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Lyapunov small-gain theorems for networks of not necessarily ISS hybrid systems[☆]

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ABSTRACT

We prove a novel Lyapunov-based small-gain theorem for networks composed of $n \geq 2$ hybrid subsystems which are not necessarily input-to-state stable. This result unifies and extends several small-gain theorems for hybrid and impulsive systems proposed in the last few years. We also show how average dwell-time (ADT) clocks and reverse ADT clocks can be used to modify the ISS Lyapunov functions for subsystems and to enlarge the applicability of the derived small-gain theorems.

1. Introduction

The study of interconnections plays a significant role in the system theory, as it allows one to establish stability for a complex system based on properties of its less complex components. In this context, small-gain theorems prove to be useful and general in analyzing feedback interconnections, which are ubiquitous in the control literature. An overview of classical small-gain theorems involving input–output gains of linear systems can be found in Desoer and Vidyasagar (2009). In Hill (1991) and Mareels and Hill (1992), the small-gain technique was extended to nonlinear feedback systems within the input–output context. The next peak in the stability analysis of interconnections was reached based on the input-to-state stability (ISS) framework proposed in Sontag (1989), which unified the notions of internal and external stability. Nonlinear small-gain theorems for general feedback interconnections of two ISS systems were introduced in Jiang, Mareels, and Wang (1996) and Jiang, Teel, and Praly (1994). Their generalization to networks composed of $n \geq 2$ ISS systems were reported in Dashkovskiy, Rüffer, and Wirth (2007, 2010), with several variations summarized in Dashkovskiy, Efimov, and Sontag (2011).

The results described above have been developed for continuous-time systems (i.e., ordinary differential equations). In the discrete-time context, small-gain theorems for general feedback interconnections of two ISS systems were established in Jiang and Wang (2001) and Laila and Nešić (2003), and their generalization to networks composed of $n \geq 2$ ISS systems can be found in Liu, Jiang, and Hill (2012). However, in modeling real-world phenomena one often has to consider interactions between continuous and discrete dynamics. A general framework for modeling such behaviors is the hybrid systems theory (Goebel, Sanfelice, & Teel, 2012; Haddad, Chellaboina, & Nersesov, 2006). In this work, we adopt the hybrid system model in Goebel et al. (2012), which proves to be natural and general from the viewpoint of Lyapunov stability theory (Cai, Teel, & Goebel, 2007, 2008). The notions of input-to-state stability and ISS Lyapunov functions were extended for this class of hybrid systems in Cai and Teel (2009).

Due to their interactive nature, many hybrid systems can be inherently modeled as feedback interconnections (Liberzon, Nešić, & Teel, 2014 Section V). During recent years, great efforts have been devoted to the development of small-gain theorems for interconnected hybrid systems. Trajectory-based small-gain theorems for interconnections of two hybrid systems were reported in Dashkovskiy and Kosmykov (2013), Karafyllis and Jiang (2007) and Nešić and Liberzon (2005), while Lyapunov-based formulations were proposed in Liberzon and Nešić (2006), Liberzon et al. (2014) and Nešić and Teel (2008). Some of these results were extended to networks composed of $n \geq 2$ ISS hybrid systems in Dashkovskiy and Kosmykov (2013).

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A more challenging problem is the study of hybrid systems in which either the continuous or the discrete dynamics is destabilizing (non-ISS). In this case, input-to-state stability is usually achieved under restrictions on the frequency of discrete events, such as dwell-time (Morse, 1996), average dwell-time (ADT) (Hespanha & Morse, 1999) and reverse average dwell-time (RADT) (Hespanha, Liberzon, & Teel, 2008). For interconnections of such hybrid subsystems, the small-gain theorems established in Dashkovskiy and Kosmykov (2013) and Liberzon et al. (2014) cannot be applied directly. The results of Liberzon et al. (2014) show that one can modify the non-ISS dynamics in subsystems by first adding auxiliary clocks and then constructing ISS Lyapunov functions for the augmented subsystems that decrease both during flow and at jumps. One advantage of this method is that it can be applied even if the non-ISS dynamics are of different types (i.e., if in some subsystems the continuous dynamics are non-ISS, and in some other ones the discrete dynamics are non-ISS). However, such modifications will lead to enlarged Lyapunov gains of subsystems, and hence make the small-gain condition more restrictive.

Another type of small-gain theorems was proposed in Dashkovskiy, Kosmykov, Mironchenko, and Naujok (2012) and Dashkovskiy and Mironchenko (2013b) for interconnected impulsive systems with continuous or discrete non-ISS dynamics. The first step in this method is to construct a candidate exponential ISS Lyapunov function for the interconnection. Provided that the non-ISS dynamics of subsystems are of the same type (i.e., when either the continuous dynamics of all subsystems or the discrete dynamics of all subsystems are ISS), the candidate exponential ISS Lyapunov function can be used to establish input-to-state stability of the interconnection under suitable ADT/RADT conditions. Compared with the previous method, this one does not require modifications of subsystems, and hence preserves the Lyapunov gains and validity of small-gain conditions. However, this method has been developed only for impulsive systems and requires candidate exponential ISS Lyapunov functions for subsystems. Moreover, it cannot be applied to interconnections of subsystems with different types of non-ISS dynamics.

In this paper, we unify the two methods above. In Section 2, we introduce the modeling framework and main definitions, followed by a Lyapunov-based sufficient condition for ISS of hybrid systems with continuous or discrete non-ISS dynamics. In Section 3, we establish a general small-gain theorem for an interconnection of $n \geq 2$ hybrid subsystems by constructing a candidate ISS Lyapunov function for the interconnection, which generalizes the Lyapunov-based small-gain theorems from Dashkovskiy and Kosmykov (2013), Dashkovskiy et al. (2012), Dashkovskiy and Mironchenko (2013b), Liberzon et al. (2014) and Nešić and Teel (2008). We also derive several implications of the general result, in particular, a small-gain theorem for interconnections of subsystems with the same type of non-ISS dynamics and also candidate exponential ISS Lyapunov functions with linear Lyapunov gains. In Section 4, we propose a version of the approach of modifying ISS Lyapunov functions for subsystems from Liberzon et al. (2014), in which fewer subsystems are affected (and hence fewer Lyapunov gains are enlarged). In Section 5, we summarize the results of this work as a unified method for establishing ISS of interconnections of hybrid subsystems and conclude the paper with an outlook on future research.

A preliminary and shortened version of the paper has been presented at the 21st International Symposium on Mathematical Theory of Networks and Systems (Mironchenko, Yang, & Liberzon, 2014).

2. Framework for hybrid systems

Let $\mathbb{R}_+ := [0, \infty)$ and $\mathbb{N} := \{0, 1, 2, \ldots\}$. For a vector $x \in \mathbb{R}^N$, denote by $|x|$ its Euclidean norm, and by $|x|_{\mathcal{A}} := \inf_{y\in\mathcal{A}}|x - y|$ its Euclidean distance to a set $\mathcal{A} \subset \mathbb{R}^N$. For n vectors $x_1, \ldots, x_n$, denote by $(x_1, \ldots, x_n) := (x_1^\top, \ldots, x_n^\top)^\top$ their concatenation. For two vectors $x, y \in \mathbb{R}^n$, we say that $x \geq y$ and $x > y$ if the corresponding inequality holds in all scalar components, and that $x \not\geq y$ if there is at least one scalar component i in which $x_i < y_i$. For a set $\mathcal{A}$, denote by $\overline{\mathcal{A}}$ and $\operatorname{int} \mathcal{A}$ its closure and interior, respectively.

Denote by id the identity function. A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{PD}$ if it is continuous and positive-definite (i.e., $\alpha(r) = 0 \Leftrightarrow r = 0$); it is of class $\mathcal{K}$ if $\alpha \in \mathcal{PD}$ and is strictly increasing; it is of class $\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and is unbounded. A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{L}$ if it is continuous, strictly decreasing and $\lim_{t\to\infty}\gamma(t) = 0$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}$ for each fixed t and $\beta(r, \cdot) \in \mathcal{L}$ for each fixed $r > 0$.

Motivated by Cai and Teel (2009), a hybrid system is modeled as the combination of a continuous flow and discrete jumps

$$
\begin{aligned}
\dot{x} &\in F(x, u), \qquad (x, u) \in \mathcal{C},\\
x^+ &\in G(x, u), \qquad (x, u) \in \mathcal{D},
\end{aligned}
\tag{1}
$$

where $x \in \mathcal{X} \subset \mathbb{R}^N$ is the state, $u \in \mathcal{U} \subset \mathbb{R}^M$ is the input, $\mathcal{C} \subset \mathcal{X}\times\mathcal{U}$ is the flow set, $\mathcal{D} \subset \mathcal{X} \times \mathcal{U}$ is the jump set, $F : \mathcal{C} \rightrightarrows \mathbb{R}^N$ is the flow map (here by $\rightrightarrows$ we mean that F is a set-valued function, which maps each element of $\mathcal{C}$ to a subset of $\mathbb{R}^N$), and $G : \mathcal{D} \rightrightarrows \mathcal{X}$ is the jump map. (In this model, the dynamics of (1) is continuous in $\mathcal{C} \setminus \mathcal{D}$ and discrete in $\mathcal{D} \setminus \mathcal{C}$. In $\mathcal{C} \cap \mathcal{D}$, it can be either continuous or discrete.) The hybrid system (1) is fully characterized by its *data* $\mathcal{H} := (F, G, \mathcal{C}, \mathcal{D}, \mathcal{X}, \mathcal{U})$.

Solutions of (1) are defined on hybrid time domains. A set $E \subset \mathbb{R}_+ \times \mathbb{N}$ is called a *compact hybrid time domain* if $E = \bigcup_{j=0}^{J}([t_j, t_{j+1}], j)$ for some finite sequence of times $0 = t_0 \leq t_1 \leq \cdots \leq t_{J+1}$. It is a *hybrid time domain* if $E\cap([0, T]\times\{0, 1, \ldots, J\})$ is a compact hybrid time domain for each $(T, J) \in E$. On a hybrid time domain, there is a natural ordering of points, that is, $(s, k) \preceq (t, j)$ if $s + k \leq t + j$, and $(s, k) \prec (t, j)$ if $s + k < t + j$.

Functions defined on hybrid time domains are called *hybrid signals*. A hybrid signal $x : \operatorname{dom} x \to \mathcal{X}$ (defined on the hybrid time domain $\operatorname{dom} x$) is a *hybrid arc* if $x(\cdot, j)$ is locally absolutely continuous for each j. A hybrid signal $u : \operatorname{dom} u \to \mathcal{U}$ is a *hybrid input* if $u(\cdot, j)$ is Lebesgue measurable and locally essentially bounded for each j. A hybrid arc $x : \operatorname{dom} x \to \mathcal{X}$ and a hybrid input $u : \operatorname{dom} u \to \mathcal{U}$ form a *solution pair* (x, u) of (1) if

- $\operatorname{dom} x = \operatorname{dom} u$ and $(x(0, 0), u(0, 0)) \in \overline{\mathcal{C}} \cup \mathcal{D}$, where $x(t, j)$ denotes the state of the hybrid system at hybrid time (t, j), that is, at time t and after j jumps;
- for each $j \in \mathbb{N}$, it holds that $(x(t, j), u(t, j)) \in \mathcal{C}$ for all $t \in \operatorname{int} I_j$ and $\dot{x}(t, j) \in F(x(t, j), u(t, j))$ for almost all $t \in I_j$, where $I_j := \{t : (t, j) \in \operatorname{dom} x\}$;
- for each $(t, j) \in \operatorname{dom} x$ such that $(t, j + 1) \in \operatorname{dom} x$, it holds that $(x(t, j), u(t, j)) \in \mathcal{D}$ and $x(t, j + 1) \in G(x(t, j), u(t, j))$.

