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Stabilization of switched linear differential-algebraic equations via time-dependent switching signals

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Abstract— **We investigate stabilizability of switched systems of differential-algebraic equations (DAEs). For such systems we introduce a parameterized family of switched ordinary differential equations that approximate the dynamic behavior of the switched DAE. A criterion for stabilizability of a switched DAE system using time-dependent switching is obtained in terms of these parameterized approximations. The tightness of the proposed criterion is analyzed.**

Keywords: differential-algebraic systems, stabilization, switched systems.

I. INTRODUCTION

In this paper we investigate the stabilizability of a system of switched differential-algebraic equations (DAEs). The system consists of m constituent linear DAEs of the form:

$$E_i\dot{x} = A_i x \tag{1}$$

where $E_i, A_i \in \mathbb{R}^{n\times n}$ for $i \in \mathscr{I} := \{1,\dots,m\}$. The dynamics (E_i, A_i) of the switched linear DAE change instantaneously according to some piecewise continuous switching rule. In this note we investigate the existence of time-dependent switching signals $\sigma : \mathbb{R}_+ \to \mathscr{I}$ such that the resulting switched linear DAE is asymptotically stable.

Whenever the consistency spaces of the constituent systems do not coincide the solution of the switched linear DAE may exhibit jumps at the switching instances. The jumps between the consistency spaces introduce new phenomena to the dynamics of switched DAE systems and make it considerably more complicated to analyze. To formalize the concept of a switched DAE system, we exploit the distributional framework introduced in [15].

In contrast to the DAE case, for stabilizability of switched ODE systems (when all $E_i = I$) a number of results is available, see [18], [4], [9], [3], [8], [12], [10]. Thus, we may wish to use the results for stabilizability of ordinary differential equations to study stabilizability of DAEs. To follow this strategy, we approximate in Section III every constituent system of the switched system by an ODE, and then study the stabilizability of a switched ODE, constructed from such ODE-approximations. In this way we derive a criterion of stabilizability of differential-algebraic equations.

Approximation methods in DAE theory have a long history. In particular, within singular perturbation approach the behavior of ODEs for small parameters is studied with the help of DAE systems [2, p. 56]. Another method has been used in [17, Theorem 6.2], where it is shown that results on the existence of common Lyapunov functions may be transferred from the ODE to the DAE case by adding a differential part to what we call here the *flow matrix*, see (13) below. Note, however, that in [17] there is no dynamic interpretation of this procedure. Approximation of switched DAEs via averaging has been studied in a recent paper [5]. However, at the time this method has been elaborated only for switched DAEs with 2 constitutive systems and commutative consistency projectors. Another interesting approach to reduce the complexity of the stability analysis of switched DAE systems can be found in [11].

While most stabilization procedures for switched linear systems propose a feedback approach, in this paper we concentrate on the stabilization by time-dependent switching.

The paper is structured as follows. In Section II we introduce the main definitions and notation. In particular, we introduce the flow matrices and consistency projectors needed in the description of solutions of switched DAEs. In all our considerations we assume that Dirac impulses do not appear in the solutions we consider. In Section III we investigate stabilizability via time-dependent switching signals. We introduce a family of parameterized switched ordinary differential equations that is used to characterize stabilizability. In particular, a uniform stabilizability property of the parameterized family is equivalent to stabilizability of the switched DAE. A counterexample shows that the parameterized family may be stabilizable in a non-uniform way, from which it cannot be concluded that the DAE is stabilizable.

II. SOLUTION AND STABILITY OF DAES

In this section we briefly review the relevant results from the existence theory of switched DAEs and introduce the necessary concepts. We begin by recalling some results on time-invariant DAEs.

Consider a linear time-invariant DAE of the form

$$E\dot{x} = Ax, \tag{2}$$

where $E, A \in \mathbb{R}^{n\times n}$. By $C_{(E,A)}$ we denote the consistency

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space of (2) given by

$$C_{(E,A)} := \{x_0 \in \mathbb{R}^n : \exists x \in C^1([0,\infty),\mathbb{R}^n) \text{ with } x(0) = x_0 \text{ which solves (2)}\}.$$

In the following we assume that the matrix pencil $(sE - A)$ defining (2) is regular, i.e. $\det(sE - A)$ is a nonzero polynomial. This ensures that $C_{(E,A)} \neq \emptyset$ and $\forall x_0 \in C_{(E,A)}$ the solution of (2) is unique [6, Theorem 2.12].

Regular DAEs can be brought into so-called quasi-Weierstraß form [15], [1]:

Theorem 1 (Quasi-Weierstraß form): If (2) is regular, then there exist invertible matrices $S, T \in \mathbb{R}^{n\times n}$ such that

$$SET = \begin{pmatrix} I & 0 \\ 0 & N \end{pmatrix}, \quad SAT = \begin{pmatrix} J & 0 \\ 0 & I \end{pmatrix},$$

where N, J are square matrices, N is nilpotent and the partitioning of both matrices is the same.

