53rd IEEE Conference on Decision and Control
December 15-17, 2014. Los Angeles, California, USA

Integral input-to-state stability of bilinear infinite-dimensional systems*

Andrii Mironchenko[1] and Hiroshi Ito[2]

Abstract— **We prove that uniform global asymptotic stability of bilinear infinite-dimensional control systems is equivalent to their integral input-to-state stability. Next we present a method for construction of iISS Lyapunov functions for such systems if the state space is a Hilbert space. Unique issues arising due to infinite-dimensionality are highlighted.**

Keywords: bilinear systems, infinite-dimensional systems, integral input-to-state stability, Lyapunov methods.

I. INTRODUCTION

The notion of input-to-state stability (ISS) unifies in a natural way two different types of stable behavior: asymptotic stability in the sense of Lyapunov and input-output stability [28]. ISS plays an important role in constructive nonlinear control [19], in particular in robust stabilization of nonlinear systems [9], design of robust (w.r.t. errors in measurements and/or quantization) nonlinear observers [20], stability of nonlinear networked control systems [17], [8] etc.

After its indisputable success in the field of ordinary differential equations, ISS theory began to spread out to other types of dynamical systems: time-delay systems, impulsive, switched and hybrid systems. Recently a development of ISS theory of partial differential equations has been started.

In [16], [6], [7], [21], ISS of infinite-dimensional systems

$$\dot{x}(t) = Ax(t) + f(x(t), u(t)),\ x(t) \in X, u(t) \in U, \quad (1)$$

has been studied via methods of semigroup theory [15], [4]. Here the state space X and the space of input values U are Banach spaces, $A : D(A) \to X$ is the generator of a C_0-semigroup over X with a domain of definition $D(A)$ and $f : X \times U \to X$ is Lipschitz w.r.t. the first argument. Many classes of evolution PDEs, such as parabolic and hyperbolic PDEs are of this kind [10], [3].

In [6] sufficient conditions, Lyapunov-based small-gain theorems and a linearization method for ISS of systems (1) have been developed, which provide tools to verify ISS in the infinite-dimensional setting via Lyapunov functions, and to construct such Lyapunov functions for interconnections of ISS systems. On the basis of [6], several results within ISS theory for impulsive infinite-dimensional systems have been proposed in [7].

In [16] and [21], special classes of systems (1) have been investigated via frequency-domain methods. In [16], relations between circle-criterion and ISS for systems (1) with sector-bounded nonlinearities have been provided and in [21] the problem of stabilization of infinite-dimensional systems via observer-based feedback has been addressed. A substantial effort has been made also in applications of ISS theory for nonlinear parabolic [22] and linear time-varying hyperbolic systems [25].

Already for finite-dimensional systems, ISS is still far too restrictive for wide classes of practical systems. Saturation and limitations in actuators, biochemical processes, population dynamics and traffic flows etc. often prevent systems from enjoying the ISS property. The state of a typical system stays bounded as long as the magnitude of the applied input remains below a specific threshold, but it becomes unbounded when the input magnitude exceeds the threshold. Here integral input-to-state stability (iISS) comes into play [27], [2]. The development of iISS framework allowed us to broaden the class of nonlinearities we can address in analysis and design of interconnected systems, but also highlighted the fundamental difference between techniques needed for iISS and ISS. Serious obstacles were encountered in extending ISS small-gain theorem to iISS systems [12], e.g. absence of ISS gain, incompatibility of signal spaces in trajectory-based approaches, and insufficiency of max-type Lyapunov functions popular in ISS Lyapunov-based approaches. Recently, breakthroughs have been made in [11], [13], [1], [18] for removing the obstacles. In particular, within the approach developed in [13], [14], Lyapunov functions for interconnected systems are explicitly constructed, which allows us to use small-gain criteria and to address external disturbances in exactly the same formulation as in the ISS small-gain theorem. Although the practical issues of saturation and limitations are not limited to finite-dimensional systems, to the best of the authors knowledge, the iISS has not been yet utilized for infinite-dimensional systems (excluding time-delay systems) in the literature.

