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Design of saturated controls for an unstable parabolic PDE

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Abstract: We derive a saturated feedback control, which locally stabilizes a linear reaction-diffusion equation. In contrast to most other works on this topic, we do not assume Lyapunov stability of the uncontrolled system, and consider general unstable systems. Using Lyapunov methods, we provide estimates for the region of attraction for the closed-loop system, given in terms of linear and bilinear matrix inequalities. We show that our results can be used with distributed as well as scalar boundary control. The efficiency of the proposed method is demonstrated by means of a numerical simulation.

Keywords: Nonlinear control systems, partial differential equations, reaction-diffusion equation, saturated control, stabilization, attraction region.

1. INTRODUCTION

Saturated control is one of the most natural nonlinearities to be considered in control theory, because physical inputs are often limited in amplitude. Neglecting this nonlinearity in the control design could lead to undesirable behavior of the solutions to the closed-loop system, and to shrinking of the attraction region of the closed-loop system (see e.g. Tarbouriech et al. (2011); Zaccarian and Teel (2011); Saberi et al. (2012)). This leads to the problem of (local or global) stabilization of control systems via inputs of a norm not exceeding a prescribed value.

In this paper, a linear unstable reaction-diffusion equation is introduced. Many results exist in the literature for the control of this class of partial differential equations (PDE), using either bounded or unbounded control operators, with or without input delays. See in particular Krstic (2009); Krstic and Smyshlyaev (2008) for the delayed control using a backstepping approach; see Fridman and Orlov (2009) where a stable heat PDE is controlled by means of a delayed bounded linear control operator.

As far as input saturations and synthesis of saturated controllers for infinite-dimensional systems are concerned, see Slemrod (1989); Lasiecka and Seidman (2003) for the first studies of PDEs with constrained controllers. In these results, the saturation map has to be understood as a limitation on the space norm of the input function, and not as as a pointwise saturation, which limits the values of the control input at each point by a prescribed value.

Other types of saturation functions can be useful in practice, see Marx et al. (2017b) for a discussion. Regarding the saturation map definition, our definition fits the one of Prieur et al. (2016) where Lyapunov methods are shown to be useful for the stability analysis of wave equations subject to saturated inputs. See also Marx et al. (2017b,a), where systems in Hilbert spaces with applications to the Korteweg–de Vries equation have been addressed.

In this conference paper, for the unstable reaction-diffusion system we derive a saturated controller for two cases: firstly, for static in-domain control (through a bounded control operator), and secondy for dynamic boundary control (through an unbounded control operator). In both cases, pointwise saturated feedback laws are designed. Moreover the developed Lyapunov approach yields numerically tractable procedures for the synthesis of the control gains, and for the estimation of the basin of attraction of the closed-loop system. Let us emphasize that, contrary to the works Slemrod (1989); Lasiecka and Seidman (2003); Prieur et al. (2016), in our paper the open-loop system is unstable, therefore controlling the system with a saturating controller only provides local asymptotic stability, paralleling what is known for finite-dimensional systems (see in particular Teel (1992)).

To solve the design problem of saturated controllers, we first isolate the finite number of unstable modes and then we apply classical finite-dimensional techniques for the stabilization of the unstable part (see e.g. Tarbouriech et al. (2011); Zaccarian and Teel (2011); Saberi et al. (2012)). This yields a Lyapunov function of the unstable state part and linear and bilinear matrix inequalities (LMIs and BMIs) to be considered for the estimation of the region of attraction of the closed-loop finite-dimensional system (see Boyd et al. (1994); Scherer and Weiland (2000); Van Antwerp and Braatz (2000) for an introduction on matrix inequalities). Then, incorporating the other part of the state of the reaction-diffusion equation, we show how asymptotic stability and estimates for the region of attraction can be obtained when closing the loop with the nonlinear saturated controller.

In this light, our approach may be useful for all infinite-dimensional systems for which there exists only a finite number of unstable modes to be controlled through a bounded control operator (e.g. for systems with input delays considered in e.g. Fiagbedzi and Pearson (1986)).

The paper is organized as follows. We first introduce the in-domain control problem for the reaction-diffusion equation in Section 2. A saturated feedback is designed in Section 3 and a region of attraction is estimated. Some numerical simulations illustrate our results in Section 4. In Section 5, we show how our results could be applied to unbounded control operator, when designing a stabilizing dynamic saturated controller.

Due to the page limitations we omit most of the proofs in this paper, which can be found in a journal version of this paper Mironchenko et al. (2019).

Notation: The Euclidean norm on $\mathbb{R}^n$ is denoted by $|\cdot|$, the operator norm induced by this norm on spaces of matrices is denoted by $\|\cdot\|$. We denote the interior of a set S in a topological space by $\operatorname{int} S$. The ball of radius ε around 0 is denoted by $B_\varepsilon(0)$. For convenience, we denote $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$. For $L > 0$, $H^k(0, L)$ denotes the Sobolev space of functions from the space $L_2(0, L)$, which have weak derivatives of order $\leqslant k$, all of which belong to $L_2(0, L)$. $H_0^k(0, L)$ is the closure of $C_0^k(0, L)$ (the continuous functions with compact support in $(0, L)$) in the norm of $H^k(0, L)$.

