Stabilization of Switched Linear Differential Algebraic Equations and Periodic Switching

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***Abstract*—We investigate the stabilizability of switched linear systems of differential-algebraic equations (DAEs). For such systems we introduce a parameterized family of switched ordinary differential equations that approximate the dynamic behavior of the switched DAE. A necessary and sufficient criterion for the stabilizability of a switched DAE system using time-dependent switching is obtained in terms of these parameterized approximations. Furthermore, we provide conditions for the stabilizability of switched DAEs via fast switching as well as using solely the consistency projectors of the constituent systems. The stabilization of switched DAEs with commuting vector fields is also analyzed.**

***Index Terms*—Differential-algebraic equations, stabilization, switched systems.**

I. Introduction

IN THIS paper, we investigate the stabilizability of a system of switched linear differential-algebraic equations (DAEs) of the form

$$E_{\sigma(t)}\dot{x} = A_{\sigma(t)}x. \tag{1}$$

The system is described by m constituent linear DAEs

$$E_i\dot{x} = A_i x \tag{2}$$

where $E_i, A_i \in \mathbb{R}^{n\times n}$ for $i \in \mathcal{I} := \{1,\dots,m\}$ and $x \in \mathbb{R}^n$. The dynamics (E_i, A_i) of the switched linear DAE change instantaneously according to the piecewise continuous switching rule σ. In this note we investigate the existence of time-dependent switching signals $\sigma : [t_0,\infty) \to \mathcal{I}$ such that the resulting time-varying linear DAE is asymptotically stable.

If E_i is not invertible for some i, then continuously differentiable solutions of the ith constituent system exist only on a subspace of $\mathbb{R}^n$. This is the consistency space, which depends on the pair (E_i, A_i). Whenever the consistency spaces of the constituent systems do not coincide, the solution of the switched linear DAE may exhibit jumps at the switching instances. These jumps introduce new phenomena to the dynamics of switched DAE systems, which complicate the analysis considerably. To formalize the concept of a switched DAE, we exploit the distributional framework introduced in [1]. Within this framework solutions corresponding to piecewise-constant switching signals are well defined, but have in general distributional components.

Despite an increased interest in switched linear DAE systems in the recent past, there are only a few stability conditions available to date. Most results exploit some commutation properties of the constituent systems, see e.g., [2]–[4]. In [5] a result is presented that reduces the dimension of the stability problem and [6] considers stabilization via state-feedback. To the best of the authors' knowledge this is the first paper on the stabilization of switched DAEs via time-dependent switching. Preliminary versions of some of our results have been presented in [7].

In contrast, a number of results are available for the stabilizability of switched systems of ordinary differential equations (ODEs), see e.g., [8]–[14], which are obtained when all $E_i = I$. While most stabilization procedures for switched linear systems propose a feedback approach, in this paper we shall concentrate on the stabilization via time-dependent switching. One of the most prominent results for the ODE case is the following: if there exists a Hurwitz matrix in the convex hull of the set of system matrices $\{A_1,\dots,A_m\} \subset \mathbb{R}^{n\times n}$, then there exists a time-dependent switching signal, stabilizing the corresponding switched ODE system. This was noted in several publications. Among the earliest is [15], devoted to stability of switched systems under periodic time-varying switching and also [8], which concentrates on the construction of state-dependent quadratically stabilizing feedbacks. The stabilization problem by the construction of time-dependent switching signals has also been studied for the case of nonlinear systems in [16]. Feedback stabilizability has been characterized in terms of the Lyapunov spectrum of a switched system in [11]. Numerical procedures that are based on this approach are presented in [9], [11]. Surveys of stabilization results concentrating on the stabilization via state feedback can be found in [13], [14].

Given the large body of related literature for the ODE case, we may wish to use the results to study the stabilizability of switched DAEs. Therefore, we approximate every constituent system (2) of the switched system (1) by an ODE, and then study the stabilizability of a switched ODE system, constructed from such ODE-approximations (see Section IV). In this way a necessary and sufficient condition for the stabilizability of the switched DAE system (1) is obtained.

Approximation methods have a long history in DAE theory. In particular, within the singular perturbation approach the behavior of ODEs for small parameters is studied using DAE systems [17, p. 56]. A further method has been used in [18, Th. 6.2], where it is shown that results on the existence of common Lyapunov functions may be transferred from the ODE to the DAE case by adding a differential part to what we call

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the *flow matrix* in this contribution (see (18) below). Note, however, that in [18] there is no dynamic interpretation of this procedure. Approximation of switched DAEs via averaging has been studied in the recent papers [19], [20]. This method applies to switched DAEs with a finite number of constituent systems with commuting consistency projectors.

The paper is structured as follows. In Section III we give a brief overview over the solution theory of switched DAEs and introduce the main concepts. In particular, the description of solutions requires the notions of flow matrices and consistency projectors. In Section IV the first main result on the stabilizability of switched DAEs via time-dependent switching signals is obtained. A family of parameterized switched ODEs is used to characterize stabilizability. It is shown in Theorem 14 that a uniform stabilizability property of the parameterized family of ODEs is equivalent to stabilizability of the switched DAE. Using this result, it follows in particular that stabilization via time-dependent signals can be obtained in a uniformly exponential way by means of a periodic signal, if it is at all possible. We provide a counterexample showing that the uniformity condition on the family of ODEs cannot be relaxed. For the special case of commuting flow matrices uniformity is implied by our stabilizability condition. We point out that in general solutions of switched DAEs may have impulsive components. Our notion of stability requires that no impulses occur for solutions starting in a consistency space, we comment below on the question of general initial values.

A particular feature of switched DAEs is that stabilization may be obtained by exploiting the discontinuities of the solutions that occur at the switching instances. In Section V we investigate this property and consider the design of stabilizing switching signals via rapid switching. Our first result may be interpreted as a generalization of the convex hull condition for ODEs in [15]. Our second result pertains to the stabilization using solely the consistency projectors.

Finally, in Section VI we return to the case of DAEs with commuting flow matrices and obtain a further condition for stabilizability using results of [4]. The problem of stabilizability can then be reduced to that of the switched ODE which is well defined on the intersection of the consistency spaces. Thus in this case the algebraic nature of the problem is of minor importance. The one major exception to this rule occurs when the intersection of the consistency spaces is merely the origin. In this circumstance stabilization in finite time is possible. We conclude in Section VII and point out further directions of research.

II. Notation

As usual, the real numbers are denoted by $\mathbb{R}$, the nonnegative half-axis is $\mathbb{R}_+ := [0,\infty)$ and the space of square matrices with real entries is denoted by $\mathbb{R}^{n\times n}$. The spectrum of a matrix $A \in \mathbb{R}^{n\times n}$ is denoted by $\operatorname{Spec}(A)$ and the spectral radius by $\rho(A)$. The commutator of $A, B \in \mathbb{R}^{n\times n}$ is denoted by $[A,B] = AB - BA$. By $C^1(\mathbb{R}_+, \mathbb{R}^n)$ we denote the space of continuously differentiable functions from $\mathbb{R}_+$ to $\mathbb{R}^n$. $x(t_0^-)$ denotes the left limit of x at the point t_0. Further, we denote by $o_n(h)$ the class of functions $f : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ for which $\lim_{h\searrow+0}(\|f(h)\|/h) = 0$; and $O_n(h)$ is the class of functions $f : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ for which there exists a $c > 0$, so that $\|f(h)\| \le ch$ for all sufficiently small $h > 0$.

III. Solutions and Stability of DAEs

In this section we review relevant results from the existence theory of switched DAEs and introduce the necessary concepts. We begin by recalling results on time-invariant DAEs.

A. Linear Time-Invariant DAEs

Consider a linear time-invariant DAE of the form

$$E\dot{x} = Ax \tag{3}$$

where $E, A \in \mathbb{R}^{n\times n}$. By $C_{(E,A)}$ we denote the consistency space of (3) given by

$$C_{(E,A)} := \left\{x_0 \in \mathbb{R}^n : \exists x \in C^1([0,\infty), \mathbb{R}^n) \text{ with } x(0) = x_0 \text{ which solves (3)}\right\}.$$

In the following we assume that the matrix pencil $(sE - A)$ defining (3) is regular, i.e., $\det(sE - A)$ is a nonzero polynomial. This ensures that for all $x_0 \in C_{(E,A)}$ the solution of (3) is unique, [21, Th. 2.12].

Regular DAEs can be brought into the so-called quasi-Weierstraß form [1], [22], which is introduced now.

Theorem 1 (Quasi-Weierstraß Form): If (3) is regular, then there exist invertible matrices $S, T \in \mathbb{R}^{n\times n}$ such that

$$SET = \begin{pmatrix} I & 0 \\ 0 & N \end{pmatrix}, \quad SAT = \begin{pmatrix} J & 0 \\ 0 & I \end{pmatrix} \tag{4}$$

where I denotes an identity matrix of an appropriate dimension, N, J are square matrices, N is nilpotent and the partitioning of both block matrices is the same.

Note that the quasi-Weierstraß form is not the Weierstraß canonical form since we do not assume that the matrices N, J are in Jordan canonical form.

The quasi-Weierstraß form provides a decoupling of (3) into a purely differential and a purely algebraic part. Moreover, the matrices S and T are real and can be efficiently computed via Wong sequences [23], [24].

Using T and the partition from Theorem 1 we introduce the so-called flow matrix A^d by

$$A^d := T\begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}T^{-1}. \tag{5}$$

The importance of A^d lies in the following result [4, Lemma 3], [25].

Lemma 2: Any solution of (3) with $x(0^-) = x_0 \in C_{(E,A)}$, is also a solution of

$$\dot{x} = A^d x, \quad x(0^-) = x_0. \tag{6}$$

A further important object is the consistency projector $\Pi = \Pi_{(E,A)}$, defined by

$$\Pi := T\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}T^{-1} \tag{7}$$

where T is taken from Theorem 1. We note that Π is a projection for A^d and we have

$$A^d\Pi = \Pi A^d = A^d. \tag{8}$$

It can be shown, [1], that Π and A^d do not depend on the matrices T and S, which bring (3) into quasi-Weierstraß form.