With proper assumptions on the data $\mathcal{H}$, one can establish local existence of solutions, which are not necessarily unique (see, e.g. Goebel et al., 2012 Proposition 2.10). A solution pair (x, u) is *maximal* if it cannot be extended, and *complete* if $\operatorname{dom} x$ is unbounded. In this paper, we only consider maximal solution pairs.

Following Cai and Teel (2009), the essential supremum norm of a hybrid signal u up to a hybrid time (t, j) is defined by

$$
\|u\|_{(t,j)} := \max\left\{ \operatorname*{esssup}_{\substack{(s,k)\in\operatorname{dom} u,\\ (s,k)\preceq(t,j)}} |u(s, k)|,\ \sup_{\substack{(s,k)\in J(u),\\ (s,k)\preceq(t,j)}} |u(s, k)| \right\},
$$

where $J(x) := \{(s, k) \in \operatorname{dom} u : (s, k + 1) \in \operatorname{dom} u\}$ is the set of jump times. In particular, the set of measure 0 of hybrid times that are ignored in computing the essential supremum norm cannot contain any jump time.

For a locally Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}$, its *Dini derivative* at $x \in \mathbb{R}^n$ in the direction $y \in \mathbb{R}^n$ is given by

$$\dot{V}(x; y) := \overline{\lim_{h\searrow 0}} \frac{V(x+hy) - V(x)}{h},$$

where $\overline{\lim}$ denotes the limit superior.

In this paper, we study input-to-state stability (ISS) properties of the hybrid system (1) using ISS Lyapunov functions. Let $\mathcal{A} \subset \mathcal{X}$ be a compact set.

Definition 1. Following Liberzon et al. (2014), we say that a set of solution pairs $\mathcal{S}$ of (1) is *pre-input-to-state stable (pre-ISS) w.r.t.* $\mathcal{A}$ if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for all $(x, u) \in \mathcal{S}$,

$$|x(t,j)|_{\mathcal{A}} \le \max\{\beta(|x(0,0)|_{\mathcal{A}}, t+j),\ \gamma(\|u\|_{(t,j)})\} \tag{2}$$

for all $(t, j) \in \operatorname{dom} x$. If $\mathcal{S}$ contains all solution pairs of (1), then we say that (1) is *pre-ISS w.r.t.* $\mathcal{A}$. In addition, if all solution pairs are complete then we say that (1) is *ISS w.r.t.* $\mathcal{A}$.

Remark 1. If (2) holds with $\gamma \equiv 0$, then the set $\mathcal{S}$ is *globally pre-asymptotically stable (pre-GAS)*, which implies that all complete solution pairs in $\mathcal{S}$ converge to $\mathcal{A}$. In addition, if all solution pairs in $\mathcal{S}$ are complete then it is *globally asymptotically stable (GAS)* (Liberzon et al., 2014).

Remark 2. In Cai and Teel (2009), ISS of hybrid systems is defined in terms of class $\mathcal{KLL}$ functions and without requiring all solution pairs to be complete, which is equivalent to our definition of pre-ISS with $\mathcal{KL}$ functions (Cai et al., 2007, Lemma 6.1).

Definition 2. For the hybrid system (1), a function $V : \mathcal{X} \to \mathbb{R}_+$ is a *candidate ISS Lyapunov function w.r.t.* $\mathcal{A}$ if it is locally Lipschitz outside $\mathcal{A}$,[1] and

1. there exist functions $\psi_1, \psi_2 \in \mathcal{K}_\infty$ such that
$$\psi_1(|x|_{\mathcal{A}}) \le V(x) \le \psi_2(|x|_{\mathcal{A}}) \qquad \forall x \in \mathcal{X}; \tag{3}$$
2. there exist a gain function $\chi \in \mathcal{K}$ and a continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}$ with $\varphi(0) = 0$ such that for all $(x, u) \in \mathcal{C}$ with $x \notin \mathcal{A}$,
$$V(x) \ge \chi(|u|) \Longrightarrow \dot{V}(x; y) \le -\varphi(V(x)), \forall y \in F(x, u); \tag{4}$$
3. there is a function $\alpha \in \mathcal{K}$ such that for all $(x, u) \in \mathcal{D}$,[2]
$$V(x) \ge \chi(|u|) \Longrightarrow V(y) \le \alpha(V(x)), \forall y \in G(x, u). \tag{5}$$

In addition, if there exist two constants $c, d \in \mathbb{R}$ so that

$$\varphi(r) \equiv cr, \quad \alpha(r) \equiv e^{-d} r \tag{6}$$

in (4) and (5), then V is a *candidate exponential ISS Lyapunov function w.r.t.* $\mathcal{A}$ with *rate coefficients* c, d.

The next lemma gives an alternative characterization of the candidate ISS Lyapunov function, which will be useful in formulating the small-gain theorems in Section 3.

Lemma 1. *For the hybrid system (1), a function $V : \mathcal{X} \to \mathbb{R}_+$ is a candidate ISS Lyapunov function w.r.t. $\mathcal{A}$ if and only if it is locally Lipschitz outside $\mathcal{A}$, and*

1. *there exist functions $\psi_1, \psi_2 \in \mathcal{K}_\infty$ such that (3) holds;*
2. *there exist a gain function $\bar{\chi} \in \mathcal{K}$ and a continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}$ with $\varphi(0) = 0$ such that for all $(x, u) \in \mathcal{C}$ with $x \notin \mathcal{A}$,*
$$V(x) \ge \bar{\chi}(|u|) \Longrightarrow \dot{V}(x; y) \le -\varphi(V(x)), \forall y \in F(x, u); \tag{7}$$
3. *there is a function $\alpha \in \mathcal{K}$ such that for all $(x, u) \in \mathcal{D}$,*
$$V(y) \le \max\{\alpha(V(x)),\ \bar{\chi}(|u|)\} \quad \forall\, y \in G(x, u). \tag{8}$$

Proof. The proof is along the lines of the proof of Dashkovskiy and Mironchenko (2013b, Proposition 1) for ISS Lyapunov functions for impulsive systems, and is omitted here. □

Exponential ISS Lyapunov functions can be characterized in a similar way. Note that the functions χ in Definition 2 and $\bar{\chi}$ in Lemma 1 are different in general.

The notion of candidate ISS Lyapunov function is defined to characterize the effect of destabilizing (non-ISS) dynamics in a hybrid system. In Definition 2, it is not required that $\varphi \in \mathcal{PD}$ or $\alpha < \text{id}$ on $(0, \infty)$, that is, V does not necessarily decrease along solutions of the hybrid system (1). If both of these conditions hold, then V becomes an *ISS Lyapunov function*, and similar analysis to the proof of Cai and Teel (2009, Proposition 2.7) can be used to show that (1) is pre-ISS (note that ISS in Cai and Teel, 2009 means pre-ISS in this paper; see Remark 2). Moreover, if only one of them holds,[3] we are still able to establish ISS for the sets of solution pairs satisfying suitable conditions on the density of jumps (i.e., the number of jumps per unit interval of continuous time).

Proposition 1. *Let V be a candidate exponential ISS Lyapunov function w.r.t. $\mathcal{A}$ for the hybrid system (1) with rate coefficients c, d. For arbitrary constants $\eta, \lambda, \mu > 0$, denote by $\mathcal{S}[\eta, \lambda, \mu]$ the set of solution pairs (x, u) so that*

$$-(d-\eta)(j-k) - (c-\lambda)(t-s) \le \mu \tag{9}$$

for all $(s, k) \preceq (t, j)$ in the hybrid time domain $\operatorname{dom} x$. Then $\mathcal{S}[\eta, \lambda, \mu]$ is pre-ISS w.r.t. $\mathcal{A}$.

Proof. The proof is along the lines of the proof of Hespanha et al. (2008, Theorem 1) for ISS of impulsive systems. Consider an arbitrary solution pair $(x, u) \in \mathcal{S}[\eta, \lambda, \mu]$. Let the function χ be as in (4) and (5). For all $(t_0, j_0) \preceq (t_1, j_1)$ in $\operatorname{dom} x$, if

$$V(x(s,k)) \ge \chi(\|u\|_{(s,k)}) \tag{10}$$

for all $(s, k) \in \operatorname{dom} x$ such that $(t_0, j_0) \preceq (s, k) \preceq (t_1, j_1)$, then (4)–(6) imply that

$$\begin{aligned} V(x(t_1, j_1)) &\le e^{-d(j_1-j_0)-c(t_1-t_0)} V(x(t_0, j_0)) \\ &\le e^{-\eta(j_1-j_0)-\lambda(t_1-t_0)+\mu} V(x(t_0, j_0)), \end{aligned} \tag{11}$$

where the last inequality follows from (9). Now consider an arbitrary $(t, j) \in \operatorname{dom} x$. If (10) holds for all $(s, k) \preceq (t, j)$ in $\operatorname{dom} x$, then (11), together with (3), implies that

$$|x(t,j)|_{\mathcal{A}} \le \beta(|x(0,0)|_{\mathcal{A}}, t+j) \tag{12}$$

with the function $\beta \in \mathcal{KL}$ defined by

$$\beta(r, l) := \psi_1^{-1}\big(e^{-l\min\{\eta,\lambda\}+\mu}\psi_2(r)\big). \tag{13}$$

[1] The Lipschitz condition here is used to ensure the existence of the Dini derivative in (4), and it can be relaxed to that the function V is locally Lipschitz on an open set containing all $x \notin \mathcal{A}$ such that $(x, u) \in \mathcal{C}$ for some $u \in \mathcal{U}$.

[2] There is no loss of generality in requiring $\alpha \in \mathcal{K}$ instead of $\alpha \in \mathcal{PD}$, as a class $\mathcal{PD}$ function can always be majorized by a class $\mathcal{K}$ one. Meanwhile, $\alpha \in \mathcal{K}$ is needed in establishing the small-gain theorems below, as explained in footnote 4.

[3] Namely, either the continuous or the discrete dynamics taken alone is ISS; see Sontag (1989) and Jiang and Wang (2001) for the definitions of ISS for continuous and discrete dynamics, respectively.

