{N}$). By $\mathbb{1}$, we denote the vector in ℓ_∞^+ whose components are all equal to 1. In any metric space (X, d), we write $\mathrm{int}(A)$ for the interior of a set A and $B_\delta(x)$ for the open ball of radius δ centered at x. A function $f : \mathbb{R}_+ \to X$ is called *piecewise right-continuous* if there are pairwise disjoint intervals $I_1 = [0, a_1)$, $I_2 = [a_1, a_2)$, $I_3 = [a_2, a_3), \ldots$ whose union equals $\mathbb{R}_+$, such that $f_{|I_n}$ is continuous for each $n \in \mathbb{N}$.

A continuous function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is positive definite, denoted by $\alpha \in \mathcal{P}$, if $\alpha(0) = 0$ and $\alpha(r) > 0$ for all $r > 0$. For the sets of comparison functions $\mathcal{K}$, $\mathcal{K}_\infty$, $\mathcal{L}$ and $\mathcal{KL}$, we refer to [16]. Given $\Gamma : \ell_\infty^+ \to \ell_\infty^+$, Γ^n denotes the n-th iterate of Γ, i.e. $\Gamma^0 = \mathrm{id}$ and $\Gamma^n = \Gamma \circ \Gamma^{n-1}$ for $n \geq 1$. By $\Gamma_i^n(s)$, we denote the i-th component of the vector $\Gamma^n(s)$. A function $\mu : \ell_\infty^+ \to [0, \infty]$ is called a *monotone, homogeneous aggregation function (MHAF)* if it has the following properties:

1) Homogeneity of degree one: $\mu(cs) = c\mu(s)$ for all $s \in \ell_\infty^+$ and $c \geq 0$ (with the convention that $0 \cdot \infty = 0$).
2) Monotonicity: $\mu(r) \leq \mu(s)$ for all $r, s \in \ell_\infty^+$ s.t. $r \leq s$.
3) Subadditivity: $\mu(r + s) \leq \mu(r) + \mu(s)$ for all $r, s \in \ell_\infty^+$ (with the convention that $\infty + \infty = \infty$).

The concept of monotone aggregation functions in the context of small-gain theory has been introduced in [5], and here it is adapted to our setting. Typical examples are $\mu(s) = \sum_{i\in\mathbb{N}} \alpha_i s_i$, $\mu(s) = \sup_{i\in\mathbb{N}} \alpha_i s_i$, or $\mu(s) = \max_{1\leq i\leq N} \alpha_i s_i + \sum_{i=N+1}^{\infty} \alpha_i s_i$, for some $(\alpha_i)_{i\in\mathbb{N}} \in \ell_\infty^+$ with $\alpha_i \geq 0$ for all $i \in \mathbb{N}$, and $N \in \mathbb{N}$.

B. Interconnections

Consider a family of control systems of the form

$$\Sigma_i : \quad \dot{x}_i = f_i(x_i, \bar{x}_i, u_i), \quad i \in \mathbb{N}.$$

This family comes with sequences $(n_i)_{i\in\mathbb{N}}$ and $(m_i)_{i\in\mathbb{N}}$ of positive integers as well as finite (possibly empty) sets $I_i \subset \mathbb{N}$, $i \notin I_i$, such that the following assumptions are satisfied:

- The *state vector* x_i is an element of $\mathbb{R}^{n_i}$.
- The *internal input vector* $\bar{x}_i$ is composed of the state vectors x_j, $j \in I_i$, and thus is an element of $\mathbb{R}^{N_i}$, where $N_i := \sum_{j\in I_i} n_j$.
- The *external input vector* u_i is an element of $\mathbb{R}^{m_i}$.
- The *right-hand side* $f_i : \mathbb{R}^{n_i} \times \mathbb{R}^{N_i} \times \mathbb{R}^{m_i} \to \mathbb{R}^{n_i}$ is a continuous function.
- For every initial state $x_{i0} \in \mathbb{R}^{n_i}$ and all essentially bounded inputs $\bar{x}_i(\cdot)$ and $u_i(\cdot)$, there is a unique (local) solution $\phi_i(t, x_{i0}, \bar{x}_i, u_i)$ of the Cauchy problem
$$\dot{x}_i(t) = f_i(x_i(t), \bar{x}_i(t), u_i(t)), \quad x_i(0) = x_{i0}.$$