B. Switched Linear DAEs

We consider the class $\mathcal{S}$ of right-continuous, piecewise constant switching signals $\sigma : [t_0, \infty) \to \mathcal{I}$. With a given switching signal $\sigma \in \mathcal{S}$ we associate the sequence $\{(t_0, i_0), \ldots, (t_k, i_k), \ldots\}$, where t_k are the points of discontinuity of σ with $t_k > t_{k-1}$ and $\sigma(t) = i_k$, $t \in [t_k, t_{k+1})$. Note that the notion of piecewise continuity includes in particular that on each compact set $K \subset [t_0, \infty)$ the switching signals have only finitely many discontinuities; in particular, $t_k \to \infty$ if there are infinitely many discontinuities. This will be assumed without further mention in the following. For a switching signal $\sigma : [t_0, \infty) \to \mathcal{I}$ we denote the weight of the mode i in the interval $[t_0, t]$ by

$$I_{\sigma,i}(t_0, t) = \mu\left(\{s \in [t, t_0] : \sigma(s) = i\}\right) \tag{9}$$

where μ is the Lebesgue measure. Given $\sigma \in \mathcal{S}$ we denote by $\mathcal{I}_\sigma := \{i \in \mathcal{I} : \exists\, t \geq 0 : I_{\sigma,i}(0, t) > 0\}$ the active index set for σ.

Given the constituent systems (2) and the switching signal $\sigma \in \mathcal{S}$ we consider the time-varying linear DAE

$$E_{\sigma(t)}\dot{x} = A_{\sigma(t)}x. \tag{10}$$

The consistency space, the consistency projector and the flow matrix of the i-th constituent system are denoted by C_i, Π_i and A_i^d, respectively. Note that the consistency spaces $C_i := C_{(E_i, A_i)}$ of the constituent systems (2) may not coincide. Thus it is natural to assume that switching in (10) may lead to a jump (i.e., a discontinuity) of the state at the switching instances t_k.

We define a solution concept for (10) by treating the switched DAE as a special case of the time-varying homogeneous DAE

$$E(t)\dot{x} = A(t)x \tag{11}$$

where E and A are piecewise constant matrix functions. Thus, in this paper solutions of (10) are to be understood as the distributional solutions of (11) in the sense of [1], [23]. In this case, the solution may include derivatives of discontinuities in the distributional sense, i.e., Dirac impulses. In our analysis discontinuous solutions are permitted, but impulses are an impediment to stability and we will impose some restrictions in our notion of stability. Note, that within the distributional framework, initial conditions at time t_0 do not yield sufficient information and we need to require that solutions extend to the left, so that a condition of the form $x(t_0^-) = x_0$ is well-defined. In the following we denote by $\varphi(\cdot, t_0, x_0)$ the solution of (11) corresponding to the initial condition $x(t_0^-) = x_0$, $t_0 \geq 0$. If $t_0 = 0$, the argument t_0 is omitted in φ.

Remark 3: To clarify the distinction between discontinuities and distributional components of the solutions, consider the two examples of systems in $\mathbb{R}^2$ given by

$$0\dot{x} = x, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \dot{x} = x.$$

Both systems are in Weierstraß form and have a consistency space equal to $\{0\}$ as can be easily checked. For the initial value $x(0^-) = e_2 = [0\ 1]^\top$, the solution of the first system is given by $x(t) = 0$, $t \geq 0$, i.e., the state jumps from e_2 to 0 at $t = 0$. For the same initial condition a Dirac impulse appears in the solution of the second system: the distributional derivative of the jump from e_2 to 0 at time $t = 0$ is $-\delta_0 e_2$, where δ_0 is the standard Dirac distribution at 0. It is then easy to check, that the solution of the second DAE is $x(t) = 0 - \delta_0(t)e_1$, where $e_1 = [1\ 0]^\top$, i.e., we do not only have a jump, but a Dirac impulse in the solution.

We model the following definition of stability of the time-varying linear system (10) given by a particular switching signal on the definition given in [1, Def. 4.3.7]. However, we have to point out a significant conceptual difference, in that in [1] switching signals and solutions are defined on $\mathbb{R}$ and the definition of stability only requires properties of these solutions. In this context, initial values at time $x(0^-)$ do not really play a role and in particular, since solutions over $\mathbb{R}$ are considered, the definition in [1] implicitly only considers values of $x(0^-)$ which already lie in some consistency space. We therefore restrict ourselves here to this situation, but point out that more general definitions of stability can be argued for.

In the following we let $U_\mathcal{I} := \bigcup_{i \in \mathcal{I}} C_i$ denote the union of all consistency spaces of the switched system.

Definition 4: Let $\sigma \in \mathcal{S}$ be fixed. The zero solution of (10) is called *asymptotically stable* if for all initial conditions $x(0^-) = x_0 \in U_\mathcal{I}$ the corresponding distributional solution of (10) does not contain Dirac impulses and if the zero solution of (10) is

1) stable: $\forall \varepsilon > 0\ \exists \delta > 0$: $\forall x_0 \in U_\mathcal{I} : |x_0| < \delta$ implies that $|\varphi(t, x_0)| < \varepsilon\, \forall\, t \geq 0$;
2) attractive: $\lim_{t \to \infty} \varphi(t, x_0) = 0$ for all $x_0 \in U_\mathcal{I}$.

The zero solution of (11) is called uniformly exponentially stable, if for all initial conditions $x(t_0^-) = x_0 \in U_\mathcal{I}$, $t_0 \geq 0$ the corresponding distributional solution of (10) does not contain Dirac impulses and if there exist $M, \beta > 0$ such that $|\varphi(t, t_0, x_0)| \leq M e^{-\beta(t - t_0)} |x_0|$ for all $x_0 \in U_\mathcal{I}$, $t \geq t_0 \geq 0$.

Remark 5: The above definition of stability still allows for Dirac impulses in solutions corresponding to initial values $x_0 \notin U_\mathcal{I}$. However, as we will see below, Dirac impulses in these solutions may occur only at $t = 0$ provided the zero solution of (10) is asymptotically stable.

Note that for linear systems (11) the attractivity of the zero solution implies its stability. Also, as is usual for linear systems, local asymptotic stability is equivalent to global asymptotic stability. We further point out that for time-varying DAEs, as in the ODE case, uniform exponential stability is a stronger notion than exponential stability, see [26] for details.

It is convenient to have an equivalent definition of asymptotic stability in terms of comparison functions. We call $f : \mathbb{R}_+ \to \mathbb{R}_+$ of class $\mathcal{L}$ (and write $f \in \mathcal{L}$) if f is continuous and strictly decreasing to 0.

Lemma 6: Let $\sigma \in \mathcal{S}$ be fixed. Assume that σ is such that all solutions of (11) corresponding to an initial value $x(t_0^-) \in U_\mathcal{I}$ do not exhibit Dirac impulses. Then the zero solution of (11) is asymptotically stable if and only if there exists a $\zeta \in \mathcal{L}$ (depending on σ) so that for all initial conditions $x(0^-) = x_0 \in U_\mathcal{I}$ and for all $t \geq 0$ it holds that

$$|\varphi(t, x_0)| \leq \zeta(t)|x_0|. \tag{12}$$

Proof: $\Rightarrow$: By linearity, the solution of (11) with initial condition $x(0^-) = x_0 \in U_\mathcal{I}$ can be written in the form $\varphi(t, x_0) = \Phi(t)x_0$ for some $\Phi : \mathbb{R}_+ \to \mathbb{R}^{n \times n}$ such that $\Phi(t)\mathrm{span}\, U_\mathcal{I} \subset U_\mathcal{I}$. Thus,

$$|\varphi(t, x_0)| \leq \left\| \Phi(t)_{|\mathrm{span}\, \mathcal{U}_\mathcal{I}} \right\| |x_0|.$$

We claim that $\|\Phi(t)_{|\mathrm{span}\, \mathcal{U}_\mathcal{I}}\| \to 0$, $t \to \infty$. Fix a basis $\{v_1, \ldots, v_\ell\} \subset U_\mathcal{I}$ of span $U_\mathcal{I}$ and $\varepsilon > 0$. Since (11) is attractive, there exists a $T > 0$ such that $|\varphi(t, v_k)| = |\Phi(t)v_k| < \varepsilon$ for all $t \geq T$ and $k = 1, \dots, \ell$. Now, $R \mapsto \max\{|Rv_k|; k = 1, \dots, \ell\}$ defines a norm on span $U_\mathcal{I}$. As all norms on finite-dimensional vector spaces are equivalent, there exists a uniform constant M such that $\|\Phi(t)_{|\text{span } U_\mathcal{I}}\| \leq M \max\{|\Phi(t)v_k|; k = 1, \dots, \ell\}$ for all $t \geq 0$. Thus the previous considerations show that $\|\Phi(t)_{|\text{span } U_\mathcal{I}}\| \leq M\varepsilon$ for all $t \geq T$. As $\varepsilon > 0$ is arbitrary this shows the claim.

Defining $\zeta \in \mathcal{L}$ by $\zeta(t) := \sup_{\tau \geq t} \|\Phi(\tau)_{|\text{span } \mathcal{U}_\mathcal{I}}\| + e^{-t}$ we obtain (12).

"$\Leftarrow$" This is clear. □

Our objective in this paper is to find conditions ensuring the existence of a time-dependent switching signal $\sigma \in \mathcal{S}$, such that the switched DAE given by the constituent systems (2) is asymptotically stable. More precisely, we define the notion of stabilizability of the switched DAE via time-dependent switching as follows.

Definition 7: The switched system given by the constituent systems (2) is called *asymptotically (exponentially) stabilizable via time-dependent switching* if there exists a switching signal $\mathcal{S} \ni \sigma : [0, \infty) \to \mathcal{I}$ such that the equilibrium of (10) is asymptotically (exponentially) stable. In this case σ is called asymptotically stabilizing, resp. exponentially stabilizing.

We will show below that asymptotic and exponential stabilizability are equivalent. For most of the paper, we will therefore only speak of asymptotic stabilizability.