In this paper we are going to develop basic tools for investigation of iISS of infinite-dimensional systems. First we introduce the notion of iISS Lyapunov functions and prove that existence of an iISS Lyapunov function for a system (1) implies iISS of this system. In Section III we investigate iISS of several classes of systems. After a short discussion of iISS theory of infinite-dimensional linear systems, which already differs from the finite-dimensional special case, we proceed to the study of bilinear systems. We prove, that uniformly globally asymptotically stable bilinear systems are necessarily iISS. First we show this result for systems which

*This work was supported by Japanese Society for Promotion of Science (JSPS).

[1] Andrii Mironchenko is with Department of Systems Design and Informatics, Kyushu Institute of Technology, 680-4 Kawazu, Iizuka, Fukuoka 820-8502, Japan andrii.mironchenko@mathematik.uni-wuerzburg.de
[2] Hiroshi Ito is with Department of Systems Design and Informatics, Kyushu Institute of Technology, 680-4 Kawazu, Iizuka, Fukuoka 820-8502, Japan hiroshi@ces.kyutech.ac.jp

state space is an arbitrary Banach space. Next we provide another proof of this result for systems whose state space is a Hilbert space. It resembles the original proof of this fact for finite-dimensional systems, given by Sontag in [27] and results into an explicit construction of an iISS Lyapunov function for bilinear systems. We illustrate our findings on an example of a parabolic system. The proofs of several results are omitted due to space constraints. They can be found in a full version of the paper [23].

We use the following notation throughout the paper. For linear normed spaces X, Y let $L(X, Y)$ be the space of bounded linear operators from X to Y and $L(X) := L(X, X)$. A norm in these spaces we denote by $\|\cdot\|$. By $C(X, Y)$ we denote the space of continuous functions from X to Y, $C(X) := C(X, X)$ and by $PC(X, Y)$ the space of piecewise right-continuous functions from X to Y. Both are equipped with the standard sup-norm.

We define $\mathbb{R} := (-\infty, \infty)$ and $\mathbb{R}_+ := [0, \infty)$. Let $\mathbb{N}$ denote the set of natural numbers. Let $L_p(0, d)$, $p \geq 1$ be a space of p-th power integrable functions $f : (0, d) \to \mathbb{R}$ with the norm $\|f\|_{L_p(0,d)} = \left(\int_0^d |f(x)|^p dx\right)^{\frac{1}{p}}$.

II. Problem formulation

Consider a system (1) and assume throughout the paper that X and U are Banach spaces and $f(0, 0) = 0$, i.e., $x \equiv 0$ is an equilibrium point of the unforced system (1). Let $\phi(t, \phi_0, u)$ denote the state of a system (1) at moment $t \in \mathbb{R}_+$ associated with an initial condition $\phi_0 \in X$ at $t = 0$, and input $u \in U_c$, where U_c is a linear normed space of admissible inputs equipped with a norm $\|\cdot\|_{U_c}$.

We use the following classes of comparison functions

$$
\begin{aligned}
\mathcal{P} \quad &:= \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous}, \\
&\qquad \gamma(0) = 0 \text{ and } \gamma(r) > 0 \text{ for } r > 0\} \\
\mathcal{K} \quad &:= \{\gamma \in \mathcal{P} \mid \gamma \text{ is strictly increasing}\} \\
\mathcal{K}_\infty \quad &:= \{\gamma \in \mathcal{K} \mid \gamma \text{ is unbounded}\} \\
\mathcal{L} \quad &:= \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous and strictly} \\
&\qquad \text{decreasing with } \lim_{t\to\infty} \gamma(t) = 0\} \\
\mathcal{KL} \quad &:= \{\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \mid \beta \text{ is continuous}, \\
&\qquad \beta(\cdot, t) \in \mathcal{K},\ \forall t \geq 0,\ \beta(r, \cdot) \in \mathcal{L},\ \forall r > 0\}
\end{aligned}
$$

We consider weak solutions of (1), i.e. solutions of the integral equation

$$x(t) = T(t)x(0) + \int_0^t T(t-s) f(x(s), u(s)) ds. \quad (2)$$

belonging to the class $C([0, \tau], X)$ for all $\tau > 0$. Here $\{T(t),\ t \geq 0\}$ is a C_0-semigroup on a Banach space X with an infinitesimal generator $A : D(A) \to X$, $Ax = \lim_{t\to+0} \frac{1}{t}(T(t)x - x)$, whose domain of definition $D(A)$ consists of those $x \in X$, for which this limit exists.