2. PROBLEM FORMULATION

We consider the stabilization problem of the heat equation by means of a distributed control $u : \mathbb{R}_+ \to \mathbb{R}^m$. Let $L > 0$. We are given m functions $b_k : [0, L] \to \mathbb{R}$ describing at which places the control input u_k is acting. We have

$$\begin{aligned} &w_t(t,x) = w_{xx}(t,x) + c(x)w(t,x) \\ &\qquad + \sum_{k=1}^{m} b_k(x)\mathrm{sat}(u_k(t)),\ t > 0,\ x \in (0,L), \\ &w(t,0) = w(t,L) = 0, \quad t > 0, \\ &w(0,x) = w^0(x), \quad x \in (0,L). \end{aligned} \tag{1}$$

We assume that the state space of this system is $X := L_2(0, L)$ and that $c, b_k \in X$, $k = 1, \dots, m$.

Here sat is a component-wise saturation function, that is, for all $k = 1, \dots, m$ and for any $v \in \mathbb{R}^m$,

$$\mathrm{sat}(v)_k := \begin{cases} v_k & \text{if } |v_k| \leqslant \ell \\ \dfrac{\ell}{|v_k|} v_k & \text{if } |v_k| \geqslant \ell, \end{cases} \tag{2}$$

where $\ell > 0$ is the given level of the saturation, which is assumed to be uniform with respect to the index k.

Remark 1. Systems of the form (1) also occur in the problem of stabilizing a linear heat equation by means of a boundary control subject to delays or saturations, see e.g. Prieur and Trélat (2019) as well as Section 5 below. ◦

Define the operator

$$A = \partial_{xx} + c(\cdot)\mathrm{id} : X \to X \tag{3}$$

with domain $D(A) = H^2(0, L) \cap H_0^1(0, L)$. Then the above control system takes the form

$$w_t(t, \cdot) = Aw(t, \cdot) + \sum_{k=1}^{m} b_k \mathrm{sat}(u_k(t)). \tag{4}$$

We note that A is selfadjoint and has compact resolvent, see (Mironchenko et al., 2019, Appendix A). Hence, the spectrum of A consists of only isolated eigenvalues with finite multiplicity (see (Curtain and Zwart, 1995, Lemma A.4.19) and (Curtain and Zwart, 1995, Example A.4.26)). Furthermore, there exists a Hilbert basis $(e_j)_{j\geqslant 1}$ of X consisting of eigenfunctions of A, associated with the sequence of eigenvalues $(\lambda_j)_{j\geqslant 1}$. Note that

$$-\infty < \cdots < \lambda_j < \cdots < \lambda_1 \quad \text{and} \quad \lambda_j \underset{j\to+\infty}{\longrightarrow} -\infty$$

and that $e_j(\cdot) \in D(A)$ for every $j \geqslant 1$.

We consider the mild solutions of the system (1) (see (Curtain and Zwart, 1995, Section 3.1)), which exist and are unique for any initial condition in X and for any u_k that is in $L_{1,loc}([0, \infty))$, for $k = 1, \dots, m$.

Every (mild) solution $w(t, \cdot) \in D(A)$ of (4) can be expanded as a series in the eigenfunctions $e_j(\cdot)$.

$$\begin{aligned} w(t,\cdot) &= \sum_{j=1}^{\infty} w_j(t) e_j(\cdot), \\ w_j(t) &:= \langle w(t,\cdot), e_j(\cdot)\rangle_{L_2(0,L)},\ j \in \mathbb{N}^*. \end{aligned} \tag{5}$$

Analogously, we can expand the coefficients b_k in the series

$$b_k(\cdot) = \sum_{j=1}^{\infty} b_{jk} e_j(\cdot), \quad b_{jk} = \langle b_k(\cdot), e_j(\cdot)\rangle_{L_2(0,L)},\ j \in \mathbb{N}^*.$$

Using the mild formulation of the problem (4), we thus get that (4) is equivalent to the infinite-dimensional control system

$$\dot{w}_j(t) = \lambda_j w_j(t) + \sum_{k=1}^{m} b_{jk} \mathrm{sat}(u_k(t)) \tag{6}$$

$$= \lambda_j w_j(t) + \mathbf{b}_j \cdot \mathrm{sat}(u(t)), \qquad j \in \mathbb{N}^*, \tag{7}$$

where '$\cdot$' is a scalar product in $\mathbb{R}^m$, $\mathrm{sat}(u(t)) \in \mathbb{R}^m$ is the vector with entries $\mathrm{sat}(u_k(t))$ and $\mathbf{b}_j$ is the row vector with entries b_{jk}, $k = 1, \dots, m$.

Let $n \in \mathbb{N}^*$ be the number of nonnegative eigenvalues of A and let $\eta > 0$ be such that

$$\forall j > n \quad \text{we have} \quad \lambda_j < -\eta < 0. \tag{8}$$

With the matrix notations

$$z := \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}, \mathbf{A} := \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}, \mathbf{B} := \begin{pmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nm} \end{pmatrix} \tag{9}$$

the n first equations of (6) form the unstable finite-dimensional control system

$$\dot{z}(t) = \mathbf{A}z(t) + \mathbf{B}\mathrm{sat}(u(t)). \tag{10}$$

3. ESTIMATION OF THE REGION OF ATTRACTION FOR SATURATED INPUTS

3.1 Decomposition of the system into stable and unstable part

We now introduce a decomposition of the state space into a finite dimensional space on which the stabilization problem has to be solved and its orthogonal complement, which is invariant under the free dynamics.