The notion of asymptotic stability in Definition 4 does not allow for Dirac impulses for initial conditions lying in some consistency space. Therefore it is convenient to have conditions guaranteeing the absence of Dirac impulses in the solution. The result in [27, pp. 71–72] provides such sufficient condition for solutions defined on $\mathbb{R}$. In terms of the initial condition $x(t_0^-)$ this means that the result in [27] holds provided that $x(t_0^-)$ is an element of some consistency space.

Theorem 8: Consider the switched DAE (10) with the class of switching signals $\mathcal{S}$. The distributional solutions of (10) are impulse-free for all $\sigma \in \mathcal{S}$ and all $x_0 \in U_\mathcal{I}$ if and only if the following impulse-freeness conditions hold:

$$E_i(I - \Pi_i)\Pi_j = 0 \qquad \forall i,\ j = 1, \dots, m. \tag{13}$$

In this case the solution of each constituent system (2) with initial condition $x(0^-) = x_0 \in U_\mathcal{I}$ is given by

$$\varphi(t, x_0) = e^{A_i^d t}\Pi_i x_0, \quad \forall t \in \mathbb{R}_+. \tag{14}$$

Note that impulses can still occur for solutions corresponding to an initial value which is not consistent with any of the given systems. Another way of interpreting the conditions for stability and stabilizability presented here is thus that for general solutions we admit an impulse at time t_0 and require an impulse free condition on (t_0, ∞).

For the remainder of this paper we assume that (13) is satisfied. Certainly, for a particular stabilizing switching signal, e.g., if we do not switch between all systems, milder conditions can be applied.

The representation (14) can be used to construct the solution of (10) for $\sigma \in \mathcal{S}$ and initial condition $x(t_0^-) = x_0 \in U_\mathcal{I}$ by

$$\varphi(t, t_0, x_0) = \Phi_\sigma^{DAE}(t, t_0)x_0 \tag{15}$$

where we set for $t \in [t_k, t_{k+1})$

$$\Phi_\sigma^{DAE}(t, t_0) := e^{A_{i_k}^d(t - t_k)}\Pi_{i_k} \times e^{A_{i_{k-1}}^d(t_k - t_{k-1})}\Pi_{i_{k-1}} \cdots e^{A_{i_0}^d(t_1 - t_0)}\Pi_{i_0}. \tag{16}$$

We sometimes omit the argument t_0 if $t_0 = 0$. As the expression for $\Phi_\sigma^{DAE}(t, t_0)$ can be extended to all of $\mathbb{R}^n$, we will in the sequel consider this expression as globally defined. In particular, this does not affect the conditions on attractivity as $\operatorname{Im} \Phi_\sigma^{DAE}(t, t_0) \subset U_\mathcal{I}$ for all $t \geq t_0$.

We point out, that (16) describes the non-impulsive part of the solution for all $x_0 \in \mathbb{R}^n$. For systems, satisfying (13), one can reformulate the notion of asymptotic stability as follows:

Proposition 9: Fix any $\sigma \in \mathcal{S}$ and assume that condition (13) holds. The zero solution of (10) is asymptotically stable if and only if the non-impulsive part of solution (15), (16) is asymptotically stable on $\mathbb{R}^n$.

Proof: Since (13) holds, Dirac impulses do not occur for $x_0 \in U_\mathcal{I}$. And asymptotic stability of (15), (16) on $\mathbb{R}^n$ implies asymptotic stability of the zero solution of (10). Conversely, let the zero solution of (10) be asymptotically stable. According to Lemma 6 there exists a $\zeta \in \mathcal{L}$ so that

$$\left|\Phi_\sigma^{DAE}(t)x_0\right| \leq \zeta(t)|x_0|$$

for all initial conditions $x_0 \in U_\mathcal{I}$ and for all $t \geq 0$. By (16) this inequality extends to $|\Phi_\sigma^{DAE}(t)x_0| \leq \|\Pi_{i_0}\|\zeta(t)|x_0|$ for all $x_0 \in \mathbb{R}^n$, which shows the claim. □

IV. Stabilization Via Time-Dependent Switching

We now turn to the main subject of this paper. Our approach in this section is to relate the stabilizability of a switched DAE to properties of the solutions of associated switched ODE approximations. Recalling the quasi-Weierstraß form (4) we define for (3) the ODE approximation

$$\dot{x} = A^\varepsilon x \tag{17}$$

with the system matrix

$$A^\varepsilon := A^d - \frac{1}{\varepsilon}(I - \Pi) = T\begin{pmatrix} J & 0 \\ 0 & -\frac{1}{\varepsilon}I \end{pmatrix}T^{-1}. \tag{18}$$

For the set of system matrices $\{A_1^\varepsilon, \dots, A_m^\varepsilon\}$ we define the switched linear ODE

$$\dot{x} = A_{\sigma(t)}^\varepsilon x. \tag{19}$$

The idea behind the additional term $-(1/\varepsilon)I$ in (18) is to model a jump between the consistency spaces by continuous but fast dynamics in the direction of the jump. The smaller ε is chosen, the faster the corresponding dynamics are, resulting in better approximations of the jump. More precisely, we have the following relation between the solutions of the systems (3) and (17).

Lemma 10: For each DAE (3) and the associated ODE (17) it holds that

$$\left(e^{A^d t}\Pi - e^{A^\varepsilon t}\right)x_0 = -e^{-\frac{t}{\varepsilon}}(I - \Pi)x_0 \tag{20}$$

for all $t > 0$ and $x_0 \in \mathbb{R}^n$. In particular, for all $t > 0$

$$\lim_{\varepsilon\to 0} \left\| e^{A^d t}\Pi - e^{A^\varepsilon t} \right\| = 0. \quad (21)$$

Proof: Computing $e^{A^d t}\Pi$ using (5) and (7) we obtain

$$e^{A^d t}\Pi = T \begin{pmatrix} e^{Jt} & 0 \\ 0 & 0 \end{pmatrix} T^{-1}.$$

Now (20) follows from:

$$e^{A^d t}\Pi - e^{A^\varepsilon t} = -e^{-\frac{1}{\varepsilon}t}(I - \Pi).$$

Clearly, for every fixed $t > 0$ the last expression tends to zero as $\varepsilon \searrow 0$. □

Having established the approximation for the constituent systems we can now relate the impulse-free solutions of the switched DAE (10) to the solutions of the corresponding switched ODE (19). Let $\Phi_\sigma^{DAE}(t, t_0)$ denote the solution operator of (10) and let

$$\Phi_\sigma^\varepsilon(t, t_0) := \exp\left(A_{i_k}^\varepsilon (t - t_k)\right) \times \exp\left(A_{i_{k-1}}^\varepsilon (t_k - t_{k-1})\right) \cdots \exp\left(A_{i_0}^\varepsilon (t_1 - t_0)\right) \quad (22)$$

denote the solution operator of (19) for the same switching signal σ.

Lemma 11: For all $\sigma \in \mathcal{S}$ and all $t > t_0 \geq 0$ we have

$$\lim_{\varepsilon\to 0} \left\| \Phi_\sigma^{DAE}(t, t_0) - \Phi_\sigma^\varepsilon(t, t_0) \right\| = 0.$$

Proof: It is sufficient to show the claim for $t_0 = 0$. Fix $\sigma \in \mathcal{S}$. For any fixed $t > 0$ there exists a k such that $t_{k+1} > t \geq t_k$. Let $\tau_{k-1} = t_k - t_{k-1}$ and $\tau_k' = t - t_k$ such that $t = \tau_k' + \tau_{k-1} + \ldots + \tau_0$. Then we have

$$\Phi_\sigma^{DAE}(t, 0) = e^{A_{i_k}^d \tau_k'}\Pi_{i_k} \cdot e^{A_{i_{k-1}}^d \tau_{k-1}}\Pi_{i_{k-1}} \cdots e^{A_{i_0}^d \tau_0}\Pi_{i_0}. \quad (23)$$

According to (20) we have

$$e^{A_i^d \tau}\Pi_i = e^{A_i^d \tau}\Pi_i - e^{A_i^\varepsilon \tau} + e^{A_i^\varepsilon \tau} = -e^{-\frac{\tau}{\varepsilon}}(I - \Pi_i) + e^{A_i^\varepsilon \tau}.$$

Substituting this expression into (23) we obtain that $\|\Phi_\sigma^{DAE}(t, 0) - \Phi_\sigma^\varepsilon(t, 0)\|$ can be represented by a finite sum of terms, all of which having a multiplier of the form $e^{-(p/\varepsilon)}$ for some $p > 0$. Taking the limit for $\varepsilon \to 0$, we obtain the claim of the lemma. □

Now we state one of the main results which shows that the stabilizability of the switched DAE (10) can be verified by analyzing the corresponding switched ODE (19).

In the following we denote the solutions of (10) and (19) at time t, corresponding to an initial condition $x(0^-) = x_0$ and a switching signal σ by $\varphi(t, x_0, \sigma)$ and $\varphi_\varepsilon(t, x_0, \sigma)$, respectively.

Theorem 12: Consider (10) and the corresponding system (19). Assume there exist $\varepsilon_0 > 0$, $s > 0$, and $c \in (0, 1)$ so that for all $0 < \varepsilon < \varepsilon_0$ there exists a periodic switching signal $\sigma_\varepsilon \in \mathcal{S}$ with period s such that

$$|\varphi_\varepsilon(s, x_0, \sigma_\varepsilon)| \leq c|x_0|, \quad \forall x_0 \in \mathbb{R}^n.$$

Moreover, assume there exists $t_d > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, for any two subsequent switching instants of σ_ε we have $|t_k^\varepsilon - t_{k+1}^\varepsilon| \geq t_d$. Then (10) is stabilizable via σ_{ε^*} for some $\varepsilon^* \in (0, \varepsilon_0)$.