For each $i \in \mathbb{N}$, we fix (arbitrary) norms on the spaces $\mathbb{R}^{n_i}$ and $\mathbb{R}^{m_i}$, respectively. For brevity, we avoid adding indices to these norms, indicating to which space they belong, and simply write $|\cdot|$ for each of them. The interconnection of systems Σ_i, $i \in \mathbb{N}$, is defined on the state space

$$X := \ell_\infty(\mathbb{N}, (n_i)) := \{x = (x_i)_{i\in\mathbb{N}} : x_i \in \mathbb{R}^{n_i},\ \sup_{i\in\mathbb{N}} |x_i| < \infty\}.$$

This space is a Banach space with the ℓ_∞-type norm $\|x\|_X := \sup_{i\in\mathbb{N}} |x_i|$. The space of admissible external input values is likewise defined as the Banach space $U := \ell_\infty(\mathbb{N}, (m_i))$, $\|u\|_U := \sup_{i\in\mathbb{N}} |u_i|$. The class of admissible external input functions is given by

$$\mathcal{U} := \{u \in L_\infty(\mathbb{R}_+, U) : u \text{ is piecewise right-continuous}\},$$

which is equipped with the L_∞-norm $\|u\|_{\mathcal{U}} := \operatorname{ess\,sup}_{t\in\mathbb{R}_+} |u(t)|_U$. Finally, the right-hand side of the interconnected system is defined by

$$f : X \times U \to \prod_{i\in\mathbb{N}} \mathbb{R}^{n_i},\ f(x,u) := (f_i(x_i, \bar{x}_i, u_i))_{i\in\mathbb{N}}.$$

Hence, the interconnected system can formally be written as the following differential equation:

$$\Sigma : \quad \dot{x} = f(x,u). \tag{1}$$

To make sense of this equation, we introduce an appropriate notion of solution. For fixed $x^0 \in X$ and $u \in \mathcal{U}$, a function $\lambda : J \to X$, where J is an interval of the form $[0,T)$ with $0 < T \le \infty$, is called a *solution* of the Cauchy problem $\dot{x}(t) = f(x(t), u(t))$, $x(0) = x^0 \in X$ provided that $s \mapsto f(\lambda(s), u(s))$ is a locally integrable X-valued function (in the Bochner integral sense) and

$$\lambda(t) = x_0 + \textstyle\int_0^t f(\lambda(s), u(s))\,\mathrm{d}s \text{ for all } t \in J.$$

Sufficient conditions for the existence and uniqueness of solutions can be found in [9, Thm. III.2]. We say that the system Σ is *well-posed* if local solutions exist and are unique. In this case, it holds that

$$\pi_i(\phi(t, x^0, u)) = \phi_i(t, x_i^0, \bar{x}_i, u_i) \text{ for all } t \in J,$$

where $\pi_i : X \to \mathbb{R}^{n_i}$ denotes the canonical projection onto the i-th component, $\bar{x}_i(\cdot) = (\pi_j(\phi(\cdot, x, u)))_{j\in I_i}$, and x_i^0, u_i denote the i-th components of x^0 and u, respectively, see [9, Sec. III].

Throughout this paper, we assume that Σ is well-posed and also has the so-called *boundedness-implies-continuation (BIC) property* [17]. The BIC property requires that any bounded solution, defined on a compact time interval, can be extended to a solution defined on a larger time interval. If Σ is well-posed, we call Σ *forward-complete* if every solution can be extended to $\mathbb{R}_+$.

C. Input-to-state stability

We study input-to-state stability of Σ via the construction of ISS Lyapunov functions.

Definition II.1 *A well-posed system Σ is called* input-to-state stable (ISS) *if it is forward complete and there are $\beta \in \mathcal{KL}$ and $\kappa \in \mathcal{K}$ such that for all $x \in X$ and $u \in \mathcal{U}$,*

$$\|\phi(t,x,u)\|_X \le \beta(\|x\|_X, t) + \kappa(\|u\|_{\mathcal{U}}) \quad \textit{for all } t \ge 0.$$

A sufficient condition for ISS is the existence of an ISS Lyapunov function [18, Thm. 1].

Definition II.2 *A function $V : X \to \mathbb{R}_+$ is called an* ISS Lyapunov function *for Σ if it satisfies the following conditions:*

(i) V is continuous.