Let X_n be the subspace of $L_2(0,L)$ spanned by $e_1(\cdot),\dots,e_n(\cdot)$ and let π_n be the orthogonal projection onto X_n, that is

$$\pi_n w(t,\cdot) := \sum_{j=1}^{n} w_j(t)e_j(\cdot). \tag{11}$$

We define also $X_n^{\perp}$ as the orthogonal complement of X_n in X. Let $\iota : \mathbb{R}^n \to X_n$ be the isomorphism defined by $\iota(e^j) = e_j(\cdot)$, where $(e^j)_{j=1,\dots,n}$ is the canonical basis of $\mathbb{R}^n$. It will be useful to use the isometric representation of $L_2(0,L)$ as $\ell^2(\mathbb{N}^*,\mathbb{R})$ obtained by the isomorphism $e_j(\cdot) \mapsto e^j$, where

$$\ell^2(\mathbb{N}^*,\mathbb{R}) := \{(x_k)_{k\in\mathbb{N}^*} : x_k \in \mathbb{R}\ \forall k \in \mathbb{N}^*, \sum_{k=1}^{\infty}|x_k|^2 < \infty\}.$$

Corresponding to the decomposition $L_2(0,L) = X_n \bigoplus X_n^{\perp}$, where "$\bigoplus$" is the orthogonal sum of subspaces, we denote $\ell^2(\mathbb{N}^*,\mathbb{R}) = \mathbb{R}^n \bigoplus \ell^2_{j>n}$, where we identify $\mathbb{R}^n$ with the sequences with support in $\{1,\dots,n\}$ and $\ell^2_{j>n}$ is the set of sequences in $\ell^2(\mathbb{N}^*,\mathbb{R})$ which are 0 in the first n entries.

Given a linear map $K : X_n \to \mathbb{R}^m$, consider the following feedback control

$$\begin{aligned} u(t) =& K\pi_n w(t,\cdot) = K\Big(\sum_{j=1}^{n} w_j(t)e_j(\cdot)\Big) = \sum_{j=1}^{n} w_j(t)Ke_j(\cdot) \\ =& \sum_{j=1}^{n} w_j(t)\mathbf{K}_j = \mathbf{K}z(t), \end{aligned} \tag{12}$$

where $\mathbf{K}_j := Ke_j(\cdot) \in \mathbb{R}^m$, $j = 1,\dots,n$, and where we use in the final step the notation from (9) and set $\mathbf{K} := (\mathbf{K}_1,\dots,\mathbf{K}_n) \in \mathbb{R}^{m\times n}$.

Hence the system (6) with the feedback (12) is equivalent to the following set of differential equations:

$$\dot{w}_j(t) = \lambda_j w_j(t) + \mathbf{b}_j \cdot \mathrm{sat}(\mathbf{K}z(t)), \quad j = 1,2,\dots \tag{13}$$

Using the notation (9), we can rewrite the first n equations of this system as

$$\dot{z}(t) = \mathbf{A}z(t) + \mathbf{B}\mathrm{sat}(\mathbf{K}z(t)). \tag{14}$$

Now, (13) can be considered as a cascade interconnection of an n-dimensional part, described by the equations (14) and of an infinite-dimensional part described by the equation

$$\dot{w}_j(t) = \lambda_j w_j(t) + \mathbf{b}_j \cdot \mathrm{sat}(\mathbf{K}z(t)), \quad j \geqslant n+1. \tag{15}$$

Next we show that the problem of exponential stabilization of the overall system (6) boils down to the exponential stabilization of the finite-dimensional unstable system (10). This latter problem will be elaborated in Section 3.2.

Definition 1. *Assume that* $\mathbf{K}$ *is chosen so that* 0 *is a locally asymptotically stable fixed point of* (14). *We say that* S *is a* region of attraction *of* 0 *if (i)* $0 \in \mathrm{int}\, S$*; (ii) for any* $z_0 \in S$ *the corresponding solution* $z(t;z_0) \to 0$ *as* $t \to \infty$*; (iii)* S *is forward invariant, i.e. for any* $z_0 \in S$ *it holds that* $z(t;z_0) \in S$ *for all* $t \geqslant 0$*. The largest set (with respect to set inclusion) with these three properties is called the* maximal region of attraction.

Definition 2. *We say that* (14) *is locally exponentially stable with region of attraction* S*, if the following two conditions are satisfied:*

(i) there exist $\varepsilon, M, a > 0$ *such that for any initial condition* $z(0) \in X$ *satisfying* $|z(0)| < \varepsilon$*, it holds*
$$|z(t)| \leqslant Me^{-at}|z(0)| \quad \forall t \geqslant 0.$$
(ii) $B_\varepsilon(0) \subset S$ *and* S *is a region of attraction of* (14).

Definitions 1 and 2 can be stated analogously for the system (13).