Proof: Given the assumptions and using (20) we have for all $x, y \in \mathbb{R}^n$ the following estimate:

$$\begin{aligned} \left| e^{A_i^d t}\Pi_i x - e^{A_i^\varepsilon t} y \right| &= \left| \left(e^{A_i^d t}\Pi_i - e^{A_i^\varepsilon t}\right) x + e^{A_i^\varepsilon t}(x - y) \right| \\ &= \left| -e^{-\frac{t}{\varepsilon}}(I - \Pi_i)x + e^{A_i^\varepsilon t}(x - y) \right| \\ &\leq e^{-\frac{t}{\varepsilon}} \|I - \Pi_i\| |x| + \left\| e^{A_i^\varepsilon t} \right\| |x - y|. \end{aligned} \quad (24)$$

Fix $\varepsilon \in (0, \varepsilon_0)$. Using the semigroup property of linear operators we have for all $x_0 \in \mathbb{R}^n$

$$\varphi\left(t_{k+1}^\varepsilon, x_0, \sigma_\varepsilon\right) = \varphi\left(t_{k+1}^\varepsilon - t_k^\varepsilon, \varphi\left(t_k^\varepsilon, x_0, \sigma_\varepsilon\right), \sigma_\varepsilon\left(\cdot + t_k^\varepsilon\right)\right)$$

and

$$\varphi_\varepsilon\left(t_{k+1}^\varepsilon, x_0, \sigma_\varepsilon\right) = \varphi_\varepsilon\left(t_{k+1}^\varepsilon - t_k^\varepsilon, \varphi_\varepsilon\left(t_k^\varepsilon, x_0, \sigma_\varepsilon\right), \sigma_\varepsilon\left(\cdot + t_k^\varepsilon\right)\right).$$

Pick any $t^* > 0$. We are going to find an estimate of

$$\left| \varphi\left(t_{k+1}^\varepsilon, x_0, \sigma_\varepsilon\right) - \varphi_\varepsilon\left(t_{k+1}^\varepsilon, x_0, \sigma_\varepsilon\right) \right|$$

on the interval $[0, t^*]$. Using (24) we obtain

$$\begin{aligned} &\left| \varphi\left(t_{k+1}^\varepsilon, x_0, \sigma_\varepsilon\right) - \varphi_\varepsilon\left(t_{k+1}^\varepsilon, x_0, \sigma_\varepsilon\right) \right| \\ &= \left| e^{A_{i_k}^d \left(t_{k+1}^\varepsilon - t_k^\varepsilon\right)}\Pi_{i_k}\varphi\left(t_k^\varepsilon, x_0, \sigma_\varepsilon\right) - e^{A_{i_k}^\varepsilon \left(t_{k+1}^\varepsilon - t_k^\varepsilon\right)}\varphi_\varepsilon\left(t_k^\varepsilon, x_0, \sigma_\varepsilon\right) \right| \\ &\leq e^{-\frac{t_{k+1}^\varepsilon - t_k^\varepsilon}{\varepsilon}} \|I - \Pi_{i_k}\| \left|\varphi\left(t_k^\varepsilon, x_0, \sigma_\varepsilon\right)\right| + \left\| e^{A_{i_k}^\varepsilon \left(t_{k+1}^\varepsilon - t_k^\varepsilon\right)} \right\| \left|\varphi\left(t_k^\varepsilon, x_0, \sigma_\varepsilon\right) - \varphi_\varepsilon\left(t_k^\varepsilon, x_0, \sigma_\varepsilon\right)\right| \\ &\leq L e^{-\frac{t_d}{\varepsilon}} |x_0| + M \left|\varphi\left(t_k^\varepsilon, x_0, \sigma_\varepsilon\right) - \varphi_\varepsilon\left(t_k^\varepsilon, x_0, \sigma_\varepsilon\right)\right| \end{aligned}$$

where

$$L := \sup_{l=1}^{m} \|I - \Pi_l\| \cdot \sup_{\tau\in[0,t^*],\varepsilon\in(0,\varepsilon_0)} \|\varphi(\tau, \cdot, \sigma_\varepsilon)\|$$

$$M := \sup_{l=1}^{m} \left(\|T_l\| \left\|T_l^{-1}\right\| \sup_{t\in[t_d,t^*]} \left(\|e^{J_l t}\| + 1\right) \right).$$

Note that the constants L and M depend neither on ε nor on the sequence $\{t_k\}$ as the following estimates hold:

$$\|I - \Pi_{i_k}\| \left|\varphi\left(t_k^\varepsilon, x_0, \sigma_\varepsilon\right)\right| \leq L|x_0|$$

and

$$\begin{aligned} \sup_{t\in[t_d,t^*]} \left\| e^{A_{i_k}^\varepsilon t} \right\| &\leq \|T_{i_k}\| \left\|T_{i_k}^{-1}\right\| \sup_{t\in[t_d,t^*]} \left\| \begin{pmatrix} e^{J_{i_k} t} & 0 \\ 0 & e^{-\frac{1}{\varepsilon}t} \end{pmatrix} \right\| \\ &\leq \|T_{i_k}\| \left\|T_{i_k}^{-1}\right\| \sup_{t\in[t_d,t^*]} \left(\|e^{J_{i_k} t}\| + \left\|e^{-\frac{1}{\varepsilon}t}\right\| \right) \\ &\leq M. \end{aligned}$$

Since $|\varphi(t_0^\varepsilon, x_0, \sigma_\varepsilon) - \varphi_\varepsilon(t_0^\varepsilon, x_0, \sigma_\varepsilon)| = 0$, we continue with the above estimates to obtain for any $x_0 \in \mathbb{R}^n$

$$\left| \varphi\left(t_{k+1}^\varepsilon, x_0, \sigma_\varepsilon\right) - \varphi_\varepsilon\left(t_{k+1}^\varepsilon, x_0, \sigma_\varepsilon\right) \right| \leq L \sum_{j=0}^{k} M^j e^{-\frac{t_d}{\varepsilon}} |x_0|. \quad (25)$$

According to the assumptions of the theorem there exist $s > 0$ and $c \in (0,1)$ such that $|\varphi_\varepsilon(s, x_0, \sigma_\varepsilon)| < c|x_0|$ for all $\varepsilon \in (0, \varepsilon_0)$.

Using (25) one can find $\varepsilon^* < \varepsilon_0$, so that

$$|\varphi(s, x_0, \sigma_{\varepsilon^*}) - \varphi_{\varepsilon^*}(s, x_0, \sigma_{\varepsilon^*})| < \frac{1-c}{2}|x_0|.$$

And thus via the triangle inequality we have for all $x_0 \in U_{\mathcal{I}}$

$$|\varphi(s, x_0, \sigma_{\varepsilon^*})| \le \frac{1+c}{2}|x_0| < |x_0|.$$

This implies, that (10) is stabilized via σ_{ε^*}. □

The above result establishes that the stabilizability of the switched DAE can be shown by considering the stabilizability of the corresponding switched ODEs.

Remark 13: (i) Note that the result requires the existence of a certain "common minimum dwell-time" t_d for all approximating ODEs with small enough ε.

(ii) It is interesting to note that by Theorem 12 there exists an $\varepsilon^* > 0$ such that the switching signal σ_{ε^*} stabilizes the approximating switched ODE as well as the switched DAE.

The above theorem helps to prove a less precise but more handy and, in particular, necessary and sufficient criterion for the stabilizability of switched DAEs.

Theorem 14: The following statements are equivalent:

1) The switched linear DAE (10) is asymptotically stabilizable via a time-dependent switching signal.
2) The switched linear DAE (10) is uniformly exponentially stabilizable via a time-dependent switching signal.
3) The switched linear DAE (10) is asymptotically stabilizable via a periodic switching signal σ_p.
4) There exists an $\varepsilon_0 > 0$ and a periodic switching signal σ_p so that the switched ODE (19) is stabilized via the (same) periodic switching signal σ_p for all $0 < \varepsilon < \varepsilon_0$, uniformly w.r.t. $\varepsilon : \exists s > 0,\ c \in (0,1)$, s.t. for all $\varepsilon \in (0, \varepsilon_0)$

$$|\varphi_\varepsilon(s, x_0, \sigma_p)| \le c|x_0|, \quad \forall x_0 \in \mathbb{R}^n.$$

Proof: 2)⇒1): This is trivial.

3)⇒2): Let $\sigma \in \mathcal{S}$ be asymptotically stabilizing and have period $p > 0$. Then $\rho(\Phi_\sigma^{DAE}(p,0)) < 1$ as otherwise $\Phi_\sigma^{DAE}(p,0)^k \not\to 0$ for $k \to \infty$. Thus there exist constants $M, \beta > 0$ such that $\|\Phi_\sigma^{DAE}(kp,0)\| = \|\Phi_\sigma^{DAE}(p,0)^k\| \le Me^{-\beta kp}$. Define

$$M_2 := \max\left\{\left\|\Phi_\sigma^{DAE}(s,0)\right\|, \left\|\Phi_\sigma^{DAE}(p,s)\right\| : s \in [0,p]\right\}.$$

Exploiting that σ is p-periodic we obtain for $t > t_0 \ge 0$, where $t_0 = kp - s,\ s \in [0,p)$ and $t = r + \ell p,\ r \in [0,p)$ with $k, \ell \in \mathbb{N},\ k \le \ell$ that

$$\begin{aligned}&\left\|\Phi_\sigma^{DAE}(t,t_0)\right\| \\ &= \left\|\Phi_\sigma^{DAE}(r+\ell p, \ell p)\Phi_\sigma^{DAE}(\ell p, kp)\Phi_\sigma^{DAE}(kp, kp-s)\right\| \\ &\le \left\|\Phi_\sigma^{DAE}(r,0)\right\| \cdot \left\|\Phi_\sigma^{DAE}(p,0)^{\ell-k}\right\| \cdot \left\|\Phi_\sigma^{DAE}(p,p-s)\right\| \\ &\le M \cdot M_2^2 e^{2p\beta} e^{-\beta(t-t_0)}.\end{aligned}$$

This shows the assertion.

1)⇒3): Let (10) be stabilizable via time-dependent switching. According to Lemma 6 there exist a switching signal σ and $\eta \in \mathcal{L}$ so that

$$\left|\Phi_\sigma^{DAE}(t)x_0\right| \le \eta(t)|x_0| \tag{26}$$

for all initial conditions $x_0 \in U_{\mathcal{I}}$ and for all $t \ge 0$. By (16) this inequality extends to $|\Phi_\sigma^{DAE}(t)x_0| \le \|\Pi_{i_0}\|\eta(t)|x_0|$ for all $x_0 \in \mathbb{R}^n$ and we denote $\zeta := \|\Pi_{i_0}\|\eta$.