Note that the quasi-Weierstraß form is not the Weierstraß canonical form since we do not assume that the matrices N, J are in a Jordan canonical form.

The quasi-Weierstraß form provides a decoupling of (2) into a purely differential and a purely algebraic part. Moreover, the matrices S and T are real and can be efficiently computed via Wong sequences [15], [19].

Using T from Theorem 1 we introduce the so-called flow matrix A^d

$$A^d := T \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix} T^{-1}. \tag{3}$$

The importance of A^d lies in the following result [7, Lemma 3], [14]:

Lemma 1: Any solution of (2) with $x(0) = x_0 \in C_{(E,A)}$, is also a solution of

$$\dot{x} = A^d x, \quad x(0) = x_0. \tag{4}$$

A further important object is the consistency projector $\Pi_{(E,A)}$, defined by

$$\Pi_{(E,A)} := T \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} T^{-1}. \tag{5}$$

We note that $\Pi_{(E,A)}$ is a reducing projection for A^d and we have

$$A^d \Pi_{(E,A)} = \Pi_{(E,A)} A^d = A^d. \tag{6}$$

It can be shown, [15], that the consistency projector does not depend on the matrices T and S, which bring (2) into quasi-Weierstraß form.

Switched DAEs

For our analysis of switched DAEs we restrict ourselves to piecewise constant switching signals $\sigma : [0,\infty) \to \mathscr{I}$. Note that the notion of piecewise continuity includes in particular that on each compact set $K \subset [0,\infty)$ the switching signals have only finitely many discontinuities. This will be assumed without further mention in the following.

Consider a switched linear DAE given by the constituent systems (1) and a time-dependent, piecewise constant switching signal σ:

$$E_{\sigma(t)}\dot{x} = A_{\sigma(t)}x. \tag{7}$$

The consistency spaces $C_i := C_{(E_i,A_i)}$ of the constituent systems may not coincide. Thus it is natural to assume that switching in (7) may lead to a jump of the state at the switching instances. One of the possibilities to define a concept of the solution of (7) is to treat a switched DAE as a special case of a time-varying homogeneous DAE:

$$E(t)\dot{x} = A(t)x, \tag{8}$$

where E and A are piecewise constant matrix functions and to consider distributional solutions of (8) in sense of [15]. In this case, solutions may include derivatives of discontinuities, i.e. Dirac impulses. In our analysis jumps are permitted, but impulses are an impediment to stability and thus excluded.

Let us denote the Euclidean norm by $|\cdot|$, and the matrix norm by $\|\cdot\|$.

Definition 1: [15, Def. 4.3.7] The zero solution of (8) is called *globally asymptotically stable (GAS)* if for all initial conditions $x(0) = x_0$ the corresponding solution of (8), does not contain Dirac impulses and it is

1) Stable: $\forall \varepsilon > 0\ \exists \delta > 0$: $\forall x_0$: $|x_0| < \delta$ it follows that $|x(t)| < \varepsilon$.
2) Attractive: $\lim_{t\to\infty} |x(t)| = 0$ for all $x_0 \in \mathbb{R}^n$.

Note that for linear systems (8) the attractivity of the zero solution implies its stability.

It is convenient to have an equivalent definition of GAS in terms of comparison functions.

We write that $f \in \mathscr{L}$, if $f : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and strictly decreasing to 0.

Lemma 2: The zero solution of (8) is GAS if and only if there exists $\delta \in \mathscr{L}$ so that for all admissible initial conditions x_0 and for all $t \geq 0$ it holds, that

$$|x(t)| \leq \delta(t)|x_0|. \tag{9}$$

Proof: $\Rightarrow$: For all admissible x_0 the solution of (8) with an initial condition $x(0) = x_0$ can be written in the form $x(t) = \Phi(t)x_0$ for some $\Phi : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$. Thus,

$$|x(t)| \leq \|\Phi(t)\||x_0|$$

Since (8) is attractive, $\|\Phi(t)\| \to 0$, $t \to \infty$. Otherwise there would exist a sequence of times $\{t_i\} \to \infty$, $i \to \infty$ and sequence of admissible states $\{x_0^i\}$, $|x_0^i| = 1$, so that $|\Phi(t_i)x_0^i| \geq C$. Since the set $\{x \in \mathbb{R}^n : |x| = 1\}$ is compact, there exists a convergent sequence $\{x_0^{i_k}\} \to x^*$. Then $|\Phi(t)x^*| \geq C$ for all t. Indeed, let $\exists t^*$: $|\Phi(t^*)x^*| < C$, then due to construction of x^* for all $\varepsilon < 0$ there exist r_ε: $|x^* - x_{r_\varepsilon}| < \varepsilon$ and at the same time $|\Phi(t^*)x_{r_\varepsilon}| \geq C$, which contradicts to the continuous dependence of (8) on initial data.