Proposition 1. *Assume* $\mathbf{K}$ *is chosen such that the subsystem* (14) *is locally exponentially stable in* 0 *with region of attraction* $S \subset \mathbb{R}^n$*. Then:*

(i) system (13) *is locally exponentially stable in* 0 *with region of attraction* $S \times \ell^2_{j>n}$.
(ii) (1) *with the controller* (12) *is locally exponentially stable in* 0 *with a region of attraction* $\iota(S) \times X_n^{\perp}$.

In addition, for any closed and bounded set $G \subset \mathrm{int}\,(\iota(S)\times X_n^{\perp})$*, there exist two positive values* M *and* a *such that for any initial condition* $w(0,\cdot)$ *in* G*, the solution* $w(\cdot)$ *to* (1) *with the controller* (12) *satisfies*

$$\|w(t,\cdot)\|_X \leqslant Me^{-at}\|w(0,\cdot)\|_X \quad \forall t \geqslant 0. \tag{16}$$

Proof. Pick a compact subset G' of $\mathrm{int}\, S$. Since we assume that (14) is locally exponentially stable with region of attraction $S \subset \mathbb{R}^n$, it is well known that there exist $M, a > 0$ so that for all $z(0) \in G'$ the solution z to (14) satisfies $|z(t)| \leqslant Me^{-at}|z(0)|$. From equations (6) and (12), we derive that for $j = n+1,\dots,\infty$, for any $t \geqslant 0$ and for any $(w_{n+1}(0), w_{n+2}(0),\dots)$ in $\ell^2_{j>n}$ it holds that

$$w_j(t) = e^{\lambda_j t}w_j(0) + \mathbf{b}_j \cdot \int_0^t e^{\lambda_j(t-s)}\mathrm{sat}(\mathbf{K}z(s))ds.$$

From (2) it follows that for all $z \in \mathbb{R}^n$ we have

$$|\mathrm{sat}(\mathbf{K}z)| \leqslant |\mathbf{K}z| \leqslant \|\mathbf{K}\||z|.$$

Also due to Cauchy-Bunyakovsky-Schwarz inequality we have that, for all $j = 1,2,\dots$,

$$|b_{jk}| = \Big|\langle b_k(\cdot), e_j(\cdot)\rangle_X\Big| \leqslant \|b_k\|_X\|e_j\|_X = \|b_k\|_X. \tag{17}$$

Thus, for all $j = n+1, n+2,\dots$, we obtain (exploiting (17)) that

$$\begin{aligned} |w_j(t)| \leqslant& e^{-\eta t}|w_j(0)| + |\mathbf{b}_j|\int_0^t e^{-\eta(t-s)}|\mathbf{K}z(s)|ds \\ \leqslant& e^{-\eta t}|w_j(0)| + |\mathbf{b}_j|\|\mathbf{K}\|\int_0^t e^{-\eta(t-s)}Me^{-as}|z(0)|ds \\ =& e^{-\eta t}|w_j(0)| + |\mathbf{b}_j|\frac{M\|\mathbf{K}\|}{\eta - a}(e^{-at} - e^{-\eta t})|z(0)|. \end{aligned}$$

The above computations have been performed for the case when $\eta \neq a$. If $a = \eta$, then it holds that

$$|w_j(t)| \leqslant e^{-\eta t}|w_j(0)| + M|\mathbf{b}_j|\|\mathbf{K}\|te^{-\eta t}|z(0)|.$$

Now, using the inequality $(a+b)^2 \leqslant 2(a^2+b^2)$ for any $(a,b) \in \mathbb{R}^2$, and the square summability of $|w_j(0)|$ and $|b_{jk}|$, $k = 1,\dots,m$, it follows that $\sum_{j=n+1}^{\infty}|w_j(t)|^2$ decays exponentially as well.

We now obtain local exponential stability of (13) by choosing G' such that $0 \in \mathbb{R}^n$ is in the interior of G' and noting that then $0 \in X$ is in the interior of $\iota(G') \times X_n^{\perp}$.

For the final statement of the proposition, pick a closed and bounded set $G \subset \mathrm{int}\,(\iota(S) \times X_n^{\perp})$. Select $G' = \iota^{-1} \circ \pi_n(G)$, then G' is a compact subset of $\mathrm{int}\, S$, the previous computations yield (16) for suitable constants M and a and for the superset $\imath(G') \times X_n^\perp$ which contains G. □

3.2 Estimate of the maximal region of attraction for the finite-dimensional part

In view of Proposition 1, it is important to study the local exponential stability and to estimate the region of attraction of the finite-dimensional system (14). We perform this task in this section. We assume here that $z \in \mathbb{R}^n$, $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$ and $\mathbf{K} \in \mathbb{R}^{m\times n}$. Recall system (14) one more time:

$$\dot{z} = \mathbf{A}z + \mathbf{B}\mathrm{sat}(\mathbf{K}z). \tag{18}$$

Remark 2. If a feedback $\mathbf{K}$ renders the closed-loop system (18) locally asymptotically stable, then also

$$\dot{z} = \mathbf{A}z + \mathbf{B}u \tag{19}$$

is locally and hence globally asymptotically stabilized by means of the feedback $u(t) := \mathbf{K}z(t)$. Thus, local asymptotic stability of (18) implies that the pair $(\mathbf{A}, \mathbf{B})$ is stabilizable.