Otherwise, let

$$(t',j') = \operatorname*{argmax}_{\substack{(s,k)\in \operatorname{dom} x,\\ (s,k)\preceq (t,j)}} \{s+k : V(x(s,k)) \le \chi(\|u\|_{(s,k)})\}.$$

Then (10) holds for all $(s,k)\in \operatorname{dom} x$ such that $(t',j') \prec (s,k) \preceq (t,j)$; thus (11) implies that

$$\begin{aligned} V(x(t,j)) &\le e^{-\eta(j-j')-\lambda(t-t')+\mu}\max\{1,\ e^{-d}\}V(x(t',j'))\\ &\le e^{\mu}\max\{1,\ e^{-d}\}\chi(\|u\|_{(t',j')})\\ &\le e^{\mu}\max\{1,\ e^{-d}\}\chi(\|u\|_{(t,j)}),\end{aligned}$$

where the term $\max\{1,\ e^{-d}\}$ is needed if $(t',j'+1)\in \operatorname{dom} x$ with $V(x(t',j')) < \chi(\|u\|_{(t',j')})$ and $V(x(t',j'+1)) > \chi(\|u\|_{(t',j'+1)})$, and the second inequality is due to $\eta,\lambda > 0$. Hence from (3), it follows that

$$|x(t,j)|_{\mathcal{A}} \le \gamma(\|u\|_{(t,j)}) \tag{14}$$

with the function $\gamma \in \mathcal{K}$ defined by

$$\gamma(r) := \psi_1^{-1}\big(e^{\mu}\max\{1,\ e^{-d}\}\chi(r)\big).$$

Combining (12) and (14), we obtain that (2) holds for all $(x,u)\in \mathcal{S}[\eta,\lambda,\mu]$ and all $(t,j)\in \operatorname{dom} x$. □

Remark 3. We observe that, if both $c, d < 0$, then the inequality (9) cannot hold for any complete solution pair, since there is always a large enough t or j such that $\eta j + \lambda t > \mu$. However, it may still hold for solution pairs defined on bounded hybrid time domains. Moreover, if $c > 0 > d$, then the claim of Proposition 1 also holds for $\eta = 0$. The proof remain unchanged except that the last inequality in (11) now becomes

$$\begin{aligned} &e^{-d(j_1-j_0)-c(t_1-t_0)}V(x(t_0,j_0))\\ &\quad\le e^{-\lambda(t_1-t_0)+\mu}V(x(t_0,j_0))\\ &\quad\le e^{(\lambda^2/c-\lambda)(t_1-t_0)-\lambda^2(t_1-t_0)/c+\mu}V(x(t_0,j_0))\\ &\quad\le e^{\lambda d(j_1-j_0)/c-\lambda^2(t_1-t_0)/c+(1+\lambda/c)\mu}V(x(t_0,j_0)),\end{aligned}$$

where the first inequality follows from (9) with $\eta = 0$, and the last one comes from the estimate

$$e^{(\lambda^2/c-\lambda)(t_1-t_0)} = e^{(\lambda/c)(\lambda-c)(t_1-t_0)} \le e^{(\lambda/c)(d(j_1-j_0)+\mu)},$$

and the definition (13) becomes

$$\beta(r,l) := \psi_1^{-1}\big(e^{-l\min\{-\lambda d/c,\ \lambda^2/c\}+(1+\lambda/c)\mu}\psi_2(r)\big).$$

Analogously, if $d > 0 > c$, then the claim of Proposition 1 also holds for $\lambda = 0$.

Remark 4. If $c > 0 \ge d$, then we can divide both sides of (9) by $-(d-\eta) > 0$ to transform it to an average dwell-time (ADT) condition (Hespanha & Morse, 1999). Analogously, if $d > 0 \ge c$, then we can divide both sides of (9) by $-(c-\lambda) > 0$ to transform it to the reverse average dwell-time (RADT) condition (Hespanha et al., 2008).

Given a candidate exponential ISS Lyapunov function with rate coefficients $c > 0$ and/or $d > 0$, we can determine pre-ISS sets of solution pairs via Proposition 1. In the following section, we investigate the formulation of such functions for interconnections of hybrid systems.

3. Interconnections and small-gain theorems

We are interested in the case where the hybrid system (1) is decomposed as

$$\begin{aligned} \dot{x}_i &\in F_i(x,u), \quad i=1,\dots,n, \qquad (x,u)\in\mathcal{C},\\ x_i^+ &\in G_i(x,u), \quad i=1,\dots,n, \qquad (x,u)\in\mathcal{D},\end{aligned} \tag{15}$$

where $x := (x_1,\dots,x_n)\in\mathcal{X}\subset\mathbb{R}^N$ with $x_i\in\mathcal{X}_i\subset\mathbb{R}^{N_i}$ is the state, $u\in\mathcal{U}\subset\mathbb{R}^M$ is the common (external) input, $\mathcal{C} := \mathcal{C}_1\times\cdots\times\mathcal{C}_n\times\mathcal{C}_u$ with $\mathcal{C}_i\subset\mathcal{X}_i$ and $\mathcal{C}_u\subset\mathcal{U}$ is the flow set, $\mathcal{D} := \mathcal{D}_1\times\cdots\times\mathcal{D}_n\times\mathcal{D}_u$ with $\mathcal{D}_i\subset\mathcal{X}_i$ and $\mathcal{D}_u\subset\mathcal{U}$ is the jump set, $F := (F_1,\dots,F_n)$ with $F_i : \mathcal{C}\rightrightarrows\mathbb{R}^{N_i}$ is the flow map, and $G := (G_1,\dots,G_n)$ with $G_i : \mathcal{D}\rightrightarrows\mathcal{X}_i$ is the jump map. The dynamics of x_i is called the ith subsystem of (15) and is denoted by Σ_i. The interconnection (15) is denoted by Σ. For each Σ_i, the states of other subsystems are treated as (internal) inputs.

Many systems with hybrid behaviors can be naturally transformed into the form of (15). As demonstrated in Liberzon et al. (2014, Section V), a networked control system can be treated as an interconnection of continuous states and hybrid errors due to the network protocol, and a quantized control system can be modeled as an interconnection of continuous states and a discrete quantizer. Moreover, the "natural decomposition" of a hybrid system (1) as an interconnection of its continuous and discrete parts is often of interest as well.

Remark 5. In (15), all the subsystems, as well as the interconnection, share the same flow set $\mathcal{C}$ and the same jump set $\mathcal{D}$, which justifies the view of (15) as an interconnection of n hybrid subsystems.

Remark 6. Based on Lemma 1 and standard considerations clarifying the influence of particular subsystems (see, e.g. Mironchenko, 2012, Lemma 2.4.1), one can show that a function $V_i : \mathcal{X}_i\to\mathbb{R}_+$ is a candidate ISS Lyapunov function w.r.t. a set $\mathcal{A}_i\subset\mathcal{X}_i$ for the subsystem Σ_i iff V_i is locally Lipschitz outside $\mathcal{A}_i$, and

1. there exist $\psi_{i1},\psi_{i2}\in\mathcal{K}_\infty$ such that
$$\psi_{i1}(|x_i|_{\mathcal{A}_i}) \le V_i(x_i) \le \psi_{i2}(|x_i|_{\mathcal{A}_i}) \quad \forall x_i\in\mathcal{X}_i; \tag{16}$$
2. there exist *internal gains* $\chi_{ij}\in\mathcal{K}$ for $j\ne i$ and $\chi_{ii}\equiv 0$, an *external gain* $\chi_i\in\mathcal{K}$, and a continuous function $\varphi_i : \mathbb{R}_+\to\mathbb{R}$ with $\varphi_i(0)=0$ such that for all $(x,u)\in\mathcal{C}$ with $x_i\notin\mathcal{A}_i$,
$$V_i(x_i) \ge \max\Big\{\max_{j=1}^{n}\chi_{ij}(V_j(x_j)),\ \chi_i(|u|)\Big\} \tag{17}$$
implies that
$$\dot{V}_i(x_i;y_i) \le -\varphi_i(V_i(x_i)) \qquad \forall y_i\in F_i(x,u); \tag{18}$$
3. there is a function $\alpha_i\in\mathcal{K}$ such that for all $(x,u)\in\mathcal{D}$,
$$V_i(y_i) \le \max\Big\{\alpha_i(V_i(x_i)),\ \max_{j=1}^{n}\chi_{ij}(V_j(x_j)),\ \chi_i(|u|)\Big\} \quad \forall y_i\in G_i(x,u). \tag{19}$$

In addition, V_i is a candidate exponential ISS Lyapunov function w.r.t. $\mathcal{A}_i$ with rate coefficients c_i, d_i iff

$$\varphi_i(r)\equiv c_i r, \quad \alpha_i(r)\equiv e^{-d_i}r. \tag{20}$$

Suppose that for each subsystem Σ_i, a candidate ISS Lyapunov function V_i is given (for discussions regarding the existence of candidate exponential ISS Lyapunov functions for hybrid systems, see Cai and Teel, 2009, Section 2, Cai et al., 2007, Theorem 8.1, and Yang, Liberzon, and Mironchenko, 2016, Remark 3). The question of whether the interconnection (15) is pre-ISS depends on properties of the *gain operator* $\Gamma : \mathbb{R}_+^n\to\mathbb{R}_+^n$ defined by

$$\Gamma(r_1,\dots,r_n) := \Big(\max_{j=1}^{n}\chi_{1j}(r_j),\dots,\max_{j=1}^{n}\chi_{nj}(r_j)\Big). \tag{21}$$

To construct a candidate ISS Lyapunov function for the interconnection (15), we adopt the notion of Ω-path (Dashkovskiy et al., 2010).

Definition 3. Given a function $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$, a function $\sigma := (\sigma_1, \dots, \sigma_n)$ with $\sigma_i \in \mathcal{K}_\infty$, $i = 1, \dots, n$ is called an *Ω-path w.r.t. Γ* if

1. all σ_i^{-1} are locally Lipschitz on $(0, \infty)$;
2. for each compact set $P \subset (0, \infty)$, there exist finite constants $K_2 > K_1 > 0$ such that for all i,
$$0 < K_1 \le (\sigma_i^{-1})' \le K_2$$
for all points of differentiability of σ_i^{-1} in P;
3. the function Γ is a contraction on $\sigma(\cdot)$, that is,
$$\Gamma(\sigma(r)) < \sigma(r) \qquad \forall\, r > 0. \tag{22}$$

Remark 7. In this paper, we consider primarily Ω-paths w.r.t. the gain operator Γ defined by (21), due to the terms $\max_{j=1}^n \chi_{ij}(V_j(x_j))$ in (17) and (19) when formulating candidate ISS Lyapunov functions for subsystems (which will be clear from the statement and proof of Theorem 2). However, there are other equivalent formulations of candidate ISS Lyapunov functions for subsystems, which will naturally lead to gain operators in different forms (see, e.g., Dashkovskiy et al., 2007, 2010). In particular, if (17) and (19) were formulated using $\sum_{j=1}^n \chi_{ij}(V_j(x_j))$ instead of $\max_{j=1}^n \chi_{ij}(V_j(x_j))$, we would arrive at the alternative gain operator $\bar\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ defined by

$$\bar\Gamma(r_1, \dots, r_n) := \left(\sum_{j=1}^n \chi_{1j}(r_j), \dots, \sum_{j=1}^n \chi_{nj}(r_j)\right).$$

Compared with (21), we see that $\Gamma(v) \le \bar\Gamma(v)$ for all $v \ne 0$; thus every Ω-path w.r.t. $\bar\Gamma$ is an Ω-path w.r.t. Γ. This alternative construction will be useful in establishing Theorem 4 for the case of linear internal gains below.

We say that a function $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ satisfies the *small-gain condition* if

$$\Gamma(v) \not\ge v \qquad \forall\, v \in \mathbb{R}^n_+ \setminus \{0\}, \tag{23}$$

or equivalently,

$$\Gamma(v) \ge v \iff v = 0.$$

As reported in Karafyllis and Jiang (2011, Proposition 2.7 and Remark 2.8) (see also Dashkovskiy et al., 2010, Theorem 5.2), if (23) holds for the gain operator Γ defined by (21), then there exists an Ω-path σ w.r.t. Γ. Furthermore, σ can be made smooth on $(0, \infty)$ via standard mollification arguments (Grüne, 2002, Appendix B.2). In this case, we construct a candidate ISS Lyapunov function for the interconnection (15) based on those for the subsystems and the corresponding Ω-path.