Thus, $|\Phi(t)x^*| \geq C$ for all t, which implied, that (8) is non-attractive and we come to a contradiction. Define $\delta(t) := \inf_{\tau\geq t} \|\Phi(\tau)\| + e^{-t}$. Clearly, $\delta \in \mathscr{L}$ and we obtain (9).

"$\Leftarrow$" This is clear. ■

Our objective in this paper is to find conditions ensuring the existence of a time-dependent switching signal, such that the switched DAE with constituent systems (1) is asymptotically stable. More precisely, we define the notion of stabilizability of the switched DAE via time-dependent switching as follows.

Definition 2: A switched system given by (1) is called *stabilizable via time-dependent switching* if there exists a piecewise continuous switching signal $\sigma : \mathbb{R}_+ \to \mathscr{I}$ such that the equilibrium of (7) is globally asymptotically stable.

A characteristic of switched DAEs is that the solution may exhibit Dirac impulses. However, such Dirac impulses have to be ruled out to render the zero solution asymptotically stable according to Definition 1. The following result provides a necessary and sufficient condition for the absence of Dirac impulses [16, Section 3.3].

Theorem 2: All distributional solutions of (7) are impulse-free and the solution of each constituent system (1) with $x(0) = x_0 \in \mathbb{R}^n$ is given by

$$x(t) = e^{A_i^d t}\Pi_i x_0, \quad \forall t \in \mathbb{R}_+.$$

if and only if the following condition

$$E_i(1-\Pi_i)\Pi_j = 0 \quad \forall i, j = 1, \ldots, m \tag{10}$$

holds.

Remark 3: Note that for index one systems the condition (10) is satisfied.

In what follows we assume that condition (10) holds. Certainly, if we do not switch between all systems, a milder condition can be required.

The above result can be used to construct the solution of (7) corresponding to the switching signal σ. Let the increasing sequence $\{t_i\}$ denote the points of discontinuity of σ such that $\sigma(t) = s_i \in \mathscr{I}$ for $t \in [t_i, t_{i+1})$. Then the solution of (7) with $x(t_0) = x_0$ can be written as

$$x(t) = e^{A_{s_i}^d (t-t_i)}\Pi_{s_i} \cdot e^{A_{i-1}^d (t_i - t_{i-1})}\Pi_{s_{i-1}} \cdots e^{A_{s_1}^d (t_1 - t_0)}\Pi_{s_1} x_0, \tag{11}$$

where $t \in [t_i, t_{i+1})$.

III. STABILIZATION VIA TIME-DEPENDENT SWITCHING

We shall now turn towards the main subject of this paper. Our approach in this section is to relate the stabilizability of a switched DAE to properties of solutions of associated switched ODE approximations.

For each of the constituent systems (1) we define the ODE approximation

$$\dot{x} = A_i^\varepsilon x \tag{12}$$

with the system matrix

$$A_i^\varepsilon := A_i^d \Pi_i - \frac{1}{\varepsilon}(I - \Pi_i) = T_i \begin{pmatrix} J_i & 0 \\ 0 & -\frac{1}{\varepsilon} I \end{pmatrix} T_i^{-1}. \tag{13}$$

For the set of system matrices A_i^ε we define the switched linear ODE:

$$\dot{x} = A_{\sigma(t)}^\varepsilon x \tag{14}$$

The idea behind the additional term $-\frac{1}{\varepsilon} I$ in (13) is to model a jump between the consistency spaces by continuous but fast dynamics in the direction of the jump. The smaller ε is chosen, the faster is the corresponding dynamics, resulting in better approximations of the jump. More precisely, we have the following relation between the solutions of the constituent systems (1) and (12):

Lemma 4: For each constituent DAE (1) and the associated ODE (12) it holds that

$$(e^{A_i^d t}\Pi_i - e^{A_i^\varepsilon t})x_0 = -e^{-\frac{t}{\varepsilon}}(I - \Pi_i)x_0 \tag{15}$$

for all $t > 0$ and $x_0 \in \mathbb{R}^n$. In particular, for all $t > 0$

$$\lim_{\varepsilon \to 0} \| e^{A_i^d t}\Pi_i - e^{A_i^\varepsilon t} \| = 0. \tag{16}$$

Proof: Computing $e^{A_i^d t}\Pi_i$ we obtain

$$e^{A_i^d t}\Pi_i = T_i e^{\begin{pmatrix} J_i & 0 \\ 0 & 0 \end{pmatrix} t} T_i^{-1} T_i \begin{pmatrix} I_i & 0 \\ 0 & 0 \end{pmatrix} T_i^{-1}$$

$$= T_i \begin{pmatrix} e^{J_i t} & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I_i & 0 \\ 0 & 0 \end{pmatrix} T_i^{-1} = T_i \begin{pmatrix} e^{J_i t} & 0 \\ 0 & 0 \end{pmatrix} T_i^{-1}.$$

The expression (15) can now be obtained as follows

$$e^{A_i^d t}\Pi_i - e^{A_i^\varepsilon t} = T_i \begin{pmatrix} 0 & 0 \\ 0 & -e^{-\frac{1}{\varepsilon}t} I \end{pmatrix} T_i^{-1} = -e^{-\frac{1}{\varepsilon}t}(I - \Pi_i).$$

Clearly, for every fixed t the last expression tends to zero as $\varepsilon \to 0$. ∎

Having established the approximation for the constituent systems we can now relate the impulse-free solutions of the switched DAE (7) to the solutions of the corresponding switched ODE (14). Let $\Phi_\sigma^{DAE}(t)$ denote the solution of (7) and let $\Phi_\sigma^\varepsilon(t)$ denote the solution of (14) for the same switching signal σ.