Definition 1: We call $f : X \times U \to X$ Lipschitz continuous on bounded subsets of X, uniformly w.r.t. the second argument if $\forall w > 0\ \exists L(w) > 0$, such that $\forall x, y : \|x\|_X \leq w,\ \|y\|_X \leq w$, $\forall v \in U$

$$\|f(y, v) - f(x, v)\|_X \leq L(w)\|y - x\|_X. \quad (3)$$

We will use the following assumption concerning nonlinearity f throughout the paper

Assumption 1: We assume that $f : X \times U \to X$ is Lipschitz continuous on bounded subsets of X, uniformly w.r.t. the second argument and that $f(x, \cdot)$ is continuous for all $x \in X$.

Assumption 1 ensures that the weak solution of (1) exists and is unique, according to a variation of a classical existence and uniqueness theorem [3, Proposition 4.3.3].

Next we introduce stability properties for the system (1).

Definition 2: System (1) is *globally asymptotically stable at zero uniformly with respect to state* (0-UGASs), if $\exists\beta \in \mathcal{KL}$, such that $\forall \phi_0 \in X$, $\forall t \geq 0$ it holds

$$\|\phi(t, \phi_0, 0)\|_X \leq \beta(\|\phi_0\|_X, t). \quad (4)$$

To study stability properties of (1) with respect to external inputs, we use the notion of input-to-state stability [6]:

Definition 3: System (1) is called *input-to-state stable (ISS) w.r.t. space of inputs* U_c, if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that the inequality

$$\|\phi(t, \phi_0, u)\|_X \leq \beta(\|\phi_0\|_X, t) + \gamma(\|u\|_{U_c}) \quad (5)$$

holds $\forall \phi_0 \in X$, $\forall u \in U_c$ and $\forall t \geq 0$.

We emphasize that the above definition does not yet exactly correspond to ISS of finite dimensional systems [28] since Definition 3 allows the flexibility in the choice of U_c. A *system* (1) *is called ISS*, without expressing the normed space of inputs explicitly, if it is ISS w.r.t. $U_c = C(\mathbb{R}_+, U)$ endowed with a usual supremum norm. This terminology follows that of ISS for finite dimensional systems (although in finite dimensional theory measurable locally essentially bounded w.r.t. time inputs are usually considered).

The following notion is central in this paper

Definition 4: System (1) is called integral input-to-state stable (iISS) if there exist $\alpha \in \mathcal{K}_\infty$, $\mu \in \mathcal{K}$ and $\beta \in \mathcal{KL}$ such that the inequality

$$\alpha(\|\phi(t, \phi_0, u)\|_X) \leq \beta(\|\phi_0\|_X, t) + \int_0^t \mu(\|u(s)\|_U) ds \quad (6)$$

holds $\forall \phi_0 \in X$, $\forall u \in U_c = C(\mathbb{R}_+, U)$ and $\forall t \geq 0$.

A useful tool for investigation of iISS is an iISS Lyapunov function:

Definition 5: A continuous function $V : X \to \mathbb{R}_+$ is called an *iISS Lyapunov function*, if there exist $\psi_1, \psi_2 \in \mathcal{K}_\infty$, $\alpha \in \mathcal{P}$ and $\sigma \in \mathcal{K}$ such that

$$\psi_1(\|x\|_X) \leq V(x) \leq \psi_2(\|x\|_X), \quad \forall x \in X \quad (7)$$

and system (1) satisfies

$$\dot{V}_u(x) \leq -\alpha(\|x\|_X) + \sigma(\|u(0)\|_U) \quad (8)$$

for all $x \in X$ and $u \in U_c$, where the Lie derivative of V corresponding to the input u is defined by

$$\dot{V}_u(x) = \overline{\lim_{t\to+0}} \frac{1}{t}(V(\phi(t, x, u)) - V(x)). \quad (9)$$

Furthermore, if

$$\lim_{\tau\to\infty} \alpha(\tau) = \infty \text{ or } \liminf_{\tau\to\infty} \alpha(\tau) \geq \lim_{\tau\to\infty} \sigma(\tau) \quad (10)$$

holds, system V is called an *ISS Lyapunov function*.