(ii) There exist $\psi_1, \psi_2 \in \mathcal{K}_\infty$ such that

$$\psi_1(\|x\|_X) \le V(x) \le \psi_2(\|x\|_X) \quad \textit{for all } x \in X.$$

(iii) There exist $\gamma \in \mathcal{K}$ and $\alpha \in \mathcal{P}$ such that for all $x \in X$ and $u \in \mathcal{U}$ the following implication holds:

$$V(x) > \gamma(\|u\|_{\mathcal{U}}) \quad \Rightarrow \quad \mathrm{D}^+V_u(x) \le -\alpha(V(x)),$$

where $\mathrm{D}^+V_u(x)$ denotes the right upper Dini orbital derivative, defined as

$$\mathrm{D}^+V_u(x) := \limsup_{t\to 0^+} \frac{V(\phi(t,x,u)) - V(x)}{t}.$$

The functions ψ_1, ψ_2 are also called coercivity bounds*, γ is called a* Lyapunov gain*, and α is called a* decay rate*.*

To construct an ISS Lyapunov function V for Σ, we follow a bottom-up approach in the sense that we exploit the interconnection structure and build V from ISS Lyapunov functions of the subsystems Σ_i under an appropriate small-gain condition. We thus make the following assumption on each subsystem Σ_i.

Assumption II.3 *For each subsystem Σ_i, there exists a continuous function $V_i : \mathbb{R}^{n_i} \to \mathbb{R}_+$, which is C^1 outside of $x_i = 0$, and satisfies the following properties:*

(L1) There exist $\psi_{i1}, \psi_{i2} \in \mathcal{K}_\infty$ such that

$$\psi_{i1}(|x_i|) \le V_i(x_i) \le \psi_{i2}(|x_i|) \quad \textit{for all } x_i \in \mathbb{R}^{n_i}.$$

(L2) There exist MHAFs μ_i, $\gamma_{ij} \in \mathbb{R}_+$ ($j \in \mathbb{N}$), where $\gamma_{ij} = 0$ for all $j \notin I_i$, and $\gamma_{iu} \in \mathcal{K}$, $\alpha_i \in \mathcal{P}$ such that for all $x = (x_j)_{j\in\mathbb{N}} \in X$ and $u = (u_j)_{j\in\mathbb{N}} \in U$ the following implication holds:

$$\begin{aligned} V_i(x_i) > \max\Big\{\mu_i\big((\gamma_{ij}V_j(x_j))_{j\in\mathbb{N}}\big), \gamma_{iu}(|u_i|)\Big\} \\ \Rightarrow \nabla V_i(x_i) f_i(x_i, \bar{x}_i, u_i) \le -\alpha_i(V_i(x_i)). \end{aligned} \tag{2}$$

The numbers γ_{ij} (which are identified with the linear functions $r \mapsto \gamma_{ij} r$) are called internal gains*, while the functions γ_{iu} are called* external gains*.*

Given a sequence of MHAFs $\mu = (\mu_i)_{i\in\mathbb{N}}$ and γ_{ij} as in (2), we introduce the gain operator $\Gamma_\mu : \ell_\infty^+ \to \ell_\infty^+$ by

$$\Gamma_\mu(s) := \Big(\mu_i\big((\gamma_{ij}s_j)_{j\in\mathbb{N}}\big)\Big)_{i\in\mathbb{N}} \quad \forall s = (s_i)_{i\in\mathbb{N}} \in \ell_\infty^+. \tag{3}$$

This definition covers linear, max-linear and mixed types of gain operators. With the choice of all μ_i in (3) either as summation or supremum, we have the following two special cases of the operator Γ_μ:

$$\Gamma_\oplus(s) := \Big(\sum_{j\in\mathbb{N}} \gamma_{ij} s_j\Big)_{i\in\mathbb{N}} \quad \text{for all } s = (s_i)_{i\in\mathbb{N}} \in \ell_\infty^+,$$

$$\Gamma_{\otimes}(s) := \Big(\sup_{j\in\mathbb{N}} \gamma_{ij} s_j\Big)_{i\in\mathbb{N}} \quad \text{for all } s = (s_i)_{i\in\mathbb{N}} \in \ell_\infty^+.$$

The following assumption guarantees that Γ_μ is well-defined and induces a discrete-time system with bounded finite-time reachability sets, cf. [19].

Assumption II.4 *The operator Γ_μ is well-defined, which is equivalent to*

$$\sup_{i\in\mathbb{N}} \mu_i\big((\gamma_{ij})_{j\in\mathbb{N}}\big) < \infty.$$

Definition II.5 *An operator $\Gamma : \ell_\infty^+ \to \ell_\infty^+$ is called*

1) *monotone, if $\Gamma(r) \le \Gamma(s)$ for all $r, s \in \ell_\infty^+$ s.t. $r \le s$.*
2) *homogeneous of degree one, if $\Gamma(cs) = c\Gamma(s)$ for all $s \in \ell_\infty^+$ and $c \ge 0$.*
3) *subadditive, if $\Gamma(r+s) \le \Gamma(r)+\Gamma(s)$ for all $r, s \in \ell_\infty^+$.*

The properties of the monotone aggregation functions μ_i immediately imply the statement of the next lemma.