Lemma 5: For all admissible σ and for all $t > 0$ it holds

$$\lim_{\varepsilon \to 0} \left\| \Phi_\sigma^{DAE}(t) - \Phi_\sigma^\varepsilon(t) \right\| = 0.$$

Proof: For any admissible switching signal σ and any fixed $t > 0$ there exists a k such that $t_{k+1} > t \geq t_k$. Let $\tau_{k-1} = t_k - t_{k-1}$ and $\tau_k' = t - t_k$ such that $t = \tau_k' + \tau_{k-1} + \tau_{k-2} + \ldots + \tau_0$. Then we have

$$\Phi_\sigma^{DAE}(t) = e^{A_{s_k}^d \tau_k'}\Pi_{s_k} \cdot e^{A_{s_{k-1}}^d \tau_{k-1}}\Pi_{s_{k-1}} \cdots e^{A_{s_0}^d \tau_0}\Pi_{s_0}, \tag{17}$$

$$\Phi_\sigma^\varepsilon(t) = e^{A_{s_k}^\varepsilon \tau_k'} \cdot e^{A_{s_{k-1}}^\varepsilon \tau_{k-1}} \cdots e^{A_{s_0}^\varepsilon \tau_0}. \tag{18}$$

According to equality (15) we have

$$e^{A_i^d \tau}\Pi_i = e^{A_i^d \tau}\Pi_i - e^{A_i^\varepsilon \tau} + e^{A_i^\varepsilon \tau} = -e^{-\frac{\tau}{\varepsilon}}(I - \Pi_i) + e^{A_i^\varepsilon \tau}.$$

Substituting this expression into (17) we obtain that $\|\Phi_\sigma^{DAE}(t) - \Phi_\sigma^\varepsilon(t)\|$ can be represented as a finite sum of terms, any of which has a multiplier of the form $e^{-\frac{p}{\varepsilon}}$ for some $p > 0$. Taking the limit for $\varepsilon \to 0$, we obtain the claim of the lemma. ∎

Now we are going to state one of the main results which shows that the stabilizability of the switched DAE can be verified by analyzing the corresponding switched ODE (14).

In the following we denote the solutions of (7) and (14), corresponding to an initial condition x_0 and a switching signal σ at time t by $\phi(t, x_0, \sigma)$ and $\phi_\varepsilon(t, x_0, \sigma)$, respectively.

Theorem 3: Assume there exist $\varepsilon_0 > 0$ and $s > 0$ so that for all $0 < \varepsilon < \varepsilon_0$ there exists a periodic switching signal σ_ε with period s such that

$$|\phi_\varepsilon(s, x_0, \sigma_\varepsilon)| \le \frac{1}{2}|x_0|, \quad \forall x_0 \in \mathbb{R}^n.$$

Moreover, assume there exists $t_d > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, for any two subsequent switches of σ_ε it holds $|t_i^\varepsilon - t_{i+1}^\varepsilon| \ge t_d$. Then (7) is stabilizable via σ_{ε^*} for some $\varepsilon^* \in (0, \varepsilon_0)$.

Proof: Let the assumptions of the theorem hold. Using (15) we have for all $x, y \in \mathbb{R}^n$ the following estimate

$$\begin{aligned} |e^{A_i^d t}\Pi_i x - e^{A_i^\varepsilon t} y| &= |(e^{A_i^d t}\Pi_i - e^{A_i^\varepsilon t})x + e^{A_i^\varepsilon t}(x-y)| \\ &= |-e^{-\frac{t}{\varepsilon}}(I - \Pi_i)x + e^{A_i^\varepsilon t}(x-y)| \\ &\le e^{-\frac{t}{\varepsilon}}\|I - \Pi_i\||x| + \|e^{A_i^\varepsilon t}\||x-y|. \end{aligned} \quad (19)$$

Pick any $\varepsilon \in (0, \varepsilon_0)$.