Remark 1: We have introduced a definition of an ISS Lyapunov function in a dissipative form. In [6] another definition of an ISS Lyapunov function was given (in a so-called implication form). For finite-dimensional systems existence of a Lyapunov function in an implicative form implies existence of a Lyapunov function in a dissipative form. For infinite-dimensional systems only partial result in this direction is available, see [23].

Next proposition underlines an importance of an iISS Lyapunov function.

Proposition 2: If there exist an iISS (resp. ISS) Lyapunov function for (1), then (1) is iISS (resp. ISS).

Proof: We omit the proof due to page limit policy. ■

III. iISS AND ISS OF SEVERAL CLASSES OF SYSTEMS

A. Linear systems

We begin with a class of linear systems (1) with $f(x(t), u(t)) := Bu(t)$:

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t),\\ x(0) &= \phi_0,\end{aligned} \tag{11}$$

where $x : \mathbb{R}_+ \to X$, $u : \mathbb{R}_+ \to U$, and $B : U \to X$ is a linear operator.

It is well-known that for linear finite-dimensional systems 0-GAS (local stability + global attractivity), 0-UGASs, ISS and iISS are identical properties [26], [27]. For infinite-dimensional systems, 0-GAS is not equivalent to 0-UGASs in general. In [6, Proposition 3] it was shown that 0-UGASs is equivalent to ISS provided that B is a bounded operator. Moreover, in [6, p. 8] and [22, p. 247] examples of 0-GAS infinite-dimensional systems are provided whose solutions go to infinity even for inputs of arbitrarily small magnitude. Next proposition shows the relations between ISS and iISS for systems (11).

Proposition 3: Let $B \in L(U, X)$. Then, (11) is 0-UGASs $\Leftrightarrow$ (11) is ISS $\Leftrightarrow$ (11) is ISS w.r.t. $L_p(\mathbb{R}_+, U)$ for some $p \geq 1$.

Proof: The proof is straightforward and is omitted. ■

From Proposition 3 it follows (by taking $\alpha := id$ and $\gamma := c \cdot id$ for large enough $c > 0$ in Definition 4):

Corollary 4: System (11) is ISS iff it is iISS.

For finite-dimensional systems, in the presence of nonlinearities which are locally Lipschitz w.r.t. state, 0-GAS implies local ISS [29, Lemma I.1], i.e., the ISS property for initial states and inputs with a sufficiently small norm. In contrast to this finite-dimensional fact, we next show an infinite-dimensional linear system illustrating that *for unbounded operator B, 0-UGASs implies neither ISS nor iISS, even if the initial state and the input is restricted to sufficiently small neighborhoods of the origin.*

Example 5: Consider the following ODE ensemble defined on the interval $(0, \pi/2)$ of the spatial variable l:

$$\dot{x}(l,t) = -x(l,t) + (\tan l)^{\frac{1}{8}} u(l,t), \quad l \in (0, \pi/2). \tag{12}$$

Let $X = C(0, \pi/2)$ be the space of bounded continuous functions on $(0, \pi/2)$. The functions $x(l,t)$ and $u(l,t)$ are scalar-valued. The input operator $B : D(B) \to X$ for (12) is defined by $(Bv)(l) = (\tan l)^{\frac{1}{8}} v(l)$ which is unbounded with a domain of definition

$$D(B) = \{v \in C(0, \pi/2) : \sup_{l \in (0,\pi/2)} |(\tan l)^{\frac{1}{8}} v(l)| < \infty\}.$$