We note that in the case $m = 1$ the situation simplifies further as then (19) is a linear diagonal system with scalar control input. The criterion for stabilizability is then that $b_{j1} \neq 0$ for all $j = 1, \dots, n$ and $\lambda_k \neq \lambda_j$ for all $k, j = 1, \dots, n$, $k \neq j$ (which is an easy exercise). In other words, the localization function b_1 should not be orthogonal to an unstable eigenfunction and all unstable eigenvalues need to be simple. ◦

Proposition 2. *Consider system* (18) *with* $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$ *and* $\mathbf{K} \in \mathbb{R}^{m\times n}$. *Assume that there exist a symmetric positive definite matrix* $P \in \mathbb{R}^{n\times n}$, *a diagonal positive definite matrix* $\mathbf{D} \in \mathbb{R}^{m\times m}$ *and a matrix* $\mathbf{C} \in \mathbb{R}^{m\times n}$ *such that*

$$M_1 := \begin{bmatrix} (\mathbf{A}+\mathbf{BK})^\top P + P(\mathbf{A}+\mathbf{BK}) & P\mathbf{B} - (\mathbf{DC})^\top \\ (P\mathbf{B})^\top - \mathbf{DC} & -2\mathbf{D} \end{bmatrix} < 0 \tag{20}$$

and

$$M_2 := \begin{bmatrix} P & (\mathbf{K}-\mathbf{C})^\top \\ \mathbf{K}-\mathbf{C} & \ell^2 I_m \end{bmatrix} \geqslant 0. \tag{21}$$

Then the finite-dimensional system (18) is locally asymptotically stable in 0 *with a region of attraction given by*

$$\mathcal{A} := \{z,\ z^\top P z \leqslant 1\}. \tag{22}$$

Moreover, in $\mathcal{A}$, *the function* V_1 *defined by* $V_1(z) := z^\top P z$, $z \in \mathbb{R}^n$, *decreases exponentially fast to* 0 *along the solutions to (18), i.e. there is a constant* $\alpha > 0$ *so that*

$$\dot{V}_1(z) \leqslant -\alpha |z|^2, \quad z \in \mathcal{A}. \tag{23}$$

The main interest of Proposition 2 lies in the following consequence for system (1).

Theorem 1. *Consider system* (1) *along with the feedback* K *of* (12). *Assume that the matrix representation* $\mathbf{K}$ *is such that the assumptions of Proposition 2 are satisfied. Then the closed-loop system*

$$\begin{aligned} w_t(t,x) &= w_{xx}(t,x) + c(x)w(t,x) \\ &\quad + \sum_{j=1}^{m} b_k(x)\mathrm{sat}((K\pi_n w(t,\cdot))_k),\ t>0,\ x \in (0,L), \\ w(t,0) &= w(t,L) = 0, \quad t > 0, \\ w(0,x) &= w^0(x), \quad x \in (0,L). \end{aligned} \tag{24}$$

is locally exponentially stable in 0 *with region of attraction* $\imath(\mathcal{A}) \times X_n^\perp$. *In addition, the constants of decay can be chosen uniformly on* $\imath(\mathcal{A}) \times X_n^\perp$.

4. NUMERICAL EXPERIMENT

In this section we use Proposition 2 to obtain estimates of the region of attraction for the unstable heat equation (1) subject to a saturated feedback controller. Let $c(\cdot)$ in the equation (1) be a constant function, and we slightly abuse the notation by saying that $c(\cdot) = c = const.$

According to (Henry, 1981, pp. 16-17) the eigenvalues of the operator $A := \partial_{xx} + c\ \mathrm{id} : X \to X$ on the domain $D(A) = H^2(0,L) \cap H_0^1(0,L)$ are given by

$$\lambda_j := -\frac{\pi^2}{L^2} j^2 + c, \quad j \in \mathbb{N}^*, \tag{25}$$

and the eigenfunctions e_j, $j \in \mathbb{N}^*$ of $(A, D(A))$, which form a basis of $L_2(0,1)$, are given by

$$e_j(x) := \left(\frac{2}{L}\right)^{1/2} \sin\frac{j\pi x}{L}, \quad j \in \mathbb{N}^*, \quad x \in (0,L). \tag{26}$$

The simulation results provided below, were obtained for the following parameter values:

$$c(x) \equiv 10, \quad L = 2, \quad \ell = 2, \quad b = e_1 + e_2,$$

where e_i are defined in (26). This choice results in the following values for the matrices $\mathbf{A}, \mathbf{B}$:

$$\mathbf{A} = \begin{pmatrix} 7.5325989 & 0. \\ 0. & 0.1303956 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The system (19) is stabilizable in view of Remark 2. Different choices of the matrix $\mathbf{K}$ for the stabilizing feedback $u(t) = \mathbf{K}z(t)$ lead to different attraction rates and different region of attractions. We demonstrate this by two examples.