Due to the semigroup property it holds for any $x_0 \in \mathbb{R}^n$

$$\phi(t_{i+1}^\varepsilon, x_0, \sigma_\varepsilon) = \phi(t_{i+1}^\varepsilon - t_i^\varepsilon, \phi(t_i^\varepsilon, x_0, \sigma_\varepsilon), \sigma_\varepsilon(\cdot + t_i^\varepsilon))$$

and

$$\phi_\varepsilon(t_{i+1}^\varepsilon, x_0, \sigma_\varepsilon) = \phi_\varepsilon(t_{i+1}^\varepsilon - t_i^\varepsilon, \phi_\varepsilon(t_i^\varepsilon, x_0, \sigma_\varepsilon), \sigma_\varepsilon(\cdot + t_i^\varepsilon)).$$

Pick any $T > 0$ and denote by $\{t_i^\varepsilon\}$ the sequence of switches of σ_ε, and $\sigma_\varepsilon(t_i^\varepsilon) = s_i = s_i(\varepsilon)$. We are going to find an estimate of

$$|\phi(t_{i+1}^\varepsilon, x_0, \sigma_\varepsilon) - \phi_\varepsilon(t_{i+1}^\varepsilon, x_0, \sigma_\varepsilon)|$$

on the time span $[0, T]$.

Using (19) we obtain:

$$\begin{aligned} &|\phi(t_{i+1}^\varepsilon, x_0, \sigma_\varepsilon) - \phi_\varepsilon(t_{i+1}^\varepsilon, x_0, \sigma_\varepsilon)| \\ &= |e^{A_{s_i}^d (t_{i+1}^\varepsilon - t_i^\varepsilon)}\Pi_{s_i}\phi(t_i^\varepsilon, x_0, \sigma_\varepsilon) - e^{A_{s_i}^\varepsilon (t_{i+1}^\varepsilon - t_i^\varepsilon)}\phi_\varepsilon(t_i^\varepsilon, x_0, \sigma_\varepsilon)| \\ &\le e^{-\frac{t_{i+1}^\varepsilon - t_i^\varepsilon}{\varepsilon}}\|I - \Pi_{s_i}\||\phi(t_i^\varepsilon, x_0, \sigma_\varepsilon)| \\ &\qquad + \|e^{A_{s_i}^\varepsilon (t_{i+1}^\varepsilon - t_i^\varepsilon)}\||\phi(t_i^\varepsilon, x_0, \sigma_\varepsilon) - \phi_\varepsilon(t_i^\varepsilon, x_0, \sigma_\varepsilon)| \\ &\le L e^{-\frac{t_d}{\varepsilon}}|x_0| + M|\phi(t_i^\varepsilon, x_0, \sigma_\varepsilon) - \phi_\varepsilon(t_i^\varepsilon, x_0, \sigma_\varepsilon)|, \end{aligned}$$

where the constants L and M do depend neither on ε nor on the sequence $\{t_i\}$ and are chosen so that

$$\begin{aligned} &\|I - \Pi_{s_i}\||\phi(t_i^\varepsilon, x_0, \sigma_\varepsilon)| \\ &\quad \le \sup_{k=1}^{m}\|I - \Pi_k\| \cdot \sup_{\tau \in [0,T], \varepsilon \in (0,\varepsilon_0)} \|\phi(\tau, \cdot, \sigma_\varepsilon)\||x_0| := L|x_0| \end{aligned}$$

and

$$\begin{aligned} \sup_{t \in [t_d, T]} \|e^{A_{s_i}^\varepsilon t}\| &\le \|T_{s_i}\|\|T_{s_i}^{-1}\| \sup_{t \in [t_d, T]} \left\| \begin{pmatrix} e^{J_{s_i} t} & 0 \\ 0 & e^{-\frac{1}{\varepsilon}t} I \end{pmatrix} \right\| \\ &\le \|T_{s_i}\|\|T_{s_i}^{-1}\| \sup_{t \in [t_d, T]} (\|e^{J_{s_i} t}\| + \|e^{-\frac{1}{\varepsilon}t} I\|) \\ &\le \sup_{k=1}^{m} (\|T_k\|\|T_k^{-1}\| \sup_{t \in [t_d, T]} (\|e^{J_k t}\| + 1)) =: M. \end{aligned}$$

Since $|\phi(t_0^\varepsilon, x_0, \sigma_\varepsilon) - \phi_\varepsilon(t_0^\varepsilon, x_0, \sigma_\varepsilon)| = 0$, we continue with the above estimates to obtain for all $x_0 \in \mathbb{R}^n$

$$|\phi(t_{i+1}^\varepsilon, x_0, \sigma_\varepsilon) - \phi_\varepsilon(t_{i+1}^\varepsilon, x_0, \sigma_\varepsilon)| \le L \sum_{j=0}^{i} M^j e^{-\frac{t_d}{\varepsilon}}|x_0|. \quad (20)$$

According to the assumptions of the theorem there exists $s > 0$ such that $|\phi_\varepsilon(s, x_0, \sigma_\varepsilon)| < \frac{1}{2}|x_0|$ for all $\varepsilon \in (0, \varepsilon_0)$.