Since $x(\cdot, t) = e^{-t} x(\cdot, 0)$ holds for $u(\cdot, t) = 0$, $\forall t \geq 0$, system (12) is 0-UGASs. But it is neither ISS nor iISS for $U = D(B)$. To verify this fact, consider an input $u(l,t) = \hat{u}_c(l)$ given by

$$\hat{u}_c(l) = \begin{cases} b & , \ 0 < l < \arctan(c^8) \\ bc(\tan l)^{-\frac{1}{8}} & , \ \arctan(c^8) \leq l < \frac{\pi}{2} \end{cases}$$

for real $b, c > 0$ (to simplify notation we do not reflect in the notation the dependence of $\hat{u}_c$ on b). It is easy to see that $\hat{u}_c \in D(B)$ and $\|\hat{u}_c\|_U = b$ from $\|B\hat{u}_c\|_X = \sup_{l \in (0,\pi/2)} |\hat{u}_c(l)(\tan l)^{\frac{1}{8}}| = bc$ and the definition of $\hat{u}_c$. The solution of (12) for $\phi_0 = 0$ is computed as $\phi(t, 0, u)(l) = \int_0^t e^{-(t-r)} \hat{u}_c(l)(\tan l)^{\frac{1}{8}} dr = (1 - e^{-t})\hat{u}_c(l)(\tan l)^{\frac{1}{8}}$. Thus, by definition, the solution satisfies

$$\sup_{l \in (0,\pi/2)} \phi(t, 0, u) = bc(1 - e^{-t}).$$

Now, assume that system (12) is iISS. From Definition 4 it follows that there exist α, $\mu \in \mathcal{K}_\infty$ satisfying $\|\phi(t, 0, u)\|_X \leq \alpha^{-1}(t\mu(b))$ for $t \geq 0$. Clearly, for any given α, $\mu \in \mathcal{K}_\infty$ and any $t > 0$, one can find $c > 0$ so that $bc(1 - e^{-t}) > \alpha^{-1}(t\mu(b))$. Since $\|\hat{u}_c\|_U = b$ is satisfied for any $c > 0$, system (12) is not iISS.

Similarly one can show that (12) is not ISS. Since we can take $\phi_0 = 0$ and $b > 0$ is arbitrary, the system (12) is neither iISS nor ISS even if the initial state and the input are restricted to arbitrarily small neighborhoods of the origin. □

In the following example the same equation (12) is studied as in Example 5. It highlights the dependency of ISS and iISS on the choice of spaces.

Example 6: Consider the system (12) again. The system is ISS if we choose $X = L_2(0, \pi/2)$, $U = L_4(0, \pi/2)$.

Define

$$V(x) := \int_0^{\pi/2} x^2(l) dl = \|x\|^2_{L_2(0,\pi/2)}$$

For the solutions $x(\cdot, t) = \phi(t, \phi_0, u)$ of (12) we obtain

$$\begin{aligned}\frac{d}{dt} V(x) &= 2 \int_0^{\pi/2} x(l,t)\Big(-x(l,t) + u(l,t)(\tan l)^{\frac{1}{8}}\Big) dl \\ &\leq -2V(x) + wV(x) + \frac{1}{w} \int_0^{\pi/2} u(l,t)^2 (\tan l)^{\frac{1}{4}} dl \\ &\leq (-2 + w)V(x) + \frac{K}{w} \|u(\cdot, t)\|^2_{L_4(0,d)},\end{aligned}$$

for any $w > 0$ (between lines 1 and 2 Young's inequality has been used). Here $K := \int_0^{\pi/2} (\tan l)^{\frac{1}{2}} dl < \infty$. Hence, taking $w < 2$, Proposition 2 proves that system (12) is ISS for $X = L_2(0, \pi/2)$ and $U = L_4(0, \pi/2)$. □

B. iISS of generalized bilinear systems

While for linear infinite-dimensional systems with bounded input operators the properties of ISS and iISS coincide, the difference between these two properties arises for bilinear systems which is one of the simplest classes of nonlinear systems. For finite-dimensional bilinear systems, Sontag [27] demonstrated that 0-GAS systems are seldom ISS[1], and that a system is 0-GAS if and only if it is iISS. To generalize this fact to infinite-dimensional systems, consider

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + C(x(t), u(t)), \\ x(0) &= \phi_0, \end{aligned} \tag{13}$$

where $B \in L(U, X)$, and $C : X \times U \to X$ is s.t. $\exists K > 0$:

$$\|C(x, u)\|_X \leq K \|x\|_X \|u\|_U. \tag{14}$$

for all $x \in X$ and all $u \in U$. This class of systems includes bilinear systems with bounded input operators.