Since $\zeta \in \mathcal{L}$, there exists a time $t^* > 0$, so that $\zeta(t^*) = 1/2$ and we have for all $x_0 \in \mathbb{R}^n$

$$\left|\Phi_\sigma^{DAE}(t^*)x_0\right| \le \frac{1}{2}|x_0|. \tag{27}$$

In particular, $\|\Phi_\sigma^{DAE}(t^*)\| \le 1/2$. Now consider the periodic signal σ_p defined by the formula

$$\sigma_p(t) = \sigma(t \bmod t^*).$$

For this switching signal and for all $x_0 \in \mathbb{R}^n$ we have

$$\left|\Phi_{\sigma_p}^{DAE}(kt^*)x_0\right| \le \frac{1}{2^k}|x_0| \tag{28}$$

and thus σ_p stabilizes the system (10).

3)⇒4): Let $\sigma_p \in \mathcal{S}$ be periodic with the period t^* and stabilize the system (10). Then there exists a $k > 0$, so that

$$\left\|\Phi_{\sigma_p}^{DAE}(kt^*)\right\| \le \frac{1}{2}.$$

The triangle inequality implies

$$\left\|\Phi_{\sigma_p}^{\varepsilon}(kt^*)\right\| \le \left\|\Phi_{\sigma_p}^{\varepsilon}(kt^*) - \Phi_{\sigma_p}^{DAE}(kt^*)\right\| + \left\|\Phi_{\sigma_p}^{DAE}(kt^*)\right\|. \tag{29}$$

By Lemma 11 there exists an $\varepsilon_0 > 0$ so that for all $\varepsilon \le \varepsilon_0$ we obtain

$$\left\|\Phi_{\sigma_p}^{\varepsilon}(kt^*) - \Phi_{\sigma_p}^{DAE}(kt^*)\right\| \le \frac{1}{4}.$$

From (29) we obtain that $\|\Phi_{\sigma_p}^{\varepsilon}(kt^*)\| \le 3/4$ for all $\varepsilon \le \varepsilon_0$ and consequently the following estimate holds:

$$\left|\Phi_{\sigma_p}^{\varepsilon}(wkt^*)x_0\right| \le \left(\frac{3}{4}\right)^w |x_0|. \tag{30}$$

Thus σ_p stabilizes (19) for all $\varepsilon \le \varepsilon_0$ uniformly w.r.t. ε.

4)⇒3): follows from Theorem 12. □

Note, that we require that the switched ODEs considered are stabilizable via the same periodic switching signal σ_p for all $\varepsilon \in (0, \varepsilon_0)$. In the following subsections we shed some light on the nature of this condition by first considering an example that illustrates the necessity of this condition. In the subsequent subsection we consider a class of systems for which this condition holds naturally.

A. A Switched DAE Which is Not Stabilizable With Stabilizable ODE-Approximations

We are going to construct a switched DAE the trajectories of which (for most initial conditions) diverge for all admissible switching signals, and such that the corresponding approximations (19) are periodically stabilizable for all $\varepsilon \in (0, \varepsilon_0)$ and small enough ε_0, but via different stabilizing signals.

Consider the system $\Sigma_1 = (E_1, A_1)$ with

$$E_1 := \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad A_1 := \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$$

and the system $\Sigma_2 = (E_2, A_2)$ with

$$E_2 := \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad A_2 := \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

The respective flow matrices and consistency projectors are given by

$$\Pi_1 = A_1^d = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}, \quad \Pi_2 = A_2^d = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

It can be easily shown that for any switching signal the solution of the switched DAE is divergent for any initial state, which does not lie in $\ker \Pi_1 \cup \ker \Pi_2$.

Now consider the corresponding approximations of the form (19) given by

$$A_1^\varepsilon = A_1^d - \frac{1}{\varepsilon}(I - \Pi_1) = \begin{pmatrix} -\frac{1}{\varepsilon} & 0 \\ -1 - \frac{1}{\varepsilon} & 1 \end{pmatrix}$$

and

$$A_2^\varepsilon = A_2^d - \frac{1}{\varepsilon}(I - \Pi_2) = \begin{pmatrix} 1 & 1 + \frac{1}{\varepsilon} \\ 0 & -\frac{1}{\varepsilon} \end{pmatrix}.$$

Consider the periodic signal σ_ε of period 2ε, defined by

$$\sigma_\varepsilon(t) = \begin{cases} 1, & t \in [0, \varepsilon) \\ 2, & t \in [\varepsilon, 2\varepsilon). \end{cases}$$

Denote

$$M_\varepsilon := e^{A_2^\varepsilon \varepsilon} e^{A_1^\varepsilon \varepsilon} = \begin{pmatrix} e^{\varepsilon - 1} - (e^\varepsilon - e^{-1})^2 & e^\varepsilon (e^\varepsilon - e^{-1}) \\ -e^{-1}(e^\varepsilon - e^{-1}) & e^{\varepsilon - 1} \end{pmatrix}.$$

The state of the system (19) at time $2\varepsilon k$, corresponding to σ_ε is given by

$$\varphi_\varepsilon(2\varepsilon k, x_0, \sigma_\varepsilon) = M_\varepsilon^k x_0.$$

Defining $y_\varepsilon(k) := \varphi_\varepsilon(2\varepsilon k, x_0, \sigma_\varepsilon)$ we obtain the discrete-time system

$$y_\varepsilon(k) = M_\varepsilon y_\varepsilon(k - 1). \tag{31}$$

An easy computation shows

$$M_0 := \lim_{\varepsilon \to 0} M_\varepsilon \approx \begin{pmatrix} -0.0316970 & 0.6321206 \\ -0.2325442 & 0.3678794 \end{pmatrix}$$

and

$$\text{Spec}\,(M_0) \approx \{0.1680912 \pm 0.3272317i\}.$$

Hence $\rho(M_0) < 1$. Since the eigenvalues of a matrix depend continuously on its entries (see, e.g., [28, Sec. A.4, p. 456]), we obtain, that $\rho(M_\varepsilon) < 1$ for all $\varepsilon \in (0, \varepsilon_0)$ for small enough ε_0.

This immediately implies, that (31) is stable for small enough ε and thus σ_ε, $\varepsilon \in (0, \varepsilon_0)$ stabilizes (19), see Fig. 1.

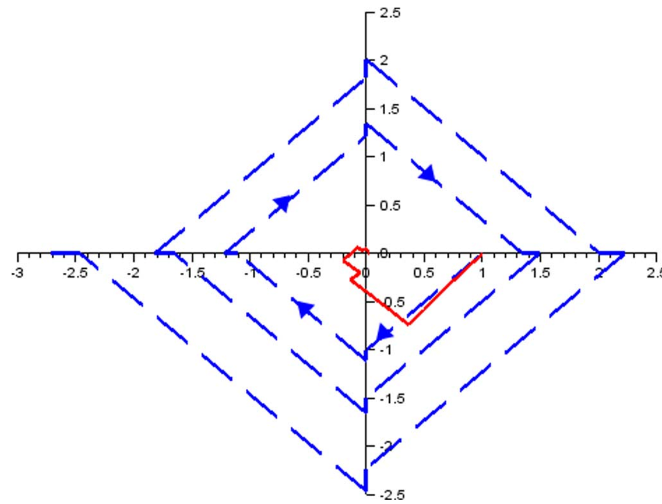

Fig. 1. A trajectory of the switched DAE given by Σ_1 and Σ_2 (blue) and of its ODE approximation (17) (red) under the control σ_ε for $x_0 = (1, 0)^T$, $\varepsilon = 0.1$, $t \in [0, 1]$.

Note that the period of σ_ε as well as the time between switches tends to 0 as $\varepsilon \to 0$, thus, the conditions of Theorem 12 are violated.

B. Commutative Constituent Systems

In this subsection we consider the special case of switched DAEs with commuting flow matrices A_i^d. As will be shown below, the requirement that all approximating switched ODEs are stabilizable via the same periodic switching signal is already implied by periodic stabilization of an approximating ODE for certain $\varepsilon_0 > 0$.

Proposition 15: Let the commutators $[A_i^d, A_j^d] = 0$, $i, j = 1, \ldots, m$ and let A_i be invertible for all $i = 1, \ldots, m$. If there exists $\varepsilon_0 > 0$ so that (19) is stabilizable for $\varepsilon := \varepsilon_0$ via the periodic switching signal $\sigma_p = \sigma_p(\varepsilon_0)$, then (19) is stabilizable via the same switching signal σ_p, uniformly w.r.t. $\varepsilon \in (0, \varepsilon_0]$.

Proof: Let A_i^d, $i = 1, \ldots, m$ commute pairwise and let A_i be invertible for all $i = 1, \ldots, m$.

From [4, Th. 7, Lemma 9] (see Theorem 28 below) it follows that $[A_i^d, \Pi_j] = 0$ and $[\Pi_i, \Pi_j] = 0$, for all $i, j = 1, \ldots, m$, which implies that $[I - \Pi_i, I - \Pi_j] = 0$ for all $i, j = 1, \ldots, m$.

Now from (18) we obtain $[A_i^{\varepsilon_i}, A_j^{\varepsilon_j}] = 0$ for all $i, j = 1, \ldots, m$ and for all $\varepsilon_i, \varepsilon_j > 0$.

Let (19) be stabilized via the periodic switching signal σ_p.