Lemma II.6 *The operator Γ_μ is monotone, subadditive and homogeneous of degree one.*

Finally, we make the following uniform boundedness assumption on the external gains, which we need to prove our small-gain theorem.

Assumption II.7 *There is $\gamma_{\max}^u \in \mathcal{K}$ such that $\gamma_{iu} \le \gamma_{\max}^u$ for all $i \in \mathbb{N}$.*

We develop a small-gain theorem ensuring ISS of the overall network Σ from the assumptions imposed on subsystems Σ_i. The small-gain condition is given in terms of a spectral radius condition. In Section III, we establish several characterizations of this condition. Then, in Section IV, we present our small-gain theorem.

III. Characterizations of the Stability of the Gain Operator

Consider an operator $\Gamma : \ell_\infty^+ \to \ell_\infty^+$ and the corresponding induced system

$$x(k+1) = \Gamma(x(k)), \quad k \in \mathbb{N}_0. \tag{4}$$

Based on [20, Prop. A.1 and Prop. 7.17], we have the following characterizations for uniform global exponential stability of the system (4) induced by a monotone, subadditive and homogeneous of degree one operator. These characterizations play a crucial role in the verification of our small-gain condition, used in Theorem IV.1 below.

Proposition III.1 *Let $\Gamma : \ell_\infty^+ \to \ell_\infty^+$ be a well-defined operator which is monotone, subadditive, and homogeneous of degree one. Then Γ is Lipschitz continuous with the Lipschitz constant $\|\Gamma(\mathbb{1})\|_{\ell_\infty}$. Furthermore, the following statements are equivalent:*

(i) The spectral radius condition

$$r(\Gamma) := \lim_{n\to\infty} \sup_{\substack{s\in\ell_\infty^+ \\ \|s\|_{\ell_\infty}=1}} \|\Gamma^n(s)\|_{\ell_\infty}^{1/n} = \lim_{n\to\infty} \|\Gamma^n(\mathbb{1})\|_{\ell_\infty}^{1/n} < 1.$$

(ii) Uniform global exponential stability (UGES) *of the discrete-time system* (4)*: there are $M > 0$ and $a \in (0,1)$ such that for all $s \in \ell_\infty^+$ and $k \in \mathbb{N}_0$*

$$\|\Gamma^k(s)\|_{\ell_\infty} \le M a^k \|s\|_{\ell_\infty}.$$

(iii) Uniform global asymptotic stability (UGAS) *of the discrete-time system* (4)*: there is $\beta \in \mathcal{KL}$ such that*

$$\|\Gamma^k(s)\|_{\ell_\infty} \le \beta(\|s\|_{\ell_\infty}, k), \quad s \in \ell_\infty^+,\ k \in \mathbb{N}_0.$$

(iv) There is a point of strict decay*, i.e., there are $\lambda \in (0,1)$ and $s^0 \in \operatorname{int}(\ell_\infty^+)$ such that*

$$\Gamma(s^0) \le \lambda s^0. \tag{5}$$

(v) It holds that

$$\|\Gamma^n(\mathbb{1})\|_{\ell_\infty} < 1 \quad \text{for some } n \in \mathbb{N}.$$

(vi) There is a linear *path of strict decay for Γ, i.e., there are $s^0 \in \operatorname{int}(\ell_\infty^+)$ and $\rho > 0$ such that $\sigma : \mathbb{R}_+ \to \ell_\infty^+$, given by*

$$\sigma(r) := r s^0 \quad \text{for all } r \ge 0,$$

satisfies

$$\Gamma(\sigma(r)) \le \sigma(r)/(1+\rho) \quad \text{for all } r \ge 0.$$

In addition, if $\Gamma = \Gamma_\otimes$, then the above properties are also equivalent to

(vii) the robust strong small-gain condition*: there are $\omega, \rho \in (0,1)$ such that for all $i, j \in \mathbb{N}$ the operator Γ_{ij}, defined by*

$$\Gamma_{ij}(s) := \Gamma(s) + \omega s_j e_i \quad \text{for all } s \in \ell_\infty^+,$$

satisfies

$$\Gamma(s) \not\ge \rho s \quad \text{for all } s \in \ell_\infty^+ \backslash \{0\}. \tag{6}$$