Theorem 2. *Consider the interconnection (15). Suppose that each subsystem Σ_i admits a candidate ISS Lyapunov function V_i w.r.t. a set $\mathcal{A}_i$ with the internal gains χ_{ij} as in (17), and the small-gain condition (23) holds for the gain operator Γ defined by (21). Let $\sigma = (\sigma_1, \dots, \sigma_n)$ be an Ω-path w.r.t. Γ which is smooth on $(0, \infty)$. Then the function $V : \mathcal{X} \to \mathbb{R}_+$ defined by*

$$V(x) := \max_{i=1}^n \sigma_i^{-1}(V_i(x_i)) \tag{24}$$

is a candidate ISS Lyapunov function w.r.t. the set $\mathcal{A} := \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ for (15).

Proof. As each $\sigma_i \in \mathcal{K}_\infty$ is smooth on $(0, \infty)$ and each V_i is locally Lipschitz outside $\mathcal{A}_i$, it follows that each $\sigma_i^{-1} \circ V_i$ is locally Lipschitz outside $\mathcal{A}_i$. Hence the function V defined by (24) is locally Lipschitz outside $\mathcal{A}$. In the following, we prove that it satisfies the conditions of Lemma 1, by combining and extending the arguments in the proofs of Dashkovskiy et al. (2010, Theorem 5.3) and Liberzon et al. (2014, Theorem III.1).

First, consider the functions ψ_1, ψ_2 defined by

$$\psi_1(r) := \min_{i=1}^n \sigma_i^{-1}(\psi_{i1}(r/\sqrt{n})), \qquad r \in \mathbb{R}_+,$$
$$\psi_2(r) := \max_{i=1}^n \sigma_i^{-1}(\psi_{i2}(r)), \qquad r \in \mathbb{R}_+$$

with ψ_{i1}, ψ_{i2} as in (16). Since $\sigma_i, \psi_{i1}, \psi_{i2} \in \mathcal{K}_\infty$, we have that $\psi_1, \psi_2 \in \mathcal{K}_\infty$. Thus (16) implies (3). In particular,

$$\psi_1(|x|_{\mathcal{A}}) \le \min_{i=1}^n \sigma_i^{-1}\left(\psi_{i1}\left(\max_{j=1}^n |x_j|_{\mathcal{A}_j}\right)\right)$$
$$\le \max_{j=1}^n \sigma_j^{-1}(\psi_{j1}(|x_j|_{\mathcal{A}_j})) \le \max_{j=1}^n \sigma_j^{-1}(V_j(x_j)) = V(x).$$

Second, consider the gain function $\bar\chi$ defined by

$$\bar\chi(r) := \max_{i=1}^n \sigma_i^{-1}(\chi_i(r)), \quad r \in \mathbb{R}_+ \tag{25}$$

with χ_i as in (17), and the function φ defined by

$$\varphi(r) := \min_{i=1}^n (\sigma_i^{-1})'(\sigma_i(r))\, \varphi_i(\sigma_i(r)), \quad r \in \mathbb{R}_+ \tag{26}$$

with φ_i as in (18). As all $\sigma_i \in \mathcal{K}_\infty$ are smooth on $(0, \infty)$, $\chi_i \in \mathcal{K}$, and φ_i are continuous with $\varphi_i(0) = 0$, it follows that $\bar\chi \in \mathcal{K}$ and φ is continuous with $\varphi(0) = 0$. Consider the sets $\mathcal{M}_i \subset \mathcal{X}$, $i = 1, \dots, n$ defined by

$$\mathcal{M}_i := \left\{x \in \mathcal{X} : \sigma_i^{-1}(V_i(x_i)) > \max_{j \ne i} \sigma_j^{-1}(V_j(x_j))\right\}.$$

The fact that all V_i and σ_i^{-1} are continuous implies that all $\mathcal{M}_i$ are open in $\mathcal{X}$, $\mathcal{M}_i \cap \mathcal{M}_j = \emptyset$ for all $j \ne i$, and $\mathcal{X} = \bigcup_{i=1}^n \overline{\mathcal{M}_i}$, where $\overline{\mathcal{M}_i}$ is the closure of $\mathcal{M}_i$ in $\mathcal{X}$. Thus for each $(x, u) \in \mathcal{C}$ with $x \notin \mathcal{A}$, there are two possibilities:

(1) There is a unique $i \in \{1, \dots, n\}$ s.t. $x \in \mathcal{M}_i$. Then

$$V(x) = \sigma_i^{-1}(V_i(x_i)), \tag{27}$$

and $x_i \notin \mathcal{A}_i$ due to $x \notin \mathcal{A}$. Hence

$$V_i(x_i) = \sigma_i(V(x)) \ge \max_{j=1}^n \chi_{ij}(\sigma_j(V(x))) \ge \max_{j=1}^n \chi_{ij}(V_j(x_j)), \tag{28}$$

where the first inequality follows from (22), and the second one follows from (24). Also, if $V(x) \ge \bar\chi(|u|)$, then $V(x) \ge \max_{j=1}^n \sigma_j^{-1}(\chi_j(|u|))$ due to (25); thus

$$V_i(x_i) = \sigma_i(V(x)) \ge \sigma_i\left(\max_{j=1}^n \sigma_j^{-1}(\chi_j(|u|))\right)$$
$$\ge \sigma_i(\sigma_i^{-1}(\chi_i(|u|))) = \chi_i(|u|). \tag{29}$$

Hence (17), and therefore (18), holds. Given an arbitrary $y = (y_1, \dots, y_n) \in F(x, u)$, as $\mathcal{M}_i$ is open, it follows that $x + hy \in \mathcal{M}_i$ for all small enough $h > 0$; thus $V(x + hy) = \sigma_i^{-1}(V_i(x_i + hy_i))$. Hence

$$\dot V(x; y) = \overline{\lim_{h \searrow 0}} \frac{V(x + hy) - V(x)}{h}$$
$$= \overline{\lim_{h \searrow 0}} \frac{\sigma_i^{-1}(V_i(x_i + hy_i)) - \sigma_i^{-1}(V_i(x_i))}{h}$$
$$= (\sigma_i^{-1})'(V_i(x_i)) \overline{\lim_{h \searrow 0}} \frac{V_i(x_i + hy_i) - V_i(x_i)}{h}$$
$$= (\sigma_i^{-1})'(V_i(x_i)) \dot V_i(x_i; y_i)$$
$$\le -(\sigma_i^{-1})'(\sigma_i(V(x)))\, \varphi_i(\sigma_i(V(x)))$$
$$\le -\varphi(V(x)),$$

where the first inequality follows from (18) and (27), and the last one follows from (26).

(2) There is a subset $I(x) \subset \{1,\dots,n\}$ of indices with the cardinality $|I(x)| \ge 2$ such that $x \in \bigcap_{i\in I(x)} \partial\mathcal{M}_i$, where $\partial\mathcal{M}_i$ denotes the boundary of $\mathcal{M}_i$ in $\mathcal{X}$ and satisfies that $\partial\mathcal{M}_i = \overline{\mathcal{M}_i} \setminus \mathcal{M}_i$ as $\mathcal{M}_i$ is open in $\mathcal{X}$. Then (27) and $x_i \notin \mathcal{A}_i$ hold for all $i \in I(x)$. Following similar arguments to those in the previous case, if $V(x) \ge \bar\chi(|u|)$, then (28) and (29), and therefore (18), hold for all $i \in I(x)$. Given an arbitrary $y = (y_1,\dots,y_n) \in F(x,u)$, as all $\mathcal{M}_i$ are open, it follows that $x + hy \in \left(\bigcap_{i\in I(x)} \partial\mathcal{M}_i\right) \cap \left(\bigcap_{i\in I(x)} \mathcal{M}_i\right)$ for all small enough $h > 0$; thus $V(x+hy) = \max_{i\in I(x)} \sigma_i^{-1}(V_i(x_i + hy_i))$. Hence

$$\begin{aligned}
\dot V(x;y) &= \overline{\lim_{h\searrow 0}} \frac{V(x+hy) - V(x)}{h} \\
&= \overline{\lim_{h\searrow 0}} \frac{1}{h}\left(\max_{i\in I(x)} \sigma_i^{-1}(V_i(x_i+hy_i)) - V(x)\right) \\
&= \overline{\lim_{h\searrow 0}} \max_{i\in I(x)} \frac{\sigma_i^{-1}(V_i(x_i+hy_i)) - \sigma_i^{-1}(V_i(x_i))}{h} \\
&= \max_{i\in I(x)} \overline{\lim_{h\searrow 0}} \frac{\sigma_i^{-1}(V_i(x_i+hy_i)) - \sigma_i^{-1}(V_i(x_i))}{h} \\
&= \max_{i\in I(x)} (\sigma_i^{-1})'(V_i(x_i)) \dot V_i(x_i;y_i) \\
&\le \max_{i\in I(x)} -(\sigma_i^{-1})'(\sigma_i(V(x)))\, \varphi_i(\sigma_i(V(x))) \\
&\le -\varphi(V(x)),
\end{aligned}$$

where the fourth equality follows partially from the continuity of all V_i and σ_i^{-1} (cf. the proof of Dashkovskiy and Mironchenko, 2013a, Theorem 4); the first inequality follows from (18) and (27) for $i \in I(x)$, and the last one follows from (26).

Hence (7) holds for each $(x,u) \in \mathcal{C}$.

Last, consider the function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\alpha(r) := \max_{i,j=1}^{n} \left\{ \sigma_i^{-1}(\alpha_i(\sigma_i(r))),\ \sigma_i^{-1}(\chi_{ij}(\sigma_j(r))) \right\} \tag{30}$$

with α_i and χ_{ij} as in (19). As all $\sigma_i \in \mathcal{K}_\infty$, $\chi_{ij} \in \mathcal{K}$ for $j \ne i$, $\chi_{ii} \equiv 0$, and $\alpha_i \in \mathcal{K}$, it follows that $\alpha \in \mathcal{K}$. Consider an arbitrary $(x,u) \in \mathcal{D}$. From (24) and (30), it follows that[4]

$$\alpha(V(x)) \ge \max_{i,j=1}^{n} \left\{ \sigma_i^{-1}(\alpha_i(V_i(x_i))),\ \sigma_i^{-1}(\chi_{ij}(V_j(x_j))) \right\}.$$

Also, (25) implies that $\bar\chi(|u|) = \max_{i=1}^n \sigma_i^{-1}(\chi_i(|u|))$. Combining the previous two equations with (19), we obtain that for all $y = (y_1,\dots,y_n) \in G(x,u)$,

$$V(y) = \max_{i=1}^n \sigma_i^{-1}(V_i(y_i)) \le \max\{\alpha(V(x)),\ \bar\chi(|u|)\}.$$

Hence (8) holds for each $(x,u) \in \mathcal{D}$.

Therefore, from Lemma 1, it follows that V is a candidate ISS Lyapunov function w.r.t. $\mathcal{A}$ for (15). □

Theorem 2 is a powerful tool in establishing ISS of interconnections of hybrid subsystems. In the following, we inspect some of its implications.

If each subsystem of (15) admits an ISS Lyapunov function, then Theorem 2 implies the following result, which generalizes (Liberzon et al., 2014, Theorem III.1) and (Dashkovskiy & Kosmykov, 2013, Theorem 3.6).