We choose the matrix $\mathbf{K}$ so that $\sigma(\mathbf{A} + \mathbf{BK}) = \{-0.1, -0.2\}$. The resulting matrix $\mathbf{K}$ is:

$$\mathbf{K} = (-7.9732782\ \ 0.0102837). \tag{27}$$

Based on Proposition 2, we solve the inequalities (20), (21) together with additional constraints: $P = P^T$ and $P > 0$ (actually, we reformulate (20), (21) in a slightly different way which is more suitable for the case if $m = 1$, see the journal version of the paper Mironchenko et al. (2019)). Additionally, we impose an optimality condition

$$(\mathbf{K} - \mathbf{C}) \cdot (\mathbf{K} - \mathbf{C})^T \to \min, \tag{28}$$

where $\cdot$ is a scalar product of vectors.

The idea behind (28) is to minimize the non-diagonal elements of the matrix M_2, which helps to obtain good approximations for the attraction region (we refer again to the journal version for details).

This algorithm is implemented in Scilab. For the solution of the matrix inequalities the LMITOOL package has been used. The resulting matrices $P, \mathbf{C}, \mathbf{D}$ are:

$$P = \mathbf{D} \cdot \tilde{P}, \quad \tilde{P} = \begin{pmatrix} 0.3108695 & -0.0054849 \\ -0.0054849 & 0.000195 \end{pmatrix}, \tag{29a}$$

$$\mathbf{C} = (-0.3053879, 0.0054754), \quad \mathbf{D} = 90.625. \tag{29b}$$

In Figure 1 one can find an elliptic region of attraction (22), subject to $P, \mathbf{D}$ given by (29) (in blue). Furthermore, in the same figure some trajectories are depicted (in black), which asymptotically converge to the origin, as shown

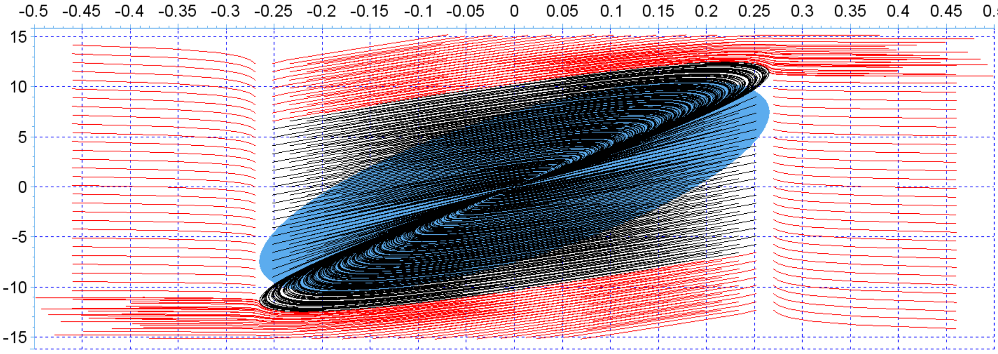

Fig. 1. Region of attraction (for the choice (27), (29)), computed via LMI technique (in blue), Trajectories of (18), attracted to the origin, computed by direct solution of ODEs (in black), diverging trajectories (in red).

by direct simulation by solving the ODE (18), as well as some diverging trajectories (in red). This provides an approximation of the maximal region of attraction of (18). One can see, that in one direction the ellipsoid obtained by our method approximates very well the actual region of attraction, but the results are not tight in the other direction.

Remark 3. (Computational costs) For this problem the elapsed time is (on a system with the specs: Intel(R) Core(TM) i5-3317U 1.70GHz, 16 GB RAM, Windows 10)
- Finding P, $\mathbf{C}$, $\mathbf{D}$ via LMIs: 0.018131 seconds
- Plotting the obtained region: 0.0129341 seconds
- Time for solving of the ODE (18) for $31^2 = 961$ distinct initial conditions on the time-interval $[0, 60]$ on a grid consisting of 600 points and for the plotting of the resulting trajectories: 91.802833 seconds.

This shows the computational efficiency of our method. ◦

5. BOUNDARY CONTROL OF HEAT EQUATION SUBJECT TO CONTROL SATURATIONS

Let us now start from a heat equation with a dynamical boundary condition

$$\begin{aligned} y_t = y_{xx} + c(x)y,\ t \geqslant 0,\ x \in (0, L),\\ y(t, 0) = 0,\ y(t, L) = y_d,\ t \geqslant 0, \end{aligned} \tag{30}$$

where y_d is the (scalar) output of the finite-dimensional dynamical system

$$\dot{x}_d = A_d x_d + B_d \mathrm{sat}(u(t)) \tag{31a}$$
$$y_d = C_d x_d. \tag{31b}$$