Next we prove the equivalence between 0-UGASs and iISS for the general system (13) in Banach spaces. For infinite-dimensions, we employ the notion of 0-UGASs instead of 0-GAS. To establish iISS from 0-GAS for finite-dimensional systems, Sontag [27] constructed Lyapunov functions for systems with Hurwitz A by means of the Lyapunov's equation. To the best of authors' knowledge, there is no generalization of this method for linear systems on Banach spaces, although for systems on Hilbert spaces such a construction exists. Thus, to allow for Banach spaces, we employ another method to prove equivalence between iISS and 0-UGASs.

Theorem 7: System (13) is iISS $\Leftrightarrow$ (13) is 0-UGASs.
We only sketch the proof. Details can be found in [23].

Proof: Clearly, for (13) iISS implies 0-UGASs. To prove the converse, assume that (13) be 0-UGASs, that is let T be an exponentially stable semigroup, generated by A.

Integrating (13), we obtain

$$x(t) = T(t)x(0) + \int_0^t T(t - r)\big(Bu(r) + C(x(r), u(r))\big) dr.$$

Since $B \in L(U, X)$, inequality (14) and exponential stability of T imply existence of $K, M, \lambda > 0$, so that

$$\begin{aligned} \|x(t)\|_X \leq Me^{-\lambda t}\|x(0)\|_X + \int_0^t Me^{-\lambda(t-r)} \\ \cdot \big(\|B\|\|u(r)\|_U + K\|x(r)\|_X\|u(r)\|_U\big) dr. \end{aligned}$$

We multiply both sides of the inequality by $e^{\lambda t}$ and define $z(t) = x(t)e^{\lambda t}$. From $\lambda > 0$ we obtain

$$\begin{aligned} \|z(t)\|_X \leq & M\Big(\|z(0)\|_X + \|B\| \int_0^t e^{\lambda r}\|u(r)\|_U dr\Big) \\ &+ \int_0^t MK\|z(r)\|_X\|u(r)\|_U dr. \end{aligned}$$

Gronwall's inequality (see e.g. [30, Lemma 2.7, p.42]) yields

$$\begin{aligned} \|z(t)\|_X \leq & M\Big(\|z(0)\|_X + \|B\| \int_0^t e^{\lambda r}\|u(r)\|_U dr\Big) \\ & \cdot e^{\int_0^t MK\|u(r)\|_U dr}. \end{aligned}$$

[1]E.g. a system $\dot{x} = -x + xu$, $x(\cdot) \in \mathbb{R}$ is not ISS.

Coming back to original variables and using $\lambda > 0$, we have

$$\begin{aligned} \|x(t)\|_X \leq & M\Big(e^{-\lambda t}\|x(0)\|_X + \|B\| \int_0^t \|u(r)\|_U dr\Big) \\ & \cdot e^{\int_0^t MK\|u(r)\|_U dr}. \end{aligned}$$

Define $\alpha \in \mathcal{K}_\infty$ by $\alpha(r) = \ln(1 + r)$, $\forall r \geq 0$. This results (after some computations) in

$$\begin{aligned} \alpha(\|x(t)\|_X) \leq & \ln\big(1 + Me^{-\lambda t}\|x(0)\|_X\big) \\ & + \ln\Big(1 + M\|B\| \int_0^t \|u(r)\|_U dr\Big) \\ & + \int_0^t MK\|u(r)\|_U dr. \end{aligned}$$

Since $\beta : (r, t) \mapsto \ln(1 + Me^{-\lambda t}r)$ is a $\mathcal{KL}$-function, the above estimate shows us that (13) is iISS. ■