[4] Note that, if α_i is of class $\mathcal{PD}$ but not increasing, then it is possible that $\sigma_i(V(x)) > V_i(x_i)$ but $\alpha_i(\sigma_i(V(x))) < \alpha_i(V_i(x_i))$ for some i; thus the inequality following this footnote may not hold. A similar issue arises in the proof of Liberzon et al. (2014, Theorem III.1) where it was overlooked, but could be fixed by majorizing the class $\mathcal{PD}$ functions λ_1, λ_2 with class $\mathcal{K}$ ones.

Corollary 3. *Consider the interconnection (15). Suppose that each subsystem Σ_i admits an ISS Lyapunov function V_i w.r.t. a set $\mathcal{A}_i$ (i.e., $\varphi_i \in \mathcal{PD}$ and $\alpha_i <$ id on $(0,\infty)$ in (18) and (19), respectively) with the internal gains χ_{ij} as in (17), and the small-gain condition (23) holds for the gain operator Γ defined by (21). Then (15) is pre-ISS w.r.t. $\mathcal{A}$.*

Proof. Following Theorem 2, the function V defined by (24) is a candidate ISS Lyapunov function w.r.t. $\mathcal{A}$ for (15). First, as all $\sigma_i \in \mathcal{K}_\infty$ are smooth on $(0,\infty)$ and $\varphi_i \in \mathcal{PD}$, the function φ defined by (26) is of class $\mathcal{PD}$. Second, (22) implies that all $\sigma_i^{-1} \circ \chi_{ij} \circ \sigma_j <$ id on $(0,\infty)$, and as all $\sigma_i \in \mathcal{K}_\infty$ and $\alpha_i <$ id on $(0,\infty)$, it follows that all $\sigma_i^{-1} \circ \alpha_i \circ \sigma_i <$ id on $(0,\infty)$; thus the function α defined by (30) satisfies that $\alpha <$ id on $(0,\infty)$. Therefore, V is an ISS Lyapunov function, and (15) is pre-ISS w.r.t. $\mathcal{A}$ following similar analysis to the proof of Cai and Teel (2009, Proposition 2.7); see also Remark 2. □

As the assumptions in Corollary 3 are quite restrictive, we now investigate the case where, for some subsystems Σ_i, either $\varphi_i \notin \mathcal{PD}$ or $\alpha_i(r) \ge r$ for some $r > 0$ (cf. footnote 3). In this case, we cannot use Corollary 3 to prove pre-ISS for the interconnection (15) directly, but rather invoke Proposition 1 to establish pre-ISS for the set of solution pairs that jump neither too fast nor too slowly. However, in general, Theorem 2 cannot provide the candidate exponential ISS Lyapunov function needed in Proposition 1. In the next theorem, we construct such a function under the assumption that each subsystem Σ_i admits a candidate exponential ISS Lyapunov function V_i, and the internal gains χ_{ij} in (17) and (19) are all linear. With a slight abuse of notation, we let all $\chi_{ij} \ge 0$ be scalars, and replace the terms $\chi_{ij}(V_j(x_j))$ in (17) and (19) with $\chi_{ij}V_j(x_j)$. Consider the *gain matrix*

$$\Gamma_M := (\chi_{ij})_{i,j=1}^n \in \mathbb{R}^{n\times n}. \tag{31}$$

Denote by $\rho(\Gamma_M)$ its spectral radius (i.e., the largest absolute value of its eigenvalues). Due to Dashkovskiy et al. (2007, p. 110), if

$$\rho(\Gamma_M) < 1, \tag{32}$$

then the small-gain condition (23) holds for the function $\bar\Gamma : \mathbb{R}_+^n \to \mathbb{R}_+^n$ defined by $\bar\Gamma(v) := \Gamma_M v$, which is the alternative gain operator in Remark 7. Consequently, there exists a linear Ω-path w.r.t. $\bar\Gamma$ (Dashkovskiy, Rüffer, & Wirth, 2006, p. 78); for more results on Ω-paths, the reader may consult Rüffer (2010).

Theorem 4. *Consider the interconnection (15). Suppose that each subsystem Σ_i admits a candidate exponential ISS Lyapunov function V_i w.r.t. a set $\mathcal{A}_i$ with rate coefficients c_i, d_i. Assume also that the internal gains χ_{ij} in (17) and (19) are all linear, and (32) holds for the gain matrix Γ_M defined by (31). Let $\sigma : r \mapsto (s_1 r,\dots,s_n r)$ with scalars $s_1,\dots,s_n$ be a linear Ω-path w.r.t. the alternative gain operator $\bar\Gamma$. Then $V : \mathcal{X} \to \mathbb{R}_+$ defined by*

$$V(x) := \max_{i=1}^n \frac{1}{s_i} V_i(x_i) \tag{33}$$

is a candidate exponential ISS Lyapunov function w.r.t. $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$ for (15) with rate coefficients

$$c := \min_{i=1}^n c_i, \quad d := \min_{i,j:j\ne i} \left\{ d_i,\ -\ln\left(\frac{s_j}{s_i}\chi_{ij}\right) \right\}. \tag{34}$$

Proof. In view of Remark 7, σ is also an Ω-path w.r.t. the gain operator defined by (21) (with all $\chi_{ij}(r_j)$ replaced by $\chi_{ij} r_j$). Following Theorem 2, the function V defined by (33) is a candidate ISS Lyapunov function w.r.t. $\mathcal{A}$ for (15). Substituting (20) into (26) and (30), we obtain

$$\varphi(r) \equiv \min_{i=1}^n c_i r, \quad \alpha(r) \equiv \max_{i,j=1}^n \left\{ e^{-d_i},\ \frac{s_j}{s_i}\chi_{ij} \right\} r.$$

Hence V is a candidate exponential ISS Lyapunov function with the rate coefficients c, d defined by (34). □

Remark 8. For the more general case with the internal gains χ_{ij} being power functions instead of linear ones, a candidate exponential ISS Lyapunov function for (15) can be constructed in a similar way; cf. Dashkovskiy and Mironchenko (2013b, Theorem 9).

The following remark provides a simpler bound for the rate coefficient d in some important cases.

Remark 9. If the gain matrix Γ_M defined by (31) is irreducible, then $\rho(\Gamma_M)$ is the Perron–Frobenius eigenvalue of Γ_M, and the corresponding eigenvector $\bar{s} = (s_1, \ldots, s_n)$ satisfies $\bar{s} > 0$ (Perron–Frobenius theorem Berman & Plemmons, 1994, Theorem 2.1.3). Hence, if (32) holds, then $\Gamma_M \bar{s} = \rho(\Gamma_M)\bar{s} < \bar{s}$; thus $\sigma : r \mapsto \bar{s}r$ is a linear Ω-path as in Theorem 4. Moreover, for all $i \in \{1, \ldots, n\}$, it holds that

$$\max_{j=1}^{n} \frac{s_j}{s_i}\chi_{ij} \le \frac{1}{s_i}\sum_{j=1}^{n} s_j\chi_{ij} = \rho(\Gamma_M);$$

thus the rate coefficient d defined by (34) satisfies that $d \ge \min\{\min_{i=1}^{n} d_i, -\ln(\rho(\Gamma_M))\}$.

Having applied Theorem 4, we can establish pre-ISS for the set of solution pairs that jump neither too fast nor too slowly via Proposition 1. However, if there are subsystems Σ_k, Σ_l for which the rate coefficients $c_k, d_l < 0$, then c, d defined by (34) are negative as well, and Proposition 1 cannot be applied to complete solution pairs (see Remark 3). In the following section, we handle such cases via the approach of modifying ISS Lyapunov functions for subsystems using ADT and RADT clocks from Liberzon et al. (2014).

4. Modifying ISS Lyapunov functions for subsystems

Suppose that each subsystem Σ_i admits a candidate exponential ISS Lyapunov function with rate coefficients c_i, d_i, and there are Σ_k, Σ_l such that $c_k, d_l < 0 < c_l, d_k$. Our goal is to construct new candidate exponential ISS Lyapunov functions with rate coefficients $\tilde{c}_i$, $\tilde{d}_i$ so that either all $\tilde{c}_i > 0$ (i.e., all continuous dynamics are ISS) or all $\tilde{d}_i > 0$ (i.e., all discrete dynamics are ISS). To accomplish this, we first derive suitable conditions on the density of jumps, then augment the corresponding subsystems with auxiliary clocks to incorporate such conditions, and finally modify the corresponding candidate exponential ISS Lyapunov functions.

4.1. Making discrete dynamics ISS

In the following, we construct candidate exponential ISS Lyapunov functions so that all rate coefficients $\tilde{d}_i > 0$.

We say that a solution pair (x, u) of (15) admits an average dwell-time (ADT) (Hespanha & Morse, 1999) $\delta > 0$ if there is an integer $N_0 \ge 1$ so that all $(s, k) \preceq (t, j)$ in $\operatorname{dom} x$ satisfy[5]

$$j - k \le \delta(t - s) + N_0. \tag{35}$$

Following Liberzon et al. (2014, Section IV.A), a hybrid time domain satisfies (35) iff it is the domain of an ADT clock τ given by

$$\begin{aligned} &\dot{\tau} \in [0, \delta], && \tau \in [0, N_0], \\ &\tau^+ = \tau - 1, && \tau_i \in [1, N_0]. \end{aligned} \tag{36}$$

[5] If (35) holds with $N_0 = 1$, then the ADT condition becomes the dwell-time condition (Morse, 1996); if it holds with $N_0 < 1$, then jumps are not allowed at all, which can be seen directly from (35) by taking $t - s$ small enough.

Remark 10. This notion of ADT clock for hybrid systems first appeared in Cai et al. (2008, Appendix), where it was defined by

$$\begin{cases} \dot{\tau} \in \eta_\delta(\tau) & \text{for } \tau \in C := [0, N_0] \\ \tau^+ = \tau - 1 & \text{for } \tau \in D := [1, N_0] \end{cases} \tag{37}$$

with $\eta_\delta(\tau) := \begin{cases} \delta & \text{for } \tau \in [0, N_0) \\ [0, \delta] & \text{for } \tau = N_0 \end{cases}$ (see also Mitra, Liberzon, and Lynch, 2008 for a related earlier construction). The ADT clocks defined by (36) and (37) are equivalent in the following sense. First, as $\tau \in [0, \delta]$, an ADT clock defined by (37) always satisfies (36). Second, given an ADT clock defined by (36) that increases on $[0, N_0)$ with a speed $\dot{\tau} < \delta$, there always exists an ADT clock defined by (37) that increases on $[0, N_0)$ with $\dot{\tau} = \delta$ but stays longer at N_0 so that their hybrid time domains are the same.