Here x_d in $\mathbb{R}^{n_d}$ is the finite-dimensional state whose dynamics is subject to saturating control, A_d, B_d and C_d are three matrices of appropriate dimension, and $u(t)$ is the scalar control for the PDE (30) and the ODE (31) that is subject to a saturation map. Then, inspired by Prieur and Trélat (2019), we introduce the following change of variable:

$$w(t, x) = y(t, x) - \frac{x}{L} y_d(t),\ t \geqslant 0,\ x \in (0, L).$$

The PDE for w reads as:

$$\begin{aligned} w_t(t,x) &= y_t(t,x) - \frac{x}{L}\dot{y}_d(t)\\ &= y_{xx}(t,x) + c(x)y(t,x) - \frac{x}{L}C_d\dot{x}_d(t)\\ &= w_{xx}(t,x) + c(x)\big(w(t,x) + \frac{x}{L}y_d(t)\big)\\ &\quad - \frac{x}{L}C_d\big(A_d x_d(t) + B_d \mathrm{sat}(u(t))\big)\\ &= w_{xx}(t,x) + c(x)\big(w(t,x) + \frac{x}{L}C_d x_d(t)\big)\\ &\quad - \frac{x}{L}C_d\big(A_d x_d(t) + B_d \mathrm{sat}(u(t))\big)\\ &= w_{xx}(t,x) + c(x)w(t,x) + \underbrace{\big(c(x)\frac{x}{L}C_d - \frac{x}{L}C_d A_d\big)}_{=:d(x)} x_d(t)\\ &\quad + \underbrace{\big(-\frac{x}{L}C_d B_d\big)}_{=:b(x)}\mathrm{sat}(u(t)). \end{aligned} \tag{32}$$

Please note that b is a scalar function, and d is a row vector function with $d(x) \in \mathbb{R}^{1\times n_d}$, $x \in [0, L]$.

The boundary conditions for the variable w take the form:

$$w(t, 0) = w(t, L) = 0,\ t \geqslant 0. \tag{33}$$

The heat equation (32), (33) has to be analyzed with the ODE (31a).

Now, performing similar computations as done in Section 2 for the PDE (1) and, using the same notation for w_j and λ_j, we get

$$\dot{w}_j(t) = \lambda_j w_j(t) + b_j \mathrm{sat}(u(t)) + d_j x_d(t),\ j = 1, 2, \ldots,$$

where b and d are defined for x in $[0, L]$ in (32) and $b_j = \langle b(\cdot), e_j(\cdot)\rangle_{L_2(0,L)}$, $d_j = \langle d(\cdot), e_j(\cdot)\rangle_{L_2(0,L)}$, for $j = 1, 2, \ldots$.

Let us consider the first n (unstable) equations together with the ODE (31) and rewrite this finite-dimensional system as follows:

$$\begin{aligned} z'(t) &= \mathbf{A}z(t) + \mathbf{B}\mathbf{K}z(t) + \mathbf{B}\phi(\mathbf{K}z(t))\\ &= \mathbf{A}z(t) + \mathbf{B}\mathrm{sat}(\mathbf{K}z(t)), \end{aligned} \tag{34}$$

where $\mathbf{K}$ in $\mathbb{R}^{1\times(n+n_d)}$ is a row vector to be designed,

$$z(t) := (x_d^T(t), \omega_1(t), \ldots, \omega_n(t))^T, \quad t \geqslant 0$$
$$\mathbf{B} := (B_d^T, b_1, \ldots, b_n)^T \in \mathbb{R}^{(n+n_d)\times 1},$$

the matrix $\mathbf{A}$ in $\mathbb{R}^{(n+n_d)\times(n+n_d)}$ is given by

$$\mathbf{A} := \begin{pmatrix} A_d & 0\\ D & \Lambda \end{pmatrix},\ D := \begin{pmatrix} d_{11} & d_{12} & \cdots & d_{1n_d}\\ \vdots & \vdots & \vdots & \vdots\\ d_{n1} & d_{n2} & \cdots & d_{nn_d} \end{pmatrix},\ \Lambda := \begin{pmatrix} \lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}.$$

Applying Proposition 2 to system (34) instead of system (18), we get sufficient conditions for $\mathbf{K}$ and for an estimation of attraction of (34). Now coming back to the infinite-dimensional systems (32) and (30), and, applying Proposition 1, we get sufficient conditions for $\mathbf{K}$ and for an estimation of attraction of (30), as done in the following:

Corollary 1. *Assume that there exist a symmetric positive definite matrix* $P \in \mathbb{R}^{(n+n_d)\times(n+n_d)}$*, a* $\mathbf{D} \in \mathbb{R}$ *and a matrix* $\mathbf{C} \in \mathbb{R}^{1\times(n+n_d)}$ *such that*

$$\begin{bmatrix} (\mathbf{A}+\mathbf{BK})^\top P + P(\mathbf{A}+\mathbf{BK}) & P\mathbf{B} - (\mathbf{DC})^\top \\ (P\mathbf{B})^\top - \mathbf{DC} & -2\mathbf{D} \end{bmatrix} < 0 \quad (35)$$

and $\begin{bmatrix} P & (\mathbf{K}-\mathbf{C})^\top \\ \mathbf{K}-\mathbf{C} & \ell^2 \end{bmatrix} \geqslant 0.$

Then, with the controller $u(t) = \mathbf{K}z(t)$*, the system (34) is locally asymptotically stable in* 0 *with a region of attraction given by* $\mathcal{A} := \{z,\ z^\top P z \leqslant 1\}$*. Moreover,*

(i) (32) *is locally exponentially stable with a region of attraction* $\imath(\mathcal{A}) \times X_n^\perp$*,*
(ii) (30) *is locally exponentially stable.*

6. CONCLUSION

A linear unstable reaction-diffusion equation has been introduced in this paper. Both boundary control and in-domain control cases have been considered. Local asymptotic stabilization problem by means of saturated control has been tackled. For this infinite-dimensional control problem, saturated feedback control laws have been designed. Moreover the region of attraction has been estimated by an appropriate Lyapunov function and LMI technique. The interest and the efficiency of our approach have been illustrated by means of numerical simulations, where it is shown that our approach yields a numerically tractable design method.