(viii) For some $n \in \mathbb{N}$, it holds that

$$\sup_{j_1,\ldots,j_{n+1}} \gamma_{j_1 j_2} \cdots \gamma_{j_n j_{n+1}} < 1. \tag{7}$$

Finally, if $\Gamma = \Gamma_\oplus$, then the properties (i)–(v) are equivalent to the following:

(ix) For some $n \in \mathbb{N}$, it holds that

$$\sup_{i\in\mathbb{N}} \sum_{j_1,\ldots,j_n} \gamma_{ij_1}\gamma_{j_1 j_2} \cdots \gamma_{j_{n-1} j_n} < 1. \tag{8}$$

As seen in Theorem IV.1 below, Proposition III.1 provides an equivalence between the small-gain condition for the verification of ISS of the infinite network (1) and exponential stability of the discrete-time monotone system (4). We particularly use a linear path of strict decay to construct the overall Lyapunov function for the network Σ as a supremum over weighted Lyapunov functions of the subsystems Σ_i, where each weight is expressed as $1/s_i^0$ with s^0 as in (5).

It has been shown in [12] that a strong small-gain condition is not sufficient to guarantee ISS of an infinite network and thus it has to be robustified, leading to the notion of *robust strong small-gain condition*. Here, condition (6) represents a specific version of such a robust strong small-gain condition when the gain functions are linear.

A. Special cases

Here, we provide a graph-theoretic description of (7) and (8). For the infinite matrix $G := (\gamma_{ij})_{i,j\in\mathbb{N}}$, we define a weighted directed graph $\mathcal{G}(G)$ with infinitely many nodes and the set of edges $\mathcal{E} = \{(i,j)|i,j \in \mathbb{N}, \gamma_{ij} > 0\}$; thus, $(i,j) \in \mathcal{E}$ represents the edge from node i to node j, and $G := (\gamma_{ij})_{i,j\in\mathbb{N}}$ is the weight matrix with γ_{ij} being the weight on edge $(i,j) \in \mathcal{E}$. A path of length n from i to j is a sequence of $n+1$ distinct nodes starting with i and ending with j. Given $\mathcal{G}(G)$, condition (7) requires that the product of weights over any path of length n has to be uniformly less than one for some $n \in \mathbb{N}$. For each node i of $\mathcal{G}(G)$, condition (8) means that the sum over the product of weights over any path of length n starting from node i has to be uniformly less than one. We stress that the two criteria (7) and (8) are, in general, not comparable, as one is obtained from a maximum formulation and the other from a sum formulation, and thus their corresponding gains γ_{ij} can be different. While, in general, it can be very hard (if possible at all) to verify such conditions for an infinite network, one can easily check conditions (7) and (8) if the network has a special interconnection topology, e.g., is symmetric. We illustrate the latter point in Section V below, where we consider a spatially invariant network.

IV. Small-gain theorem

Now we are able to present our small-gain theorem which provides a Lipschitz continuous ISS Lyapunov function for the infinite interconnection Σ.

Theorem IV.1 *Consider the infinite network Σ as in* (1)*, and let the following assumptions hold:*

(i) *Σ is well-posed and satisfies the BIC property.*
(ii) *Assumption II.3 is satisfied. Moreover, there exist $\psi_1, \psi_2 \in \mathcal{K}_\infty$ such that*
$$\psi_1 \le \psi_{i1} \quad \text{and} \quad \psi_{i2} \le \psi_2 \quad \text{for all } i \in \mathbb{N}.$$
(iii) *Assumptions II.4 and II.7 are satisfied.*
(iv) *The spectral radius condition $r(\Gamma_\mu) < 1$ holds.*
(v) *For each $R > 0$, there is a constant $L(R) > 0$ such that*
$$|V_i(x_i) - V_i(y_i)| \le L(R)|x_i - y_i|$$
for all $i \in \mathbb{N}$ and $x_i, y_i \in B_R(0) \subset \mathbb{R}^{n_i}$.
(vi) *There exists $\tilde{\alpha} \in \mathcal{P}$ such that $\alpha_i \ge \tilde{\alpha}$ for all $i \in \mathbb{N}$.*

Then there exists $s^0 \in \operatorname{int}(\ell_\infty^+)$ such that the following function is an ISS Lyapunov function for Σ:
$$V(x) := \sup_{i\in\mathbb{N}} \frac{1}{s_i^0} V_i(x_i) \quad \text{for all } x = (x_i)_{i\in\mathbb{N}} \in X.$$
In particular, Σ is ISS. Moreover, V satisfies a Lipschitz condition on every bounded subset of X.