Denote by $I_d := \{i : d_i < 0\}$ the index set of subsystems with non-ISS discrete dynamics. Let $z_i := x_i \in \mathcal{X}_i =: \mathcal{Z}_i$ for $i \notin I_d$ and $z_i := (x_i, \tau_i) \in \mathcal{X}_i \times [0, N_{0i}] =: \mathcal{Z}_i$ with an integer $N_{0i} \ge 1$ for $i \in I_d$. Consider the augmented interconnection $\tilde{\Sigma}$ with state $z := (z_1, \ldots, z_n) \in \mathcal{Z}_1 \times \cdots \times \mathcal{Z}_n =: \mathcal{Z}$ and input $u \in \mathcal{U}$ modeled by

$$\begin{aligned} &\dot{z}_i \in \tilde{F}_i(z, u), \quad i = 1, \ldots, n, \qquad (z, u) \in \tilde{\mathcal{C}}, \\ &z_i^+ \in \tilde{G}_i(z, u), \quad i = 1, \ldots, n, \qquad (z, u) \in \tilde{\mathcal{D}}, \end{aligned} \tag{38}$$

where $\tilde{\mathcal{C}} := \tilde{\mathcal{C}}_1 \times \cdots \times \tilde{\mathcal{C}}_n \times \mathcal{C}_u$ with $\tilde{\mathcal{C}}_i := \mathcal{C}_i$ for $i \notin I_d$ and $\tilde{\mathcal{C}}_i := \mathcal{C}_i \times [0, N_{0i}]$ for $i \in I_d$, $\tilde{\mathcal{D}} := \tilde{\mathcal{D}}_1 \times \cdots \times \tilde{\mathcal{D}}_n \times \mathcal{D}_u$ with $\tilde{\mathcal{D}}_i := \mathcal{D}_i$ for $i \notin I_d$ and $\tilde{\mathcal{D}}_i := \mathcal{D}_i \times [1, N_{0i}]$ for $i \in I_d$, $\tilde{F} := (\tilde{F}_1, \ldots, \tilde{F}_n)$ with $\tilde{F}_i(z, u) := F_i(x, u)$ for $i \notin I_d$ and $\tilde{F}_i(z, u) := F_i(x, u) \times [0, \delta_i]$ for $i \in I_d$, and $\tilde{G} := (\tilde{G}_1, \ldots, \tilde{G}_n)$ with $\tilde{G}_i(z, u) := G_i(x, u)$ for $i \notin I_d$ and $\tilde{G}_i(z, u) := G_i(x, u) \times \{\tau_i - 1\}$ for $i \in I_d$. Then (38) is a hybrid system with the data $\tilde{\mathcal{H}} := (\tilde{F}, \tilde{G}, \tilde{\mathcal{C}}, \tilde{\mathcal{D}}, \mathcal{Z}, \mathcal{U})$. The dynamics of z_i is called the ith augmented subsystem of (38) and is denoted by $\tilde{\Sigma}_i$.

In the following proposition, we apply the modification technique from Liberzon et al. (2014, Proposition IV.1) to construct a candidate exponential ISS Lyapunov function for each augmented subsystem $\tilde{\Sigma}_i$ based on the candidate exponential ISS Lyapunov function for the subsystem Σ_i of the original interconnection (15) and the ADT clock τ_i.

Proposition 5. *Consider a subsystem Σ_i of the original interconnection (15). Suppose that it admits a candidate exponential ISS Lyapunov function V_i w.r.t. a set $\mathcal{A}_i$ with rate coefficients c_i, d_i. For a scalar $L_i \ge 0$, the function $W_i : \mathcal{Z}_i \to \mathbb{R}_+$ defined by*

$$W_i(z_i) := \begin{cases} V_i(x_i) & \text{if } i \notin I_d; \\ e^{L_i\tau_i}V_i(x_i) & \text{if } i \in I_d \end{cases}$$

is a candidate exponential ISS Lyapunov function w.r.t.

$$\tilde{\mathcal{A}}_i := \begin{cases} \mathcal{A}_i & \text{if } i \notin I_d; \\ \mathcal{A}_i \times [0, N_{0i}] & \text{if } i \in I_d \end{cases}$$

for the subsystem $\tilde{\Sigma}_i$ of (38) with rate coefficients

$$\begin{cases} \tilde{c}_i := c_i,\ \tilde{d}_i := d_i & \text{if } i \notin I_d; \\ \tilde{c}_i := c_i - L_i\delta_i,\ \tilde{d}_i := d_i + L_i & \text{if } i \in I_d. \end{cases} \tag{39}$$

More specifically,

1. *there exist functions $\tilde{\psi}_{i1}, \tilde{\psi}_{i2} \in \mathcal{K}_\infty$ such that*

$$\tilde{\psi}_{i1}(|z_i|_{\tilde{\mathcal{A}}_i}) \le W_i(z_i) \le \tilde{\psi}_{i2}(|z_i|_{\tilde{\mathcal{A}}_i}) \quad \forall z_i \in \mathcal{Z}_i; \tag{40}$$

2. *there exist internal gains $\tilde{\chi}_{ij} \in \mathcal{K}$, $j \ne i$ defined by*

$$\tilde{\chi}_{ij}(r) := \begin{cases} \chi_{ij}(r) & \text{if } i \notin I_d; \\ e^{L_iN_{0i}}\chi_{ij}(r) & \text{if } i \in I_d \end{cases} \tag{41}$$

with χ_{ij} as in (17) and $\tilde{\chi}_{ii} \equiv 0$, and an external gain $\tilde{\chi}_i \in \mathcal{K}$ such that for all $(z, u) \in \tilde{\mathcal{C}}$ with $z_i \notin \tilde{\mathcal{A}}_i$,

$$W_i(z_i) \geq \max\left\{\max_{j=1}^{n} \tilde{\chi}_{ij}(W_j(z_j)),\ \tilde{\chi}_i(|u|)\right\} \tag{42}$$

implies that

$$\dot{W}_i(z_i; y_i) \leq -\tilde{c}_i W_i(z_i) \qquad \forall\, y_i \in \tilde{F}_i(z, u); \tag{43}$$

3. *for all $(z, u) \in \tilde{\mathcal{D}}$,*

$$W_i(y_i) \leq \max\left\{e^{-\tilde{d}_i} W_i(z_i), \max_{j=1}^{n} \tilde{\chi}_{ij}(W_j(z_j)), \tilde{\chi}_i(|u|)\right\} \quad \forall\, y_i \in \tilde{G}_i(z, u). \tag{44}$$

Proof. If $i \notin I_d$, then the claim follows directly from the assumption that V_i is a candidate exponential ISS Lyapunov function with rate coefficients c_i, d_i. Therefore, we only consider the case $i \in I_d$ in the following proof. As V_i is locally Lipschitz outside $\mathcal{A}_i$ and the map $\tau_i \mapsto e^{L_i \tau_i}$ is smooth, W_i is also locally Lipschitz outside $\tilde{\mathcal{A}}_i$.

First, consider the functions $\tilde{\psi}_{i1}, \tilde{\psi}_{i2} \in \mathcal{K}_\infty$ defined by

$$\tilde{\psi}_{i1}(r) := \psi_{i1}(r), \quad \tilde{\psi}_{i2}(r) := e^{L_i N_{0i}} \psi_{i2}(r)$$

with ψ_{i1}, ψ_{i2} as in (16). Then (40) follows from (16).

Second, consider the function $\tilde{\chi}_i \in \mathcal{K}$ defined by

$$\tilde{\chi}_i(r) := e^{L_i N_{0i}} \chi_i(r) \tag{45}$$

with χ_i as in (17). For each $(z, u) \in \tilde{\mathcal{C}}$ with $z_i \notin \tilde{\mathcal{A}}_i$, if (42) holds, then

$$\begin{aligned} V_i(x_i) &= e^{-L_i \tau_i} W_i(z_i) \geq e^{-L_i N_{0i}} \max_{j=1}^{n} \tilde{\chi}_{ij}(W_j(z_j)) \\ &= \max_{j=1}^{n} \chi_{ij}(W_j(z_j)) \geq \max_{j=1}^{n} \chi_{ij}(V_j(x_j)), \end{aligned}$$

and $V_i(x_i) = e^{-L_i \tau_i} W_i(z_i) \geq e^{-L_i N_{0i}} \tilde{\chi}_i(|u|) = \chi_i(|u|)$. Hence (17), and therefore (18), holds. For all $y_i \in \tilde{F}_i(z, u)$, let $y_i = (y_{i1}, y_{i2})$ be such that $y_{i1} \in F_i(x, u)$ and $y_{i2} \in [0, \delta_i]$. Following (18), (20), and (39),

$$\begin{aligned} \dot{W}_i(z_i; y_i) &= e^{L_i \tau_i} \dot{V}_i(x_i; y_{i1}) + L_i e^{L_i \tau_i} V_i(x_i) y_{i2} \\ &\leq -c_i e^{L_i \tau_i} V_i(x_i) + L_i \delta_i e^{L_i \tau_i} V_i(x_i) = -\tilde{c}_i W_i(z_i). \end{aligned}$$

Finally, consider an arbitrary $(z, u) \in \tilde{\mathcal{D}}$. For all $y_i \in \tilde{G}_i(z, u)$, let $y_i = (y_{i1}, y_{i2})$ be such that $y_{i1} \in G_i(x, u)$ and $y_{i2} = \tau_i - 1$. From (39), it follows that

$$e^{-\tilde{d}_i} W_i(z_i) = e^{-d_i - L_i + L_i \tau_i} V_i(x_i) = e^{L_i y_{i2} - d_i} V_i(x_i),$$

and from (41) and (45), it follows that $\tilde{\chi}_{ij}(W_j(z_j)) = e^{L_i N_{0i}} \chi_{ij}(W_j(z_j)) \geq e^{L_i y_{i2}} \chi_{ij}(V_j(x_j))$ for all j and $\tilde{\chi}_i(|u|) = e^{L_i N_{0i}} \chi_i(|u|) \geq e^{L_i y_{i2}} \chi_i(|u|)$, respectively. Substituting the previous equations into (19) gives (44).

Therefore, W_i is a candidate exponential ISS Lyapunov function w.r.t. $\tilde{\mathcal{A}}_i$ for the augmented subsystem $\tilde{\Sigma}_i$ of (38) with the rate coefficients $\tilde{c}_i, \tilde{d}_i$ defined by (39). □

Proposition 5 shows that it is possible to make all $\tilde{d}_i > 0$ by choosing large enough scalars L_i, $i \in I_d$, at the cost of decreasing the convergence rates of continuous dynamics (as $\tilde{c}_i = c_i - L_i \delta_i$ in (39) above), and increasing the internal gains (as $\tilde{\chi}_{ij}(r) = e^{L_i N_{0i}} \chi_{ij}(r)$ in (41) above). Consequently, for large enough integers N_{0i}, it is possible that the small-gain condition (23) holds for the gain operator Γ defined by (21), but not for $\tilde{\Gamma} : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ defined by[6]

$$\tilde{\Gamma}(r_1, \ldots, r_n) := \left(\max_{j=1}^{n} \tilde{\chi}_{1j}(r_j), \ldots, \max_{j=1}^{n} \tilde{\chi}_{nj}(r_j)\right).$$

To see the consequence of this fact clearer, consider for simplicity an interconnection of two subsystems Σ_1, Σ_2, and their candidate exponential ISS Lyapunov functions V_1, V_2 with rate coefficients $c_1, d_2 > 0 > d_1, c_2$ and linear internal gains $\chi_{12}, \chi_{21} > 0$. After we augment Σ_1 with an ADT clock $\delta_1 \in [0, N_{01}]$, the matrix $\tilde{\Gamma}_M$ is given by

$$\tilde{\Gamma}_M = \begin{bmatrix} 0 & \tilde{\chi}_{12} \\ \tilde{\chi}_{21} & 0 \end{bmatrix} = \begin{bmatrix} 0 & e^{L_1 N_{01}} \chi_{12} \\ \chi_{21} & 0 \end{bmatrix},$$

and $\rho(\tilde{\Gamma}_M) < 1$ holds iff $\chi_{12}\chi_{21} < e^{-L_1 N_{01}}$. In order to make the rate coefficient $\tilde{d}_1 = d_1 + L_1 > 0$, we need to choose a scalar $L_1 > -d_1$. Also, the integer $N_{01} \geq 1$. Hence we cannot apply Theorem 4 to the augmented interconnection unless the original internal gains χ_{12}, χ_{21} satisfy $\chi_{12}\chi_{21} \leq e^{d_1} < 1$.