C. Lyapunov functions for generalized bilinear systems

This section develops a method to construct an iISS Lyapunov function for the infinite-dimensional system (13) analogous to the finite-dimensional case [27]. For this purpose, in this section, let X be a Hilbert space with a scalar product $\langle \cdot, \cdot \rangle$, and assume that A generate an analytic semigroup on X. Note that if (13) is 0-UGASs, the operator A generates exponentially stable semigroup [6, Lemma 1]. Since X is a Hilbert space, the exponential stability of this semigroup is equivalent to existence of a positive self-adjoint operator $P \in L(X)$ satisfying the Lyapunov equation

$$\langle Ax, Px \rangle + \langle Px, Ax \rangle = -\|x\|_X^2, \quad \forall x \in D(A) \tag{15}$$

see [4, Theorem 5.1.3, p. 217]. Recall that a self-adjoint operator $P \in L(X)$ is called positive if $\langle Px, x \rangle > 0$ holds for all $x \in X \setminus \{0\}$. A positive operator $P \in L(X)$ is called coercive if there exists $k > 0$ such that

$$\langle Px, x \rangle \geq k\|x\|_X^2 \quad \forall x \in D(P).$$

Next theorem shows a method for construction of iISS Lyapunov functions for bilinear systems

Theorem 8: Consider a system (13) over a Hilbert space X. Assume that A generate an analytic semigroup on X, and there exists a coercive positive self-adjoint operator $P \in L(X)$ satisfying (15). Then system (13) is iISS and its iISS Lyapunov function can be constructed as

$$W(x) = \ln\Big(1 + \langle Px, x \rangle\Big). \tag{16}$$

As usual we omit detailed computations.

Proof: Let assumptions of the theorem hold. Consider a function $V : x \mapsto \langle Px, x \rangle$. Since P is bounded and coercive, for some $k > 0$ it holds

$$k\|x\|_X^2 \leq V(x) \leq \|P\|\|x\|_X^2, \quad \forall x \in X,$$

and property (7) is verified. Let us compute the Lie derivative of V with respect to system (13). For $x \in D(A)$ we have

$$\begin{aligned} \dot{V}(x) = & \langle P(Ax), x \rangle + \langle Px, Ax \rangle \\ & + \langle P(Bu + C(x, u)), x \rangle + \langle Px, Bu + C(x, u) \rangle. \end{aligned}$$

From $\langle P(Ax), x\rangle = \langle Ax, Px\rangle$, (14) and (15) with the help of Cauchy-Schwarz inequality, we obtain

$$\dot{V}(x) \le -\|x\|_X^2 + 2K\|P\|\|x\|_X^2\|u\|_U + 2\|P\|\|B\|\|x\|_X\|u\|_U.$$

Let $\varepsilon > 0$. Using Young's inequality

$$2\|x\|_X\|u\|_U \le \varepsilon\|x\|_X^2 + \frac{1}{\varepsilon}\|u\|_U^2,$$

we can continue the above estimates as

$$\dot{V}(x) \le -\Big(1 - \varepsilon\|P\|\|B\|\Big)\|x\|_X^2 + 2K\|P\|\|x\|_X^2\|u\|_U + \frac{\|P\|\|B\|}{\varepsilon}\|u\|_U^2.$$

Defining W as in (16) yields

$$\dot{W}(x) \le -(1 - \varepsilon\|P\|\|B\|)\frac{\|x\|_X^2}{1 + \|P\|\|x\|_X^2} + \frac{2K\|P\|}{k}\|u\|_U + \frac{\|P\|\|B\|}{\varepsilon}\|u\|_U^2. \tag{17}$$

These derivations hold for $x \in D(A) \subset X$. If $x \notin D(A)$, then for all admissible u the solution $x(t) \in D(A)$ and $t \to W(x(t))$ is a continuously differentiable function for all $t > 0$ (these properties follow from the properties of solutions $x(t)$, see Theorem 3.3.3 in [10]). Therefore, by the mean-value theorem, $\forall t > 0\ \exists t_* \in (0, t)$

$$\frac{1}{t}(W(x(t)) - W(x)) = \dot{W}(x(t_*)),$$

where $x = x(0)$. Taking the limit when $t \to +0$ we obtain that (17) holds for all $x \in X$. Pick $\varepsilon > 0$ such that $\varepsilon < 1/(\|P\|\|B\|)$. According to Proposition 2, system (13) is iISS and W is an iISS Lyapunov function. ■