Theorem IV.1 shows ISS of the network Σ under the small-gain condition $r(\Gamma_\mu) < 1$, which by Proposition III.1 is equivalent to the existence of a point of strict decay, and thus to the existence of a linear path of strict decay with a linear decay rate. Note that in [13] ISS of the network has been shown under the assumption of the existence of a nonlinear path of strict decay with a nonlinear decay rate. Whether these conditions are equivalent for homogeneous of degree one and subadditive gain operators is an open question that is left to future investigations.

V. Example

In this section, we illustrate the effectiveness of conditions (7) and (8) to verify the small-gain condition. In particular, we allow the gain functions in part to be larger than the identity.

Consider scalar linear subsystems of the form

$$\Sigma_i : \begin{cases} \dot{x}_i = -b_{ii}x_i + b_{i(i-1)}x_{i-1} + g(b_{i(i+1)}x_{i+1}, b_{i(i+2)}x_{i+2}) \\ \qquad\qquad\qquad\qquad\qquad \text{for } i = 2k+1,\ k \in \mathbb{N}, \\ \dot{x}_i = -b_{ii}x_i + g(b_{i(i+1)}x_{i+1}, b_{i(i+2)}x_{i+2}) \\ \qquad\qquad\qquad\qquad\qquad \text{for } i = 2k,\ k \in \mathbb{N}, \end{cases}$$

where either $g(s) =: g_\oplus(s) = s_1 + s_2$ or $g(s) =: g_\otimes(s) = \max\{s_1, s_2\}$ for $s = (s_1, s_2) \in \mathbb{R}^2$, $x_i \in \mathbb{R}$, $b_{ii} > 0$ and $b_{i(i-1)}, b_{i(i+1)}, b_{i(i+2)} \in \mathbb{R}$ with $b_{i0} = 0$.

To guarantee that the network Σ of subsystems Σ_i is well-posed and satisfies the BIC property, it suffices to assume that all the coefficients b_{ij} are uniformly upper-bounded by some $\bar{b} > 0$ over i (the simple proof will be omitted).

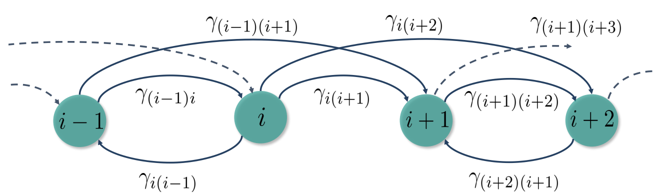

Fig. 1. The graph $\mathcal{G}(G)$ associated with G. Dashed arrows imply the continuation of the graph with the same structure.

Take $V_i(x_i) := \frac{1}{2}x_i^2$. The following estimates are obtained by using Young's inequality and homogeneity of g:

$$\begin{aligned} \nabla V_i(x_i) f_i(x_i, \bar{x}_i) \le & -\alpha_i V_i(x_i) + \frac{b_{i(i-1)}^2}{2\varepsilon_i} V_{i-1}(x_{i-1}) \\ & + g\Big(\frac{b_{i(i+1)}^2}{2\delta_i} V_{i+1}(x_{i+1}), \frac{b_{i(i+2)}^2}{2\delta_i'} V_{i+2}(x_{i+2})\Big), \end{aligned} \tag{9}$$

with $\alpha_i := 2(b_{ii} - \varepsilon_i - \delta_i - \delta_i')$, and appropriately small $\delta_i, \delta_i', \varepsilon_i > 0$. If i is even, then $b_{i(i-1)} = 0$ and neither the second term nor ε_i on the right-hand side of (9) is present. Additionally, if $g = g_\otimes$, then $\delta_i' = \delta_i$ in g and $\alpha_i := 2(b_{ii} - \varepsilon_i - \delta_i)$. From this estimate, we need to derive appropriate gains for an ISS Lyapunov function. Let