The observation above hints that it may be better to make all $\tilde{c}_i > 0$ (instead of making all $\tilde{d}_i > 0$ as in this subsection). See Yang et al. (2016) for a case-by-case study comparing the two schemes.

4.2. Making continuous dynamics ISS

In the following, we construct candidate exponential ISS Lyapunov functions so that all rate coefficients $\tilde{c}_i > 0$.

We say that a solution pair (x, u) of (15) admits a reverse average dwell-time (RADT) (Hespanha et al., 2008) $\delta^* > 0$ if there is an integer $N_0^* \geq 1$ so that all $(s, k) \preceq (t, j)$ in $\operatorname{dom} x$ satisfy

$$t - s \leq \delta^*(j - k) + N_0^* \delta^*. \tag{46}$$

Following Cai et al. (2008, Appendix) and Liberzon et al. (2014, Section IV.B), a hybrid time domain satisfies (46) iff it is the domain of an RADT clock τ defined by

$$\begin{aligned} \dot{\tau} &= 1, & \tau &\in [0, N_0^* \delta^*], \\ \tau^+ &= \max\{0,\ \tau - \delta^*\}, & \tau &\in [0, N_0^* \delta^*]. \end{aligned}$$

Denote by $I_c := \{i : c_i < 0\}$ the index set of subsystems with non-ISS continuous dynamics. Let $z_i := x_i \in \mathcal{X}_i =: \mathcal{Z}_i$ for $i \notin I_c$ and $z_i := (x_i, \tau_i) \in \mathcal{X}_i \times [0, N_{0i}^* \delta_i^*] =: \mathcal{Z}_i$ with an integer $N_{0i} \geq 1$ for $i \in I_c$. Consider the augmented interconnection $\tilde{\Sigma}$ with state $z := (z_1, \ldots, z_n) \in \mathcal{Z}_1 \times \cdots \times \mathcal{Z}_n =: \mathcal{Z}$ and input $u \in \mathcal{U}$ modeled by (38), where $\tilde{\mathcal{C}} := \tilde{\mathcal{C}}_1 \times \cdots \times \tilde{\mathcal{C}}_n \times \mathcal{C}_u$ with $\tilde{\mathcal{C}}_i = \mathcal{C}_i$ for $i \notin I_c$ and $\tilde{\mathcal{C}}_i = \mathcal{C}_i \times [0, N_{0i}^* \delta_i^*]$ for $i \in I_c$, $\tilde{\mathcal{D}} := \tilde{\mathcal{D}}_1 \times \cdots \times \tilde{\mathcal{D}}_n \times \mathcal{D}_u$ with $\tilde{\mathcal{D}}_i = \mathcal{D}_i$ for $i \notin I_c$ and $\tilde{\mathcal{D}}_i = \mathcal{D}_i \times [0, N_{0i}^* \delta_i^*]$ for $i \in I_c$, $\tilde{F} := (\tilde{F}_1, \ldots, \tilde{F}_n)$ with $\tilde{F}_i(z, u) := F_i(x, u)$ for $i \notin I_c$ and $\tilde{F}_i(z, u) := F_i(x, u) \times \{1\}$ for $i \in I_c$, and $\tilde{G} := (\tilde{G}_1, \ldots, \tilde{G}_n)$ with $\tilde{G}_i(z, u) := G_i(x, u)$ for $i \notin I_c$ and $\tilde{G}_i(z, u) := G_i(x, u) \times \{\max\{0, \tau_i - \delta_i^*\}\}$ for $i \in I_c$. Then (38) is a hybrid system with the data $\tilde{\mathcal{H}} := (\tilde{F}, \tilde{G}, \tilde{\mathcal{C}}, \tilde{\mathcal{D}}, \mathcal{Z}, \mathcal{U})$. The dynamics of z_i is called the ith augmented subsystem of (38) and is denoted by $\tilde{\Sigma}_i$.

In the following proposition, we apply the modification technique from Liberzon et al. (2014, Proposition IV.4) to construct a candidate exponential ISS Lyapunov functions for each augmented subsystem $\tilde{\Sigma}_i$ based on the candidate exponential ISS Lyapunov function for the subsystem Σ_i of the original interconnection (15) and the RADT clock τ_i.

Proposition 6. *Consider a subsystem Σ_i of the original interconnection (15). Suppose that it admits a candidate exponential ISS Lyapunov function V_i w.r.t. a set $\mathcal{A}_i$ with rate coefficients c_i, d_i. For a scalar $L_i \geq 0$, the function $W_i : \mathcal{Z}_i \to \mathbb{R}_+$ defined by*

$$W_i(z_i) := \begin{cases} V_i(x_i) & \text{if } i \notin I_c; \\ e^{-L_i \tau_i} V_i(x_i) & \text{if } i \in I_c \end{cases} \tag{47}$$

is a candidate exponential ISS Lyapunov function w.r.t.

$$\tilde{\mathcal{A}}_i := \begin{cases} \mathcal{A}_i & \text{if } i \notin I_c; \\ \mathcal{A}_i \times [0, N_{0i}^* \delta_i^*] & \text{if } i \in I_c \end{cases}$$

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[6] However, if all the original internal gains χ_{ij} are linear, and the gain matrix Γ_M defined by (31) is a triangular matrix (i.e., if (15) is a cascade interconnection), then (23) always holds for $\tilde{\Gamma}$, as all the cyclic gains equal zero.

for the augmented subsystem $\tilde{\Sigma}_i$ *of* (38) *with rate coefficients*

$$\begin{cases} \tilde{c}_i := c_i,\ \tilde{d}_i := d_i & \text{if } i \notin I_c; \\ \tilde{c}_i := c_i + L_i,\ \tilde{d}_i := d_i - L_i\delta_i^* & \text{if } i \in I_c. \end{cases} \tag{48}$$

More specifically,

1. *there exist functions* $\tilde{\psi}_{i1}, \tilde{\psi}_{i2} \in \mathcal{K}_\infty$ *so that* (40) *holds;*
2. *there exist internal gains* $\tilde{\chi}_{ij} \in \mathcal{K}$, $j \neq i$ *defined by*[7]

$$\tilde{\chi}_{ij}(r) := \begin{cases} \chi_{ij}(r) & \text{for } j \notin I_c; \\ \chi_{ij}(e^{L_j N_{0j}^* \delta_j^*} r) & \text{for } j \in I_c \end{cases} \tag{49}$$

with χ_{ij} *as in* (17) *and* $\tilde{\chi}_{ii} \equiv 0$, *and an external gain* $\tilde{\chi}_i \in \mathcal{K}$ *such that for all* $(z, u) \in \tilde{\mathcal{C}}$ *with* $z_i \notin \tilde{\mathcal{A}}_i$, (42) *implies* (43);
3. *for all* $(z, u) \in \tilde{\mathcal{D}}$, (44) *holds.*

Proof. If $i \notin I_c$, then the claim follows directly from the assumption that V_i is a candidate exponential ISS Lyapunov function with rate coefficients c_i, d_i. Therefore, we only consider the case $i \in I_c$ in the following proof. As V_i is locally Lipschitz outside $\mathcal{A}_i$ and the map $\tau_i \mapsto e^{-L_i\tau_i}$ is smooth, W_i is locally Lipschitz outside $\tilde{\mathcal{A}}_i$.

First, consider the functions $\tilde{\psi}_{i1}, \tilde{\psi}_{i2} \in \mathcal{K}_\infty$ defined by

$$\tilde{\psi}_{i1}(r) := e^{-L_i N_{0i}^* \delta_i^*} \psi_{i1}(r), \quad \tilde{\psi}_{i2}(r) := \psi_{i2}(r)$$

with ψ_{i1}, ψ_{i2} as in (16). Then (40) follows from (16).

Second, consider the function $\tilde{\chi}_i \in \mathcal{K}$ defined by

$$\tilde{\chi}_i(r) := \chi_i(r) \tag{50}$$

with χ_i as in (17). For each $(z, u) \in \tilde{\mathcal{C}}$ with $z_i \notin \tilde{\mathcal{A}}_i$, if (42) holds, then

$$\begin{aligned} V_i(x_i) &= e^{L_i\tau_i} W_i(z_i) \geq W_i(z_i) \geq \max_{j=1}^{n} \tilde{\chi}_{ij}(W_j(z_j)) \\ &= \max_{j=1}^{n} \chi_{ij}(e^{L_j N_{0j}^* \delta_j^*} W_j(z_j)) \geq \max_{j=1}^{n} \chi_{ij}(V_j(x_j)), \end{aligned}$$

and $V_i(x_i) = e^{L_i\tau_i} W_i(z_i) \geq W_i(z_i) \geq \tilde{\chi}_i(|u|) \geq \chi_i(|u|)$. Hence (17), and therefore (18), holds. For all $y_i \in \tilde{F}_i(z, u)$, let $y_i = (y_{i1}, y_{i2})$ be such that $y_{i1} \in F_i(x, u)$ and $y_{i2} = 1$. Following (18), (20), and (48),

$$\begin{aligned} \dot{W}_i(z_i; y_i) &= e^{-L_i\tau_i} \dot{V}_i(x_i; y_{i1}) - L_i e^{-L_i\tau_i} V_i(x_i) y_{i2} \\ &\leq -c_i e^{-L_i\tau_i} V_i(x_i) - L_i e^{-L_i\tau_i} V_i(x_i) = -\tilde{c}_i W_i(z_i). \end{aligned}$$

Finally, consider an arbitrary $(z, u) \in \tilde{\mathcal{D}}$. For all $y_i \in \tilde{G}_i(z, u)$, let $y_i = (y_{i1}, y_{i2})$ be such that $y_{i1} \in G_i(x, u)$ and $y_{i2} = \max\{0,\ \tau_i - \delta_i^*\}$. From (48), it follows that

$$e^{-\tilde{d}_i} W_i(z_i) = e^{-d_i + L_i\delta_i^* - L_i\tau_i} V_i(x_i) \geq e^{-L_i y_{i2} - d_i} V_i(x_i),$$

and from (49) and (50), it follows that $\tilde{\chi}_{ij}(W_j(z_j)) = \chi_{ij}(e^{L_j N_{0j}^* \delta_j^*} W_j(z_j)) \geq e^{-L_i y_{i2}} \chi_{ij}(V_j(x_j))$ for all j, and $\tilde{\chi}_i(|u|) = \chi_i(|u|) \geq e^{-L_i y_{i2}} \chi_i(|u|)$, respectively. Substituting the previous equations into (19) gives (44).

Therefore, W_i is a candidate exponential ISS Lyapunov function w.r.t. $\tilde{\mathcal{A}}_i$ for the augmented subsystem $\tilde{\Sigma}_i$ of (38) with the rate coefficients $\tilde{c}_i, \tilde{d}_i$ defined by (48). □

4.3. Example

We demonstrate the approach of modifying ISS Lyapunov functions in a case where we cannot apply Theorem 2 and Proposition 1 to establish stability directly.