For $t = \sum_{l=1}^k \tau_l$ and all $x_0 \in \mathbb{R}^n$ we have

$$\begin{aligned} \varphi_\varepsilon(t, x_0, \sigma_p) &= e^{A_{i_k}^\varepsilon \tau_k} \cdots e^{A_{i_1}^\varepsilon \tau_1} x_0 \\ &= e^{\sum_{l=1}^k \left(A_{i_l}^\varepsilon - A_{i_l}^{\varepsilon_0}\right)\tau_l} e^{A_{i_k}^{\varepsilon_0} \tau_k} \cdots e^{A_{i_1}^{\varepsilon_0} \tau_1} x_0 \\ &= e^{\sum_{j=1}^m \left(A_j^\varepsilon - A_j^{\varepsilon_0}\right) I_{\sigma_p, j}(t)} e^{A_{i_k}^{\varepsilon_0} \tau_k} \cdots e^{A_{i_1}^{\varepsilon_0} \tau_1} x_0 \\ &= \prod_{j=1}^m e^{\left(A_j^\varepsilon - A_j^{\varepsilon_0}\right) I_{\sigma_p, j}(0,t)} \varphi_{\varepsilon_0}(t, x_0, \sigma_p) \end{aligned} \tag{32}$$

where $I_{\sigma_p, j}(0, t)$ is defined as in (9). From (18) we see that

$$A_j^\varepsilon - A_j^{\varepsilon_0} = \left(\frac{1}{\varepsilon_0} - \frac{1}{\varepsilon}\right)(I - \Pi_j).$$

Due to this inequality and since $(1/\varepsilon_0) - (1/\varepsilon) < 0$, we obtain

$$\begin{aligned}\left\| \prod_{j=1}^{m} e^{\left(A_j^{\varepsilon} - A_j^{\varepsilon_0}\right) I_{\sigma_p, j}(0,t)} \right\| &\le \prod_{j=1}^{m} \left\| e^{\left(\frac{1}{\varepsilon_0} - \frac{1}{\varepsilon}\right)(I-\Pi_j) I_{\sigma_p,j}(0,t)} \right\| \\ &= \prod_{j=1}^{m} \left\| T_j \exp \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \left(\frac{1}{\varepsilon_0} - \frac{1}{\varepsilon} \right) I_{\sigma_p,j}(0,t) T_j^{-1} \right\| \\ &= \prod_{j=1}^{m} \left\| \Pi_j + e^{\left(\frac{1}{\varepsilon_0} - \frac{1}{\varepsilon}\right) I_{\sigma_p,j}(0,t)} (I - \Pi_j) \right\| \\ &\le \prod_{j=1}^{m} \left(\|\Pi_j\| + \|I - \Pi_j\| \right) =: K\end{aligned}$$

where K is independent of ε. Now from (32) we obtain

$$|\varphi_\varepsilon(t, x_0, \sigma_p)| \le K \, |\varphi_{\varepsilon_0}(t, x_0, \sigma_p)| \,, \quad \forall x_0 \in \mathbb{R}^n.$$

Take $s > 0$ such that $|\varphi_{\varepsilon_0}(s, x_0, \sigma_p)| \le (\frac{1}{2K})|x_0|$ for all $x_0 \in \mathbb{R}^n$. We obtain

$$|\varphi_\varepsilon(s, x_0, \sigma_p)| \le \frac{1}{2}|x_0|, \quad \forall x_0 \in \mathbb{R}^n$$

which immediately implies that σ_p stabilizes (19) for all $\varepsilon \in (0, \varepsilon_0]$, uniformly w.r.t. ε. □

V. Stabilization via Fast Switching

Consider the switched linear ODE system

$$\dot{x}(t) = A_{\sigma(t)} x(t), \quad x(t) \in \mathbb{R}^n \tag{33}$$

where $\mathcal{S} \ni \sigma : [t_0, \infty) \to \mathcal{I}$. It is well-known that the existence of a Hurwitz matrix in the convex hull of the matrices $A_1, \ldots, A_m$ is sufficient for the stabilizability of (33) via a time-dependent switching signal σ, see [15]. We are going to generalize this construction of stabilizers to the case of switched DAEs of the form (10).

A. The Role of Continuous Dynamics in Fast Switching

We need the following lemma.

Lemma 16: Let $P \in \mathbb{R}^{n \times n}$ be diagonalizable. For all $S \in \mathbb{R}^{n \times n}$, $\eta > 0$ and all analytic matrix-valued functions $Q : (-\eta, \eta) \to \mathbb{R}^{n \times n}$, with $Q(0) = Q'(0) = 0$, i.e., $Q \in O_n(h^2)$ it holds that

$$\rho\left(P + Sh + Q(h)\right) = \rho(P + Sh) + o_1(h). \tag{34}$$

Proof: We prove this lemma via the reduction process described in [29, II, 2.3, p. 81]. The terminology is adopted from the same reference.

Define analytic matrix-valued functions by $T_1(h) := P + Sh$ and $T_2(h) := P + Sh + Q(h)$.

Take any $\lambda \in \mathrm{Spec}(P)$ such that $\rho(P) = |\lambda|$. All such λ are semisimple since P is diagonalizable. Let V be the eigenprojection of P onto the eigenspace, corresponding to λ.

If λ is a multiple eigenvalue it may split for small h into a group of eigenvalues, for the perturbed operators $T_1(h)$ and $T_2(h)$. This is the so-called λ-group, which may of course be different for T_1 and T_2. We now describe the so-called reduction process to obtain expansions of the elements in those groups.

For $T_1(h)$ and $T_2(h)$, respectively, denote by $V_1(h)$ and $V_2(h)$ the total projections on the sums of the generalized eigenspaces of this group of eigenvalues.

Define for $i = 1, 2$

$$\tilde{T}_i(h) := \frac{1}{h} \left(T_i(h) - \lambda I\right) V_i(h).$$

Due to equation (2.38) in [29, p. 81] we have that $\tilde{T}_i(h)$ may be continuously extended by $\tilde{T}_i(0) = VSV$, $i = 1, 2$, by which $\tilde{T}_i$ is analytic in 0. We denote the non-zero eigenvalues of VSV by μ_j, $j = 1, 2, \ldots, \ell_m$. Following [29, p. 82] we assume without loss of generality that the number ℓ_m of non-zero eigenvalues of VSV (counting multiplicities) coincides with the multiplicity of the eigenvalue λ of P. The eigenvalues μ_j split under the perturbation of $\tilde{T}_i$ in eigenvalues μ_{jk}^i (for suitable indices k).

Now denote the elements in the λ-groups of T_i by $s_{jk}^i(h)$, $i = 1, 2$, where the indices j and k correspond to the eigenvalues μ_j and their splitting for $\tilde{T}_i$.

According to equation (2.40) in [29, p. 81] the eigenvalues (in the λ-group) of the operators $T_i(h)$ have an asymptotic expansion of the form

$$s_{jk}^i(h) = \lambda + h\mu_j + h^{1+\frac{1}{p_j}} \alpha_{jk}^i + \text{higher order terms in } h.$$

Note, that the constants p_j only depend on the multiplicities of the non-zero eigenvalues λ_j of VSV. Now for small enough h we have

$$\begin{aligned}|\rho\left(T_2(h)\right) - \rho\left(T_1(h)\right)| &\le \max_{j,k} \left\{ \left| |s_{jk}^2| - |s_{jk}^1| \right| \right\} \\ &\le \max_{j,k} \left\{ |s_{jk}^2 - s_{jk}^1| \right\} \\ &\le h^{1+\frac{1}{p}} \max_{j,k} |\alpha_{jk}^2 - \alpha_{jk}^1| + O\left(h^{1+\frac{1}{p}}\right).\end{aligned}$$

with $p = \max_j p_j$. This implies (34). □

The next theorem provides a sufficient condition for stabilizability of system (10) via fast switching.

Theorem 17: Consider the switched DAE given by the constituent systems (2). Let sequences $i_j \in \{1, \ldots, m\}$, $j = 0, \ldots, k$ and $\alpha_j \in (0, 1]$, $j = 0, \ldots, k$ with $\sum_{j=0}^{k} \alpha_j = 1$ be given. Define

$$P := \Pi_{i_k} \cdots \Pi_{i_0}, \; S := \sum_{j=0}^{k} \Pi_{i_k} \cdots \Pi_{i_{j+1}} \alpha_j A_{i_j}^d \Pi_{i_{j-1}} \cdots \Pi_{i_0}.$$

If P is diagonalizable and if there exist $h_0, c > 0$ so that for all $h \in (0, h_0)$ it holds that

$$\rho(P + Sh) < 1 - ch \tag{35}$$

then the switched system (10) is stabilizable via the periodic switching signal σ_p with a period h^* for some $0 < h^* < h_0$, defined by

$$\sigma_p(t) = i_j, \quad t \in [t_j, t_{j+1}) \tag{36}$$

with $t_0 := 0$, $t_{j+1} := h^* \sum_{l=0}^{j} \alpha_l$, $j = 0, \ldots, k$.

Proof: Pick $k \in \mathbb{N}$ and $\alpha_j \in (0,1]$ as in the statement of the theorem and for any $F_0, \ldots, F_k \in \mathbb{R}^{n \times n}$ denote $\prod_{i=k}^{0} F_i := F_k \cdots F_0$.

We have the following set of equalities (in the second one we use the property $A^d_{s_i} \Pi_{s_i} = A^d_{s_i}$):

$$\begin{aligned} & e^{\alpha_k A^d_{i_k} h} \Pi_{i_k} \cdots e^{\alpha_0 A^d_{i_0} h} \Pi_{i_0} \\ &\quad = \prod_{j=k}^{0} \left(I + \alpha_j A^d_{i_j} h + O_n(h^2) \right) \Pi_{i_j} \\ &\quad = \prod_{j=k}^{0} \left(\Pi_{i_j} + \alpha_j A^d_{i_j} h + O_n(h^2) \right) \\ &\quad = \prod_{j=k}^{0} \Pi_{i_j} + \left(\sum_{j=0}^{k} \Pi_{i_k} \cdots \Pi_{i_{j+1}} \alpha_j A^d_{i_j} \Pi_{i_{j-1}} \cdots \Pi_{i_0} \right) h \\ &\quad\quad + O_n(h^2) \\ &\quad = P + Sh + O_n(h^2). \end{aligned}$$

By assumption there exist $h_0 > 0$ and $c > 0$ such that $\rho(P + Sh) < 1 - ch$ for all $h \in (0, h_0)$.