$\tilde{\gamma}_{i(i-1)} := b^2_{i(i-1)}/2\varepsilon_i$, $\tilde{\gamma}_{i(i+1)} := b^2_{i(i+1)}/2\delta_i$, $\tilde{\gamma}_{i(i+2)} := b^2_{i(i+2)}/2\delta'_i$ and assume that

$$V_i(x_i) \geq \frac{1}{a_i}\tilde{\mu}\big((\tilde{\gamma}_{i(i+k)}V_{i+k}(x_{i+k}))_{k=-1,1,2}\big), \tag{10}$$

where in line with the function g we have either $\tilde{\mu}(s) = 2\max_{i=1,2,3} s_i$ or $\tilde{\mu}(s) = \sum_{i=1}^{3} s_i$, and $a_i > 0$ is specified later. From (9), it follows that

$$\nabla V_i(x_i)f_i(x_i,\bar{x}_i) \leq -\alpha_i V_i(x_i) \\ + \tilde{\mu}(\tilde{\gamma}_{i(i-1)}V_{i-1}(x_{i-1}),\tilde{\gamma}_{i(i+1)}V_{i+1}(x_{i+1}),\tilde{\gamma}_{i(i+2)}V_{i+2}(x_{i+2})).$$

Hence, (10) implies $\nabla V_i(x_i)f_i(x_i,\bar{x}_i) \leq -(\alpha_i - a_i)V_i(x_i)$. Let us pick $a_i := \alpha_i/2$. In that way, for $g = g_\oplus$, non-zero gains γ_{ij} in $\Gamma_\oplus$ are computed as

$$\gamma_{i(i-1)} = \frac{b^2_{i(i-1)}}{2\varepsilon_i w_i},\ \gamma_{i(i+1)} = \frac{b^2_{i(i+1)}}{2\delta_i w_i},\ \gamma_{i(i+2)} = \frac{b^2_{i(i+2)}}{2\delta'_i w_i},$$

where $w_i := b_{ii} - \varepsilon_i - \delta_i - \delta'_i$. On the other hand, if $g = g_\otimes$, non-zero gains γ_{ij} in $\Gamma_\otimes$ are obtained as

$$\gamma_{i(i-1)} = \frac{b^2_{i(i-1)}}{\varepsilon_i q_i},\ \gamma_{i(i+1)} = \frac{b^2_{i(i+1)}}{\delta_i q_i},\ \gamma_{i(i+2)} = \frac{b^2_{i(i+2)}}{\delta_i q_i},$$

with $q_i = b_{ii} - \varepsilon_i - \delta_i$. Obviously, Assumptions (i), (ii), and (v) of Theorem IV.1 are satisfied. To verify Assumption (vi), it is necessary to impose a uniform lower bound on the number b_{ii}, i.e., $b_{ii} \geq \underline{b}$ for all $i \in \mathbb{N}$. Furthermore, we require uniform bounds on the numbers δ_i, δ'_i, ε_i so that $\underline{\delta} \leq \delta_i \leq \overline{\delta}$, $\underline{\delta}' \leq \delta'_i \leq \overline{\delta}'$, $\underline{\varepsilon} \leq \varepsilon_i \leq \overline{\varepsilon}$, and $\underline{b} - \overline{\varepsilon} - \overline{\delta} - \overline{\delta}' > 0$. Then for both sum and maximum formulations, we can estimate $\gamma_{ij} \leq 2\overline{b}^2/\big(\underline{\varepsilon}(\underline{b} - \overline{\varepsilon} - \overline{\delta} - \overline{\delta}'\big)$ for all $i,j \in \mathbb{N}$. Hence, Assumptions (iii) and (vi) of Theorem IV.1 are satisfied.

It only remains to verify the small-gain condition. To do this, we respectively use the criteria (7) and (8) for $g = g_\otimes$ and $g = g_\oplus$. Following arguments of Section III-A, we use the weighted directed graph $\mathcal{G}(G)$ which is (partially) depicted in Fig. 1. As can be seen, the graph is symmetrical due to the spatial invariance of the overall network. Take $n = 2$ in (7). From the structure of the network and the choice of n, condition (7) reduces to the following set of conditions:

$$\begin{aligned} \gamma_{(i-1)(i+1)}\gamma_{(i+1)(i+k)} < 1, \quad k \in \{2,3\}, \\ \gamma_{(i-1)i}\gamma_{i+k} < 1, \quad k \in \{-1,1,2\}, \end{aligned} \tag{11}$$

for all $i \in \mathbb{N}$. On the other hand, by taking node $i-1$, condition (8) yields

$$\gamma_{(i-1)(i+1)}\gamma_{(i+1)(i+3)} + \gamma_{(i-1)(i+1)}\gamma_{(i+1)(i+2)} \\ + \gamma_{(i-1)i}\gamma_{i(i-1)} + \gamma_{(i-1)i}\gamma_{i(i+1)} + \gamma_{(i-1)i}\gamma_{i(i+2)} < 1. \tag{12}$$

Clearly, any other nodes give the same condition as that in (12). From either (11) or (12), for sufficiently small $\gamma_{(i-1)i}$ and $\gamma_{(i-1)(i+1)}$, conditions (11) and (12) are satisfied. Given $b_{ii}, \varepsilon_i, \delta_i, \delta'_i$, as γ_{ij} directly depend on b_{ij}, to satisfy any of (11) and (12), b_{ij} should be small enough.