Using (20) one can find $\varepsilon^* < \varepsilon_0$, so that

$$|\phi(s, x_0, \sigma_{\varepsilon^*}) - \phi_{\varepsilon^*}(s, x_0, \sigma_{\varepsilon^*})| < \frac{1}{4}|x_0|.$$

And thus via the triangle inequality we have:

$$|\phi(s, x_0, \sigma_{\varepsilon^*})| \le \frac{3}{4}|x_0|, \quad \forall x_0 \in \mathbb{R}^n.$$

This implies, that (7) is stabilizable via σ_{ε^*}. ∎

The above result establishes that the stabilizability of the switched DAE can be shown by considering the stabilizability of corresponding switched ODEs.

Remark 6: Note that the result requires the existence of a certain “common minimum dwell-time” t_d for all approximating ODEs with small enough ε.

Remark 7: For the stabilization of the switched DAE we can use the same switching signal σ_{ε^*} that stabilizes one of the approximating switched ODEs.

The above theorem helps to prove a less precise but more handy and in particular necessary and sufficient criterion for the stabilizability of switched DAEs.

Theorem 4: The following statements are equivalent:

1) The switched linear DAE (7) is stabilizable via a time-dependent switching signal.
2) The switched linear DAE (7) is stabilizable via the periodic switching signal σ_p.
3) There exists an $\varepsilon_0 > 0$ so that the switched ODE (14) is stabilizable via the (same) periodic switching signal σ_p for all $0 < \varepsilon < \varepsilon_0$, uniformly w.r.t. ε: $\exists s > 0$, s.t. for all $\varepsilon \in (0, \varepsilon_0)$

$$|\phi_\varepsilon(s, x_0, \sigma_p)| \le \frac{1}{2}|x_0|, \forall x_0 \in \mathbb{R}^n.$$

Proof: 2⇒1 is trivial.

1⇒2: Let (7) be stabilizable via time-dependent switching. According to Lemma 2 there exist a switching signal σ and $\zeta \in \mathscr{L}$ so that

$$|\Phi_\sigma^{DAE}(t)x_0| \le \zeta(t)|x_0| \quad (21)$$

for all initial conditions x_0 and for all $t \ge 0$.

Since $\zeta \in \mathscr{L}$, there exists a time $T > 0$, so that $\zeta(T) = \frac{1}{2}$ and we have for all $x_0 \in \mathbb{R}^n$

$$\left|\Phi_\sigma^{DAE}(T)x_0\right| \le \frac{1}{2}|x_0|. \quad (22)$$

In particular, $\|\Phi_\sigma^{DAE}(T)\| \le \frac{1}{2}$. Now consider the periodic signal σ_p defined by the formula

$$\sigma_p(t) = \sigma(t \mod T).$$

For this control we have

$$\left|\Phi_{\sigma_p}^{DAE}(kT)x_0\right| \le \frac{1}{2^k}|x_0|, \quad (23)$$

and thus σ_p stabilizes the system (7).

2⇒3: Let σ_p be periodic with the period T and stabilize the system (7). Then there exists a $k > 0$, so that

$$\left\|\Phi_{\sigma_p}^{DAE}(kT)\right\| \leq \frac{1}{2}.$$

The triangle inequality implies

$$\left\|\Phi_{\sigma_p}^{\varepsilon}(kT)\right\| \leq \left\|\Phi_{\sigma_p}^{\varepsilon}(kT) - \Phi_{\sigma_p}^{DAE}(kT)\right\| + \left\|\Phi_{\sigma_p}^{DAE}(kT)\right\|. \quad (24)$$

By Lemma 5 there exists an $\varepsilon_0 > 0$ so that for all $\varepsilon \leq \varepsilon_0$ it holds

$$\left\|\Phi_{\sigma_p}^{\varepsilon}(kT) - \Phi_{\sigma_p}^{DAE}(kT)\right\| \leq \frac{1}{4}.$$

From (24) we obtain that $\left\|\Phi_{\sigma_p}^{\varepsilon}(kT)\right\| \leq \frac{3}{4}$ for all $\varepsilon \leq \varepsilon_0$ and consequently the following estimate holds

$$\left|\Phi_{\sigma_p}^{\varepsilon}(wkT)x_0\right| \leq \left(\frac{3}{4}\right)^{w} |x_0|. \quad (25)$$

Thus σ_p stabilizes (14) for all $\varepsilon \leq \varepsilon_0$ uniformly w.r.t. ε.

3⇒2: follows from Theorem 3. ■

Note, that we require that the switched ODEs considered are stabilizable via the same periodic switching signal σ_p for all $\varepsilon \in (0, \varepsilon_0)$. In the following subsections we shed some light on the nature of this condition by first considering an example that illustrates the necessity of this condition. In the subsequent subsection we consider a class of systems for which this condition holds naturally.

A. Counterexample

We are going to construct a switched DAE whose trajectories (of most initial conditions) diverge for all admissible switching signals. The corresponding approximations (14) are uniformly periodically stabilizable for all $\varepsilon \in (0, \varepsilon_0)$ and small enough ε_0, but via different stabilizing signals.