Consider an interconnection of two hybrid subsystems with the state $x = (x_1, x_2)$ modeled by

$$\begin{aligned} \dot{x}_1 &= x_1 + x_2^2, \quad \dot{x}_2 = -3x_2 + 0.1\sqrt{|x_1|}, && x \in \mathcal{C}, \\ x_1^+ &= e^{-1}x_1, \quad x_2^+ = e x_2, && x \in \mathcal{D}, \end{aligned}$$

where $\mathcal{C} = \mathcal{D} = \mathbb{R}^2$. It can be represented in the form of the general interconnection (15) without the external input u by letting $n = 2$, $F_1(x) = x_1 + x_2^2$, $F_2(x) = -3x_2 + 0.1\sqrt{|x_1|}$, $G_1(x) = e^{-1}x_1$, and $G_2(x) = e x_2$. As $\mathcal{C} = \mathcal{D} = \mathbb{R}^2$, the system may flow or jump at any point in $\mathbb{R}^2$, and all solutions are complete. Hence the notions of pre-ISS and ISS coincide, and so do the notions of pre-GAS and GAS. The x_1-subsystem Σ_1 has stabilizing discrete dynamics but non-ISS continuous dynamics, while the x_2-subsystem Σ_2 has ISS continuous dynamics but destabilizing discrete dynamics. Thus we cannot apply Theorem 2 and Proposition 1 to establish pre-GAS of the interconnection directly.

Consider the functions $V_1, V_2 : \mathbb{R} \to \mathbb{R}_+$ defined by

$$V_1(x_1) := |x_1|, \quad V_2(x_2) := |x_2|,$$

and the functions $\chi_{12}, \chi_{21} : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\chi_{12}(r) := r^2/a, \quad \chi_{21}(r) := \sqrt{r}/b$$

with some scalars $a, b > 0$. From

$$\begin{aligned} V_1(x_1) \geq \chi_{12}(V_2(x_2)) &\Longrightarrow \dot{V}_1(x_1) \leq (a+1)V_1(x_1), \\ V_2(x_2) \geq \chi_{21}(V_1(x_1)) &\Longrightarrow \dot{V}_2(x_2) \leq (0.1b - 3)V_2(x_2), \end{aligned}$$

and[8]

$$V_1(x_1^+) \leq e^{-1}V_1(x_1), \quad V_2(x_2^+) \leq e V_2(x_2)$$

for all $x = (x_1, x_2) \in \mathbb{R}^2$, it follows that V_1 and V_2 are candidate exponential ISS Lyapunov functions w.r.t. $\{0\}$ for the subsystems Σ_1 and Σ_2 with the internal gains χ_{12} and χ_{21}, respectively. Since the discrete dynamics of the Σ_2 is destabilizing, we invoke the modification scheme from Section 4.1. Consider a solution x : $\operatorname{dom} x \to \mathbb{R}^2$ admitting an ADT $\delta_2 > 0$, that is, there exists an integer $N_{02} \geq 1$ such that all $(s, k) \preceq (t, j)$ in $\operatorname{dom} x$ satisfy

$$j - k \leq \delta_2(t - s) + N_{02}. \tag{51}$$

The corresponding ADT clock τ_2 is defined by

$$\begin{aligned} \dot{\tau}_2 &\in [0, \delta_2], && \tau_2 \in [0, N_{02}], \\ \tau_2^+ &= \tau_2 - 1, && \tau_2 \in [1, N_{02}]. \end{aligned}$$

Let $z_1 := x_1$ and $z_2 := (x_2, \tau_2)$. Following Proposition 5, the function $W_2 : \mathbb{R} \times [0, N_{02}] \to \mathbb{R}_+$ defined by

$$W_2(z_2) := e^{L_2\tau_2} V_2(x_2)$$

is a candidate exponential ISS Lyapunov function w.r.t. $\tilde{\mathcal{A}}_2 := \{0\} \times [0, N_{02}]$ for the augmented subsystem $\tilde{\Sigma}_2$ with the internal gain $\tilde{\chi}_{21} \in \mathcal{K}$ defined by

$$\tilde{\chi}_{21}(r) := e^{L_2 N_{02}} \chi_{21}(r) = e^{L_2 N_{02}} \sqrt{r}/b.$$

More specifically, for all $(z_1, z_2) \in \mathbb{R}^2 \times [0, N_{02}]$, if

$$W_2(z_2) \geq \tilde{\chi}_{21}(V_1(z_1))$$

then

$$\begin{aligned} \dot{W}_2(z_2; y_2) &= e^{L_2\tau_2} \dot{V}_2(x_2) + L_2 e^{L_2\tau_2} V_2(x_2) \dot{\tau}_2 \\ &\leq (0.1b - 3) e^{L_2\tau_2} V_2(x_2) + L_2\delta_2 e^{L_2\tau_2} V_2(x_2) \\ &= (0.1b - 3 + L_2\delta_2) W_2(z_2) \end{aligned}$$

for all $y_2 \in \{-3x_2 + 0.1\sqrt{|x_1|}\} \times [0, \delta_2]$. Furthermore,

$$W_2(e x_2, \tau_2 - 1) = e^{L_2(\tau_2 - 1) + 1} V_2(x_2) \leq e^{1 - L_2} W_2(z_2)$$

(see also footnote 8). To make the discrete dynamics of $\tilde{\Sigma}_2$ ISS, we set

$$L_2 > 1. \tag{52}$$

[7] Note that in (41), the forms of the internal gains $\tilde{\chi}_{ij}$ depend on whether $i \in I_d$, while in (49), the forms of $\tilde{\chi}_{ij}$ depend on whether $j \in I_c$.

[8] Note that the discrete dynamics of both subsystems are autonomous, and hence we can ignore the terms corresponding to internal gains χ_{12}, χ_{21} in (8). Similar simplifications will be made when we apply Proposition 5 and Theorem 2.

Following (21), the gain operator $\tilde{\Gamma} : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ after modification is defined by

$$\tilde{\Gamma}(r_1, r_2) = (\chi_{12}(r_2), \tilde{\chi}_{21}(r_1));$$

thus the small-gain condition (23) holds for $\tilde{\Gamma}$ iff $\chi_{12}(\tilde{\chi}_{21}(r)) < r$ for all $r > 0$, or equivalently,

$$L_2 < \frac{\ln(ab^2)}{2N_{02}}. \tag{53}$$

Let a scalar $s > 0$ be such that $e^{L_2 N_{02}}/b < 1/s < \sqrt{a}$. Then $\sigma := (\sigma_1, \sigma_2)$ with $\sigma_1(r) := r$, $\sigma_2(r) := \frac{\sqrt{r}}{s}$ is an Ω-path w.r.t. the gain operator $\tilde{\Gamma}$. Following Theorem 2, the function $W : \mathbb{R}^2 \times [0, N_{02}] \to \mathbb{R}_+$ defined by

$$\begin{aligned} W(z) &:= \max\{\sigma_1^{-1}(V_1(z_1)),\ \sigma_2^{-1}(W_2(z_2))\} \\ &= \max\{V_1(z_1),\ s^2 W_2(z_2)^2\} \end{aligned}$$

is a candidate Lyapunov function w.r.t. $\tilde{\mathcal{A}} := \{(0,0)\} \times [0, N_{02}]$ for the augmented interconnection with state $z := (z_1, z_2) \in \mathbb{R}^2 \times [0, N_{02}] =: \mathcal{Z}$. More specifically, for all $z \in \mathcal{Z}$,

$$\dot{W}(z; y) \le -cW(z)$$

for all $y \in \{-x_1 + x_2^2\} \times \{-3x_2 + 0.1\sqrt{|x_1|}\} \times [0, \delta_2]$ with

$$c := \min\{-(a+1),\ 2(3 - 0.1b - L_2\delta_2)\} < 0,$$

where the inequality follows from $a > 0$. Furthermore,

$$W(e^{-1}x_1, ex_2, \tau_2 - 1) \le e^{-d} W(z)$$

with $d := \min\{1,\ 2(L_2 - 1)\} > 0$, which follows from (52). Thus W is a candidate exponential Lyapunov function for the augmented interconnection with rate coefficients c, d. Consider the set of solutions $x : \operatorname{dom} x \to \mathbb{R}^2$ admitting the ADT δ_2 and also an RADT $\delta^* > 0$, that is, in addition to (51), there also exists an integer $N_0^* \ge 1$ such that all $(s, l) \preceq (t, j)$ in $\operatorname{dom} x$ satisfy

$$t - s \le \delta^*(j - k) + \delta^* N_0^*. \tag{54}$$

Following Proposition 1 and Remark 4, this set of solutions is GAS provided that

$$0 < \delta^* < \frac{d}{-c} = \frac{\min\{1,\ 2(L_2 - 1)\}}{\max\{a + 1,\ 2(0.1b - 3 + L_2\delta_2)\}}$$

and (53) hold. For example, if $a = 1$, $b = 5$, and $L_2 = 1.5$, then the set of solutions satisfying the ADT condition (51) with $\delta_2 = 2.25$ and $N_{02} = 1$, and also the RADT condition (54) with $\delta^* = 0.45$ and $N_0^* = 1$ is GAS.

5. Conclusion and future research

We have proved several small-gain theorems for interconnections of hybrid subsystems which yield candidate ISS Lyapunov functions for the interconnections. These results unify several Lyapunov-based small-gain theorems for hybrid systems (Dashkovskiy & Kosmykov, 2013; Liberzon et al., 2014; Nešić & Teel, 2008) and impulsive systems (Dashkovskiy et al., 2012; Dashkovskiy & Mironchenko, 2013b), and pave the way to the following general scheme for establishing ISS of interconnections of hybrid subsystems:

1. Construct a candidate exponential ISS Lyapunov function V_i for each subsystem Σ_i with rate coefficients c_i, d_i and linear internal gains.
2. Compute the index sets I_d, I_c of non-ISS dynamics.
3. Modify the candidate exponential ISS Lyapunov functions V_i *either for all* $i \in I_d$ via Proposition 5 *or for all* $i \in I_c$ via Proposition 6.
4. Invoke Theorem 4 to construct a candidate exponential ISS Lyapunov function W for the augmented interconnection $\tilde{\Sigma}$ with rate coefficients c, d.
5. Derive the conditions for ISS of $\tilde{\Sigma}$ via Proposition 1.
6. Summarize the conditions for ISS of the original interconnection Σ from those in Steps 3 and 5.

As we observed in Section 4, the modification of candidate ISS Lyapunov functions in Step 3 leads to enlarged internal gains. Therefore, a considerable improvement of this scheme above lies in the fact that only the candidate ISS Lyapunov functions with indices from I_d or those with indices from I_c would be modified, instead of all those with indices from $I_d \cup I_c$ as it was done in Liberzon et al. (2014). If either $I_d = \emptyset$ or $I_c = \emptyset$, then no subsystem needs to be modified at all. Moreover, this scheme also applies to arbitrary interconnections composed of $n \ge 2$ subsystems.

In the scheme above, it is assumed that all V_i are candidate exponential ISS Lyapunov functions with linear internal gains. However, the modification also works for candidate exponential Lyapunov functions with *nonlinear* internal gains, and Theorem 2 was proved for arbitrary candidate ISS Lyapunov functions with nonlinear internal gains. If Proposition 1 were extended to the case of non-exponential ISS Lyapunov functions, one could apply the scheme above for V_i with nonlinear internal gains as well. Such theorems have been proved in Dashkovskiy and Mironchenko (2013b, Theorems 1 and 3) for impulsive systems, and we believe that they can be generalized to hybrid systems as well. This is one of the possible directions for future research.

The more challenging questions are whether one can establish ISS of an interconnection in the presence of destabilizing dynamics in subsystems without enlarging the internal gains, or without modifying ISS Lyapunov functions at all. At the time these questions remain open.

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