Now P is diagonalizable and the matrix exponential is analytic, so the expressions $O_n(h^2)$ above represent analytic functions. Thus Lemma 16 implies (34), which together with (35) leads to

$$\rho\big(P+Sh+O_n(h^2)\big) = \rho(P+Sh)+o_1(h) \leq 1-ch+o_1(h) < 1$$

for small enough $h < h_0$. Thus, for small enough h^* we have in addition that

$$\rho\left(e^{\alpha_k A^d_{i_k} h^*} \Pi_{i_k} \cdots e^{\alpha_0 A^d_{i_0} h^*} \Pi_{i_0} \right) < 1$$

holds and consequently the periodic switching signal σ with the period h^*, defined by (36) stabilizes the system (10). □

Remark 18: If P is not diagonalizable, then (34) does not hold and the claim of the theorem is not true in general.

Note that the given sequences α_j and i_j are used to define the periodic switching signal (36) where α_j indicates the duty-cycle of the subsystem i_j within one period. Thus $\alpha_j = 0$ has to be excluded.

Note further, that S is in the convex hull of the matrices A_i if $\Pi_i = I$ for all i. This yields the classical result for the stabilization of switched ODEs [15] as a special case of Theorem 17.

Corollary 19: Consider the switched ODE (33). If there exists a Hurwitz matrix in the convex hull of the matrices $A_1, \ldots, A_m$, then (33) is stabilizable via a time-dependent periodic signal with sufficiently small period.

Proof: We check the conditions of Theorem 17 for system (33). For any system (33) we can take $k \leq m$ and $S = \sum_{i=1}^{k} \alpha_i A^d_i$ is simply a convex combination of A^d_i without loss of generality. According to the assumptions S can be chosen to be Hurwitz for suitable α_i, $i = 1, \ldots, k$.

Further we have $P = I$ and $\mathrm{Spec}(I + Sh) = 1 + \mathrm{Spec}(S)h$. We can find some $c > 0$ such that $\rho(I + Sh) < 1 - ch$ for $h < h_0$, where h_0 is small enough. Thus Theorem 17 implies that (33) is stabilizable via a periodic switching signal with sufficiently small period. □

Remark 20: A necessary condition for (35) to hold is the requirement $\rho(\Pi_{i_k} \cdots \Pi_{i_1}) \leq 1$. Otherwise continuity of the spectral radius implies $\rho(\Pi_{i_k} \cdots \Pi_{i_1} + Sh) > 1$ for small h. On the other hand, $\rho(\Pi_{i_k} \cdots \Pi_{i_1}) < 1$ automatically implies (35) for small enough h due to continuity of the spectral radius, see Section V-B for more details. Thus, the continuous ("slow") dynamics play an important role in the stabilization of (10) when $\rho(\Pi_{i_k} \cdots \Pi_{i_1}) = 1$.

Remark 21: In [19], [20] averaging techniques are studied for switched DAEs. Assuming commutativity of the consistency projectors the trajectory of the averaged system (suitably defined) can be approximated in a pointwise sense by fast switching of the DAE.

If we assume that the averaged system is exponentially stable, then one might hope that by this approximating property a fast switching signal for the DAE can be constructed that is also stabilizing. Namely, if the evolution operator of the averaged system has norm below 1 for some $t^* > 0$, then with some serendipity the evolution operator of the switched DAE with fast switching also has norm smaller 1 at t^* and this switching signal could then be extended with period t^*.

For this approach to work it would be necessary to study the error in the approximation of the evolution operators. Something which should not be too hard using the results in [19], [20], but which is not yet done.

Without an exact formulation of this approach it is hard to judge how close it is related to the design of a stabilizer by means of fast switching, developed in Section V. On the other hand, the advantage of an averaging approach is that the requirement of commuting consistency projectors is weaker than the requirement of commuting vector fields that we study in Sections IV-B and VI.

We finish this subsection with an illustrating example.

Example 22: Consider the switched system which consists of two constituent systems. The first of these is given by

$$\Pi_1 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad A^d_1 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{37}$$

and the second system is defined by

$$\dot{x} = \begin{pmatrix} 3 & 4 \\ 0 & -1 \end{pmatrix} x. \tag{38}$$

Both these systems are unstable. We now apply Theorem 17 to prove that this system is stabilizable via time-dependent switching. Since the second system is an ODE, we have

$$\Pi_2 = I, \quad A^d_2 = \begin{pmatrix} 3 & 4 \\ 0 & -1 \end{pmatrix}.$$

We compute the matrices P and S from Theorem 17 as

$$P = \Pi_1 \Pi_2 = \Pi_1.$$

Then P is a diagonal operator and

$$S = \alpha A^d_1 \Pi_2 + (1-\alpha) \Pi_1 A^d_2 = \begin{pmatrix} 0 & 0 \\ 0 & 2\alpha - 1 \end{pmatrix}.$$

Then

$$\rho(P + Sh) = \rho\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 + (2\alpha - 1)h \end{pmatrix} \right) = |1 + (2\alpha - 1)h|$$

and taking any $\alpha \in (0, (1/2))$, we see that (35) holds true.

According to Theorem 17 the system under consideration is stabilizable via a time-dependent switching signal.

B. Stabilization via Projections

In this section we investigate under which conditions the consistency projectors by themselves (i.e., the "fast" dynamics) can introduce a stabilizing effect.

To this end we need the notion of a semigroup generated by a set of matrices. Let $\mathcal{J} \neq \emptyset$ be an index set and consider a family of matrices $S := \{S_i : i \in \mathcal{J}\} \subset \mathbb{R}^{n\times n}$. The set of all possible finite products of matrices in S, denoted by $\mathrm{Pr}(S)$, is defined by

$$\mathrm{Pr}(S) := \{S_{i_1}\cdots S_{i_k} : k \in \mathbb{N}, i_1,\ldots,i_k \in \mathcal{J}\}.$$

Define the set of all consistency projections of system (10) by $S_\Pi := \{\Pi_1,\ldots,\Pi_m\}$.

Definition 23: We call (10) stabilizable via projections, if there exists an operator $R \in \mathrm{Pr}(S_\Pi)$ with $\rho(R) < 1$.

Remark 24: We note that stabilizability via projections imposes strong conditions on the set of consistency projections S_Π. If the images of the consistency projectors Π_i have nontrivial intersections, then stabilizability via projections is not possible. Indeed, if $x \neq 0$ satisfies

$$x \in \bigcap_{i=1}^m \mathrm{im}\,\Pi_i$$

then $Rx = x$ for all $R \in \mathrm{Pr}(S_\Pi)$.

Proposition 25: If (10) is stabilizable via projections, then for all $\tau > 0$ and all $\delta \in (0,1)$ there exists a τ-periodic stabilizing switching signal σ_τ for the system (10), so that the trajectories of (10) satisfy

$$|\varphi(\tau, x_0, \sigma_\tau)| \le \delta|x_0|, \quad \forall x_0 \in \mathbb{R}^n. \tag{39}$$

Proof: Let (10) be stabilizable via projections. Then there exist $k > 0$ and $i_1,\ldots,i_k \in \{1,\ldots,m\}$ so that $\rho(\Pi_{i_k}\cdots\Pi_{i_1}) < 1$.

Take any $h > 0$ and define σ_h on $[0, kh)$ by

$$\sigma_h(t) = i_j, \quad t \in [(j-1)h, jh), \quad j = 1,\ldots,k.$$

Extend σ_h periodically to $[0,\infty)$ by setting $\sigma_h(t) := \sigma_h(t - kh)$ for all $t \ge kh$.

For $\ell \in \mathbb{N}$ the solution (15) of system (10) corresponding to σ_h and the initial condition x_0 satisfies

$$\varphi(\ell kh, x_0, \sigma_h) = \left(e^{A^d_{i_k}h}\Pi_{i_k}\cdots e^{A^d_{i_1}h}\Pi_{i_1}\right)^\ell x_0.$$

As by assumption $\rho(\Pi_{i_k}\cdots\Pi_{i_1}) < 1$, we obtain using the continuity of the spectral radius that there exists $h_0 > 0$ small enough so that for all $h \in (0, h_0)$ we have

$$\rho\left(e^{A^d_{i_k}h}\Pi_{i_k}\cdots e^{A^d_{i_1}h}\Pi_{i_1}\right) < 1.$$

Now it is clear, that for all $\delta > 0$ and all $\tau > 0$ there exists $h > 0$ so that inequality (39) holds under the switching signal σ_h. □

Remark 26: It can be shown that stabilizability of (10) via projections is equivalent to the stabilizability of the discrete-time switched system

$$x(t+1) = \Pi_{\sigma(t)}x(t) \tag{40}$$

via a switching signal $\sigma : \mathbb{N} \to \{1,\ldots,m\}$. Indeed, system (40) is stabilizable if and only if the joint subradius $\check\rho(S_\Pi)$ of the set S_Π is less than 1, see [30]. Define $S^k_\Pi := \{M_1\cdots M_k : M_j \in S_\Pi, j = 1,\ldots,k\}$. It is known [30, Th. 1.1, p.14] that

$$\check\rho(S_\Pi) = \lim_{k\to\infty}\inf\left\{(\rho(M))^{\frac{1}{k}} : M \in S^k_\Pi\right\}.$$

In other words, the condition $\check\rho(S_\Pi) < 1$ is equivalent to the existence of a $k > 0$ such that for some $M \in S^k_\Pi$ we have $\rho(M) < 1$. This is precisely stabilizability of (10) via projections.

Assume that the dimensions of the consistency spaces $\dim C_{(E_i,A_i)} = 1$ for all $i = 1,\ldots,m$. From (5) we have $J_i \in \mathbb{R}$ and $A^d_i = J_i\Pi_i$, $e^{A^d_i t}\Pi_i = e^{J_i t}\Pi_i$.

The solution (15) of (10) with $x(0^-) = x_0$ reduces to

$$\varphi(t, x_0, \sigma) = \exp\left(\sum_{k=1}^m J_k I_{\sigma,k}(0,t)\right)\Pi_{r_i}\cdots\Pi_{r_1}x_0. \tag{41}$$

Below we show, that stabilizability via projections is the only nontrivial possibility to stabilize the switched system (10) with one-dimensional consistency spaces of the constituent systems.