Our results can be extended in a variety of different directions. On the one hand, the anti-windup technique Zaccarian and Teel (2011) can be used to improve the performance of the closed-loop system. On the other hand, using the methods of infinite-dimensional input-to-state stability theory Mironchenko and Wirth (2018); Dashkovskiy and Mironchenko (2013); Prieur and Mazenc (2012); Tanwani et al. (2017), one can look for the analysis robustness of the proposed saturated controllers. Some of these extensions are discussed in details in the journal version of this paper Mironchenko et al. (2019).

REFERENCES

Boyd, S., El Ghaoui, L., Feron, E., and Balakrishnan, V. (1994). *Linear Matrix Inequalities in System and Control Theory.* Siam.

Curtain, R.F. and Zwart, H. (1995). *An Introduction to Infinite-Dimensional Linear Systems Theory.* Springer-Verlag, New York.

Dashkovskiy, S. and Mironchenko, A. (2013). Input-to-state stability of infinite-dimensional control systems. *Mathematics of Control, Signals, and Systems*, 25(1), 1–35.

Fiagbedzi, Y. and Pearson, A. (1986). Feedback stabilization of linear autonomous time lag systems. *IEEE Transactions on Automatic Control*, 31(9), 847–855.

Fridman, E. and Orlov, Y. (2009). Exponential stability of linear distributed parameter systems with time-varying delays. *Automatica*, 45(1), 194–201.

Henry, D. (1981). *Geometric Theory of Semilinear Parabolic Equations.* Springer-Verlag, Berlin.

Krstic, M. (2009). Control of an unstable reaction-diffusion PDE with long input delay. *Systems & Control Letters*, 58(10-11), 773–782.

Krstic, M. and Smyshlyaev, A. (2008). *Boundary Control of PDEs: A Course on Backstepping Designs.* SIAM, Philadelphia, PA, USA.

Lasiecka, I. and Seidman, T.I. (2003). Strong stability of elastic control systems with dissipative saturating feedback. *Systems & Control Letters*, 48(3), 243–252.

Marx, S., Andrieu, V., and Prieur, C. (2017a). Cone-bounded feedback laws for m-dissipative operators on Hilbert spaces. *Mathematics of Control, Signals, and Systems*, 29(4), 18.

Marx, S., Chitour, Y., and Prieur, C. (2017b). Stability results for infinite-dimensional linear control systems subject to saturations. In *European Control Conference (ECC'18).* Limassol, Cyprus.

Mironchenko, A., Prieur, C., and Wirth, F. (2019). Local stabilization of an unstable parabolic equation via saturated controls. *Available at arXiv.*

Mironchenko, A. and Wirth, F. (2018). Characterizations of input-to-state stability for infinite-dimensional systems. *IEEE Transactions on Automatic Control*, 63(6), 1602–1617.

Prieur, C. and Mazenc, F. (2012). ISS-Lyapunov functions for time-varying hyperbolic systems of balance laws. *Mathematics of Control, Signals, and Systems*, 24(1-2), 111–134.

Prieur, C., Tarbouriech, S., and Gomes da Silva Jr, J.M. (2016). Wave equation with cone-bounded control laws. *IEEE Transactions on Automatic Control*, 61(11), 3452–3463.

Prieur, C. and Trélat, E. (2019). Feedback stabilization of a 1D linear reaction-diffusion equation with delay boundary control. *IEEE Transactions on Automatic Control*, 64(4), 1415–1425.

Saberi, A., Stoorvogel, A.A., and Sannuti, P. (2012). *Control of Linear Systems with Regulation and Input Constraints.* Springer Science & Business Media.

Scherer, C. and Weiland, S. (2000). Linear matrix inequalities in control. *Lecture Notes, Dutch Institute for Systems and Control, Delft, The Netherlands*, 3, 2.

Slemrod, M. (1989). Feedback stabilization of a linear control system in Hilbert space with an a priori bounded control. *Mathematics of Control, Signals, and Systems*, 2(3), 265–285.

Tanwani, A., Prieur, C., and Tarbouriech, S. (2017). Disturbance-to-state stabilization and quantized control for linear hyperbolic systems. *arXiv preprint arXiv:1703.00302.*

Tarbouriech, S., Garcia, G., Gomes da Silva Jr, J.M., and Queinnec, I. (2011). *Stability and Stabilization of Linear Systems with Saturating Actuators.* Springer Science & Business Media.

Teel, A. (1992). Global stabilization and restricted tracking for multiple integrators with bounded controls. *Systems & Control Letters*, 18, 165–171.

Van Antwerp, J.G. and Braatz, R.D. (2000). A tutorial on linear and bilinear matrix inequalities. *Journal of Process Control*, 10(4), 363–385.

Zaccarian, L. and Teel, A. (2011). *Modern Anti-windup Synthesis: Control Augmentation for Actuator Saturation.* Princeton Series in Applied Mathematics. Princeton University Press.