Consider the system $\Sigma_1 = (E_1, A_1)$ with

$$E_1 := \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad A_1 := \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

and the system $\Sigma_2 = (E_2, A_2)$ with

$$E_2 := \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad A_2 := \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

Their flow matrices and consistency projectors are as follows:

$$\Pi_1 = A_1^d = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}, \quad \Pi_2 = A_2^d = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

Figure 1 shows a typical trajectory of the switched DAE given by Σ_1 and Σ_2 for the initial value $x_0 = (1,0)^T$. It can be easily shown, that the solution of the switched DAE is divergent for any initial state, which does not lie in $\ker \Pi_1 \cup \ker \Pi_2$ and for any switching signal.

Now consider the corresponding approximations of the form (14) with

$$A_1^{\varepsilon} = A_1^d \Pi_1 - \frac{1}{\varepsilon}(I - \Pi_1) = \begin{pmatrix} -\frac{1}{\varepsilon} & 0 \\ -1 - \frac{1}{\varepsilon} & 1 \end{pmatrix}$$

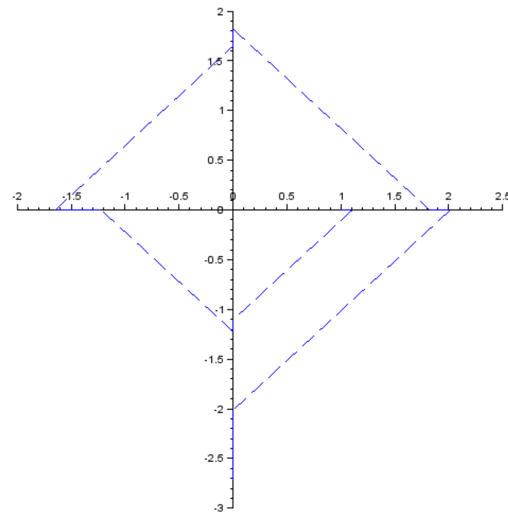

Fig. 1. Typical trajectory of a switched DAE from Section III-A

and

$$A_2^{\varepsilon} = A_2^d \Pi_2 - \frac{1}{\varepsilon}(I - \Pi_2) = \begin{pmatrix} 1 & 1 + \frac{1}{\varepsilon} \\ 0 & -\frac{1}{\varepsilon} \end{pmatrix}.$$

Consider the periodic signal σ_ε of period 2ε, defined by

$$\sigma_\varepsilon(t) = \begin{cases} 1, & t \in [0, \varepsilon), \\ 2, & t \in [\varepsilon, 2\varepsilon). \end{cases}$$

Denote

$$M_\varepsilon := e^{A_2^\varepsilon \varepsilon} e^{A_1^\varepsilon \varepsilon} = \begin{pmatrix} e^{\varepsilon - 1} - (e^\varepsilon - e^{-1})^2 & e^\varepsilon (e^\varepsilon - e^{-1}) \\ -e^{-1}(e^\varepsilon - e^{-1}) & e^{\varepsilon - 1} \end{pmatrix}.$$

The state of the system (14) at time $2\varepsilon k$, corresponding to σ_ε is given by

$$\phi_\varepsilon(2\varepsilon k, x_0, \sigma_\varepsilon) = M_\varepsilon^k x_0.$$

Define $y_\varepsilon(k) := \phi_\varepsilon(k 2\varepsilon, x_0, \sigma_\varepsilon)$. We obtain a discrete-time system

$$y_\varepsilon(k) = M_\varepsilon y_\varepsilon(k-1). \quad (26)$$

It is easy to compute, that

$$M_0 \approx \begin{pmatrix} -0.0316970 & 0.6321206 \\ -0.2325442 & 0.3678794 \end{pmatrix}$$

and

$$Spec(M_0) \approx \{0.1680912 \pm 0.3272317i\}.$$

This means, that $\rho(M_0) < 1$. Since the eigenvalues of a matrix depend continuously on its entries (see, e.g. [13, Section A.4, p. 456]), we obtain, that $\rho(M_\varepsilon) < 1$ for $\varepsilon \in (0, \varepsilon_0)$ for small enough ε_0.

This immediately implies, that (26) is stable for small enough ε and thus σ_ε, $\varepsilon \in (0, \varepsilon_0)$ stabilizes (14) uniformly w.r.t. ε.

Note that the period of σ_ε as well as the time between switches tend to 0 as $\varepsilon \to 0$, thus, the conditions of Theorem 3 are violated.

B. Commutative constituent systems

In this subsection we consider the special case of switched DAEs with commuting flow matrices A_i^d. As will be shown below, the requirement that all approximating switched ODEs are stabilizable via the same periodic switching signal is already implied by periodic stabilization of an approximating ODE for certain ε_0.

Lemma 8: Let A_i^d, $i = 1,\dots,m$ commute pairwise. If (14) is stabilizable via the periodic switching signal σ_p for a certain ε_0, then it is stabilizable via the same switching signal σ_p, uniformly w.r.t. $\varepsilon \in (0, \varepsilon_0)$.