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References

[1] N. Noroozi, A. Mironchenko, and F. R. Wirth, "A relaxed small-gain theorem for discrete-time infinite networks," in *59th IEEE Conf. Decision Control*, (Jeju Island), pp. 3102–3107, 2020.

[2] P. Barooah, P. G. Mehta, and J. P. Hespanha, "Mistuning-based control design to improve closed-loop stability margin of vehicular platoons," *IEEE Trans. Autom. Control*, vol. 54, no. 9, pp. 2100–2113, 2009.

[3] B. Bamieh, F. Paganini, and M. A. Dahleh, "Distributed control of spatially invariant systems," *IEEE Trans. Autom. Control*, vol. 47, no. 7, pp. 1091–1107, 2002.

[4] T. Liu, Z.-P. Jiang, and D. J. Hill, *Nonlinear control of dynamic networks*. Boca Raton: Taylor & Francis, 2014.

[5] S. N. Dashkovskiy, B. S. Rüffer, and F. R. Wirth, "Small gain theorems for large scale systems and construction of ISS Lyapunov functions," *SIAM J. Control Opt.*, vol. 48, no. 6, pp. 4089–4118, 2010.

[6] A. Mironchenko and C. Prieur, "Input-to-state stability of infinite-dimensional systems: recent results and open questions," *SIAM Rev.*, vol. 62, no. 3, pp. 529–614, 2020.

[7] I. Karafyllis and M. Krstic, *Input-to-state stability for PDEs*. Springer, 2019.

[8] J. Zheng and G. Zhu, "Input-to-state stability with respect to boundary disturbances for a class of semi-linear parabolic equations," *Automatica*, vol. 97, pp. 271–277, 2018.

[9] C. Kawan, A. Mironchenko, A. Swikir, N. Noroozi, and M. Zamani, "A Lyapunov-based small-gain theorem for infinite networks," *IEEE Trans. Automat. Control*, 2020.

[10] N. Noroozi, A. Mironchenko, C. Kawan, and M. Zamani, "Set stability of infinite networks: ISS small-gain theory and its applications," *IFAC-PapersOnLine*, vol. 54, no. 9, pp. 72–77, 2021.

[11] S. Dashkovskiy and S. Pavlichkov, "Stability conditions for infinite networks of nonlinear systems and their application for stabilization," *Automatica*, vol. 112, p. 108643, 2020.

[12] S. Dashkovskiy, A. Mironchenko, J. Schmid, and F. Wirth, "Stability of infinitely many interconnected systems," in *11th IFAC Symp. Nonlinear Control Syst.*, pp. 937–942, 2019.

[13] C. Kawan, A. Mironchenko, and M. Zamani, "A Lyapunov-based ISS small-gain theorem for infinite networks of nonlinear systems," *arXiv preprint arXiv:2103.07439*, 2021. Submitted.

[14] J. Glück and A. Mironchenko, "Stability criteria for positive linear discrete-time systems," *Appeared online in Positivity, https://doi.org/10.1007/s11117-021-00853-2*, 2021.

[15] A. Mironchenko, N. Noroozi, C. Kawan, and M. Zamani, "ISS small-gain criteria for infinite networks with linear gain functions," *arXiv e-prints*, p. arXiv:2103.06694, 2021.

[16] C. M. Kellett, "A compendium of comparison function results," *Math. Control Signals Syst.*, vol. 26, no. 3, pp. 339–374, 2014.

[17] I. Karafyllis and Z.-P. Jiang, *Stability and stabilization of nonlinear systems*. London: Springer, 2011.

[18] S. Dashkovskiy and A. Mironchenko, "Input-to-state stability of infinite-dimensional control systems," *Math. Control Signals Syst.*, vol. 25, no. 1, pp. 1–35, 2013.

[19] A. Mironchenko and F. Wirth, "Characterizations of input-to-state stability for infinite-dimensional systems," *IEEE Trans. Autom. Control*, vol. 63, no. 6, pp. 1602–1617, 2018.

[20] A. Mironchenko, C. Kawan, and J. Glück, "Nonlinear small-gain theorems for input-to-state stability of infinite interconnections," *Math. Control Signals Syst.*, pp. 1–43, 2021.