Proposition 27: Let $\dim C_{(E_i,A_i)} = 1$ for all $i = 1,\ldots,m$. Then (10) is stabilizable via a time-dependent switching signal if and only if one of the following conditions holds:

(a) One of the constituent systems of (10) is asymptotically stable (i.e., the corresponding $J_i < 0$).
(b) (10) is stabilizable via projections.

Proof: Sufficiency of the conditions (i) or (ii) is clear. Assume that neither (i) nor (ii) hold. Then for all $k > 0$ and for all $M \in S^k_\Pi$ it follows that $\|M\| \ge 1$. As all $J_i \ge 0$ we see from (41) that for all $\sigma \in \mathcal{S}$, all $t \ge 0$, we have $\|\Phi^{DAE}_\sigma(t)\| \ge 1$, and according to Lemma 6 there is no stabilizing switching signal.

VI. Stabilization of DAEs With Commuting Vector Fields

In this section, we extend our investigation of switched DAEs with commuting flows and derive a further simplification of the stabilization problem. In particular, we are going to show for such systems, that stabilizability of (10) on the whole space $\mathbb{R}^n$ is equivalent to stabilizability of (10) on the intersection of the consistency spaces $C := C_1 \cap \ldots \cap C_m$.

Denote by $t \mapsto \Phi_i(t) = e^{A^d_i t}\Pi_i$ the flow map of the i-th constituent system of (10). Throughout this section we assume that the flow maps of distinct constituent systems of (10) commute, i.e., $[\Phi_i(t), \Phi_j(s)] = 0$ for all $i \neq j$ and all $t, s \ge 0$.

Recall the following result from [4, Th. 7, Lemma 9].

Theorem 28: Let $i \neq j$ and A_i and A_j be invertible. The following conditions are equivalent:

(a) $[\Phi_i(t), \Phi_j(s)] = 0$ for all $t, s \ge 0$, $i, j \in \mathcal{I}$,
(b) $[A^d_i, A^d_j] = 0$ for all $i, j \in \mathcal{I}$.

If one (and hence both) of these conditions hold, then we have $[\Pi_i, \Pi_j] = 0$ as well as $[A^d_i, \Pi_j] = 0$ for all $i, j \in \mathcal{I}$.

We will exploit the following simple fact, which may also be found in [20, Lemma 1].

Lemma 29: Let $P_1, \ldots, P_k$ be projections. If $[P_i, P_j] = 0$ for all i, j, then $P = P_1 \cdots P_k$ is also a projection and $\operatorname{Im} P = \operatorname{Im} P_1 \cap \ldots \cap \operatorname{Im} P_k$.

Proof: $P^2 = P_1^2 \cdots P_k^2 = P$, thus P is a projection.

Since $P = P_1 \cdots P_k$, we have $\operatorname{Im} P \subset \operatorname{Im} P_1$, and since $[P_i, P_j] = 0$ for all i, j, we see that $\operatorname{Im} P \subset \operatorname{Im} P_j$ for all $j = 1, \ldots, k$. Since $Px = x$ for all $x \in \operatorname{Im} P_1 \cap \ldots \cap \operatorname{Im} P_k$, we have $\operatorname{Im} P = \operatorname{Im} P_1 \cap \ldots \cap \operatorname{Im} P_k$. □

If $[\Phi_i(t), \Phi_j(s)] = 0$ for all $i \neq j$ and all $t, s \geq 0$ we can reorder the factors in (15) with the help of Theorem 28 to obtain a representation of the solution of system (10). Starting at $x(0^-) = x_0$ and using $\sigma \in \mathcal{S}$ it takes for all $t \geq 0$ the form

$$\varphi(t, x_0, \sigma) = \Phi_\sigma^{DAE}(t) x_0 \tag{42}$$

$$= \exp\left(\sum_{i=1}^m A_i^d I_{\sigma,i}(0,t)\right) \prod_{i=1}^m (\Pi_i)^{\operatorname{sign} I_{\sigma,i}(0,t)} x_0 \tag{43}$$

where $\operatorname{sign}(0) = 0$, $\operatorname{sign}(r) = 1$, $r > 0$. This remark has the following implication for the dynamics on intersection of the consistency spaces $C_i, i \in \mathcal{I}_\sigma$.

Lemma 30: Assume the commutativity assumption (i) of Theorem 28 holds. For $\sigma \in \mathcal{S}$ denote

$$C_\sigma := \operatorname{Im} \prod_{i \in \mathcal{I}_\sigma} \Pi_i = \bigcap_{i \in \mathcal{I}_\sigma} \operatorname{Im} \Pi_i. \tag{44}$$

Then C_σ is invariant under $\Phi_i(t)$ and $e^{A_i^d t}$ for all $t \geq 0$, $i \in \mathcal{I}_\sigma$.

Proof: By Theorem 28 we have for all $x_0 \in C_\sigma$, $i \in \mathcal{I}_\sigma$ that

$$e^{A_i^d t} x_0 = \Phi_i(t) x_0 = e^{A_i^d t} \prod_{j \in \mathcal{I}_\sigma} \Pi_j x_0 = \prod_{j \in \mathcal{I}_\sigma} \Pi_j e^{A_i^d t} x_0 \in C_\sigma.$$

This implies the assertion. □

For $\sigma \in \mathcal{S}$ and $i \in \mathcal{I}_\sigma$ let $A_{i,\sigma}^d$ denote the operator induced by A_i^d on the space C_σ. Note that if $[A_i^d, A_j^d] = 0$, $i, j \in \mathcal{I}_\sigma$ then also $[A_{i,\sigma}^d, A_{j,\sigma}^d] = 0$ on C_σ as C_σ is a common invariant subspace of A_i^d, A_j^d.

The following theorem gives a criterion of stabilizability of (10) under switching signal σ.

Theorem 31: Assume the commutativity assumption (i) of Theorem 28 holds. Let $\sigma \in \mathcal{S}$ and C_σ be given by (44). Then (10) is asymptotically stable under the time-dependent switching signal σ if and only if the system

$$\dot{x}(t) = A_{\sigma(t),\sigma}^d x(t), \quad x_0 \in C_\sigma \tag{45}$$

is asymptotically stable under the same σ.

In particular, if $C_\sigma = \{0\}$, then all solutions of (10) reach zero in finite time under the time-dependent switching signal σ.

Proof: Take $t^* > 0$ sufficiently large so that $I_{\sigma,i}(0, t^*) \neq 0$ for all $i \in \mathcal{I}_\sigma$. For $t \geq t^*$ the solution (42) takes the form

$$\varphi(t, x_0, \sigma) = \Phi_\sigma^{DAE}(t) x_0 = \exp\left(\sum_{i=1}^m A_i^d I_{\sigma,i}(0,t)\right) \prod_{j \in \mathcal{I}_\sigma} \Pi_j x_0. \tag{46}$$

Since $[\Pi_i, \Pi_j] = 0$ for all i, j, it follows from Lemma 29 that $\Pi := \prod_{j \in \mathcal{I}_\sigma} \Pi_j$ is a projection with $\operatorname{Im} \Pi = C_\sigma$. Thus, $\Pi x_0 \in C_\sigma$ and Lemma 30 imply $\varphi(t, x_0, \sigma) \in C_\sigma$ for all $t \geq t^*$.

Moreover, since C_σ is invariant under $e^{A_i^d s}$ for all $s \geq 0$, $i \in \mathcal{I}_\sigma$, all solutions of (10) for $x_0 \in C_\sigma$ are also solutions of the switched ODE

$$\dot{x}(t) = A_{\sigma(t),\sigma}^d x(t). \tag{47}$$

Clearly, in (46) the solution $\varphi(t, x_0, \sigma) \to 0$ for all $x_0 \in \mathbb{R}^n$ if and only if $\exp(\sum_{i=1}^m A_i^d I_{\sigma,i}(0,t)) y_0 \to 0$ for all $y_0 \in \operatorname{Im} \prod_{j \in \mathcal{I}_\sigma} \Pi_j = C_\sigma$. This is equivalent to the asymptotic stability of (47) on C_σ. □

Remark 32: Under the assumptions of Theorem 31, the stabilizability of (10) is equivalent to the stabilizability of (45) on C, as we may always use a stabilizing signal $\sigma \in \mathcal{S}$ with $\mathcal{I}_\sigma = \mathcal{I}$.

VII. Conclusion

In this paper stabilizability of switched DAE systems is investigated. It is shown that the stabilizability of the switched DAE can be inferred from stabilizability properties of an approximating parameterized switched ODE. In particular, it is required that the approximating switched ODE is stabilizable in a uniform sense such that the same switching signal stabilizes any approximation of the switched DAE up to a certain level of precision ε_0. We provide a counterexample that shows that this type of uniformity is necessary in order to infer stabilizablity of the switched DAE via the approximating switched ODE. For the class of commuting DAEs this condition does not impose any restriction as the uniformity requirement is always satisfied by the approximating switched ODE provided it is stabilizable at all.

We further investigate stabilizability of a switched DAE via fast switching. We provide a sufficient condition which highlights the importance of the interaction between projections and continuous dynamics of the constituent systems for the stabilizability of switched DAEs.

This result can be regarded as a generalization of the well-known theorem, that a switched ODE system is stabilizable if there exists a Hurwitz matrix in a convex hull of the flow matrices of constituent systems. For the special case where the consistency spaces of all subsystems have dimension one, stabilization via projections is the only non-trivial possibility to stabilize the switched DAE.

For our final results we return to the special case of DAEs with commuting vector fields. For this system class the stabilization problem is equivalent to stabilizability of a system on the intersection of the consistency spaces of the constituent systems. On this invariant intersection space the dynamics of the switched DAE can be represented by those of a switched ODE. Again, this reduction simplifies the stabilization problem greatly.

The relations between stabilizability of switched DAEs and that of switched ODEs do not only simplify their analysis but also allow to employ many conditions for the stabilization of switched ODEs, obtained in the past. Of particular interest for future work may be the quadratic (state feedback) stabilization of DAEs and its connection to the results obtained for switched ODEs.

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