Proof: Let A_i^d, $i = 1,\dots,m$ commute. Then from (13) it follows that also A_i^ε, $i = 1,\dots,m$ commute for all ε.

Now let (14) be stabilizable via the periodic switching signal σ_p.

Define $I_j(t) = \mu(\{s \le t : \sigma_p(s) = j\})$, where μ is the Lebesgue measure.

For $t = \sum_{i=1}^k \tau_i$ and all $x_0 \in \mathbb{R}^n$ we have

$$\begin{aligned}
\phi_\varepsilon(t, x_0, \sigma_p) &= e^{A^\varepsilon_{s_k}\tau_k} \cdots e^{A^\varepsilon_{s_1}\tau_1} x_0 \\
&= e^{A^\varepsilon_{s_k}\tau_k} \cdots e^{A^\varepsilon_{s_1}\tau_1} e^{-A^{\varepsilon_0}_{s_1}\tau_1} \cdots e^{-A^{\varepsilon_0}_{s_k}\tau_k} e^{A^{\varepsilon_0}_{s_k}\tau_k} \cdots e^{A^{\varepsilon_0}_{s_1}\tau_1} x_0 \\
&= e^{\sum_{i=1}^k (A^\varepsilon_{s_i} - A^{\varepsilon_0}_{s_i})\tau_i} e^{A^{\varepsilon_0}_{s_k}\tau_k} \cdots e^{A^{\varepsilon_0}_{s_1}\tau_1} x_0 \\
&= e^{\sum_{j=1}^m (A^\varepsilon_j - A^{\varepsilon_0}_j) I_j(t)} e^{A^{\varepsilon_0}_{s_k}\tau_k} \cdots e^{A^{\varepsilon_0}_{s_1}\tau_1} x_0 \\
&= \prod_{j=1}^m e^{(A^\varepsilon_j - A^{\varepsilon_0}_j) I_j(t)} \phi_{\varepsilon_0}(t, x_0, \sigma_p). \qquad (27)
\end{aligned}$$

From (13) we see that

$$A^\varepsilon_j - A^{\varepsilon_0}_j = \left(\frac{1}{\varepsilon_0} - \frac{1}{\varepsilon}\right)(I - \Pi_j).$$

Due to this inequality and since $\frac{1}{\varepsilon_0} - \frac{1}{\varepsilon} < 0$, we obtain:

$$\begin{aligned}
\|\prod_{j=1}^m e^{(A^\varepsilon_j - A^{\varepsilon_0}_j) I_j(t)}\| &\le \prod_{j=1}^m \|e^{(\frac{1}{\varepsilon_0} - \frac{1}{\varepsilon})(I-\Pi_j) I_j(t)}\| \\
&= \prod_{j=1}^m \|T_j e^{\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}(\frac{1}{\varepsilon_0} - \frac{1}{\varepsilon}) I_j(t)} T_j^{-1}\| \\
&= \prod_{j=1}^m \|T_j \begin{pmatrix} I & 0 \\ 0 & e^{(\frac{1}{\varepsilon_0} - \frac{1}{\varepsilon}) I_j(t)} \end{pmatrix} T_j^{-1}\| \\
&= \prod_{j=1}^m \|\Pi_j + e^{(\frac{1}{\varepsilon_0} - \frac{1}{\varepsilon}) I_j(t)} (I - \Pi_j)\| \\
&\le \prod_{j=1}^m (\|\Pi_j\| + \|I - \Pi_j\|) =: K,
\end{aligned}$$

with constant K.

Now from (27) we obtain

$$|\phi_\varepsilon(t, x_0, \sigma_p)| \le K |\phi_{\varepsilon_0}(t, x_0, \sigma_p)|, \quad \forall x_0 \in \mathbb{R}^n.$$

Take s s.t. $|\phi_{\varepsilon_0}(s, x_0, \sigma_p)| \le \frac{1}{2K}|x_0|$. We obtain

$$|\phi_\varepsilon(s, x_0, \sigma_p)| \le \frac{1}{2}|x_0|, \quad \forall x_0 \in \mathbb{R}^n,$$

which immediately implies that σ stabilizes (14) for all $\varepsilon \in (0, \varepsilon_0)$, uniformly w.r.t. ε. ■

IV. Conclusions

In this paper we investigated the stabilizability of switched DAE systems. We show that the stabilizability of the switched DAE can be inferred from stabilizability properties of an approximating switched ODE. Our two main results show that the approximating switched ODE has to be stabilizable in a uniform sense. A counterexample establishes that these additional requirements cannot be dropped. For a certain class of systems our conditions are not restrictive.

The relations between the stabilizability of switched DAEs and that of switched ODEs allow to utilize conditions for stabilization of switched ODEs, where a number of results has been obtained. Of particular interest in this regard may be the relation to results on quadratic stabilization obtained for switched ODEs.

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