IV. An example

In this concluding section we illustrate our findings on an example of an iISS parabolic system. Let $c > 0$ and $L > 0$. Consider the following reaction-diffusion system

$$\begin{cases} \dfrac{\partial x}{\partial t}(l,t) = c\dfrac{\partial^2 x}{\partial l^2}(l,t) + \dfrac{x(l,t)}{1+|l-1|x(l,t)^2}u(l,t), \\ x(0,t) = x(L,t) = 0; \end{cases} \tag{18}$$

on the region $(l, t) \in (0, L) \times (0, \infty)$ of the $\mathbb{R}$-valued functions $x(l, t)$ and $u(l, t)$.

Let $X = L_2(0, L)$ and $U = C(0, L)$. It is easy to see that the above system is bilinear since its nonlinearity satisfies inequality (14). Clearly, this system is 0-UGASs, therefore it is iISS for any $L > 0$. Below we give an explicit construction of an iISS Lyapunov function for this system. Afterwards we will prove that this system is ISS for $L < 1$.

Define

$$W(x) = \int_0^L x^2(l)dl = \|x\|_{L_2(0,L)}^2.$$

Since $1 + |l - 1|x(l, t)^2 \ge 1$, we obtain

$$\begin{aligned}\dot{W}(x) =& 2\int_0^L x(l)\Big(c\frac{\partial^2 x}{\partial l^2}(l,t) + \frac{x(l,t)}{1+|l-1|x(l,t)^2}u(l,t)\Big)dl \\ \le& -2c\int_0^L \left(\frac{\partial x}{\partial l}(l,t)\right)^2 dl + 2\int_0^L x^2(l,t)|u(l,t)|dl.\end{aligned}$$

Using the Friedrich's inequality (see e.g. [24, p. 67]) in the first term, we continue estimates:

$$\dot{W}(x) \le -2c\left(\frac{\pi}{L}\right)^2 W(x) + 2W(x)\|u\|_{C(0,L)}$$

Choosing

$$V(x) = \ln(1 + W(x)) \tag{19}$$

yields

$$\begin{aligned}\dot{V}(x) \le& -2c\Big(\frac{\pi}{L}\Big)^2\frac{W(x)}{1+W(x)} + 2\frac{W(x)}{1+W(x)}\|u\|_{C(0,L)} \\ \le& -2c\Big(\frac{\pi}{L}\Big)^2\frac{\|x\|_{L_2(0,L)}^2}{1+\|x\|_{L_2(0,L)}^2} + 2\|u\|_{C(0,L)}, \qquad (20) \\ =& -\alpha(\|x\|_{L_2(0,L)}) + \sigma(\|u\|_{C(0,L)}),\end{aligned}$$

where

$$\alpha(s) = 2c\Big(\frac{\pi}{L}\Big)^2\frac{s^2}{1+s^2}, \quad \sigma(s) = 2s. \tag{21}$$

Thus, Proposition 2 establishes iISS of (18) irrespective of a value of L.

Remark 9: Since $x(\cdot, t) \in L_2(0, L)$, the spatial derivative of x above may not exist. However, the above derivations hold for smooth enough functions x, and the general result for all $x(\cdot, t) \in L_2(0, L)$ will follow due to the density argument, see [5, Section 2.2.1] for details.

Interestingly, when $L < 1$, the system (18) is ISS for the input space $U = C(0, L)$ as well as $U = L_2(0, L)$. To verify this, we first note that for all $l < 1$

$$\sup_{s\in\mathbb{R}}\left|\frac{s}{1 + |l-1|s^2}\right| = \frac{1}{2\sqrt{1-l}}. \tag{22}$$

Let $L < 1$. Using the same Lyapunov function W we have

$$\begin{aligned}\dot{W}(x) \le& 2\int_0^L x(l,t)c\frac{\partial^2 x}{\partial l^2}(l,t)dl \\ &+ 2\int_0^L \frac{1}{2\sqrt{1-l}}|x(l,t)u(l,t)|dl \\ \le& -2c\left(\frac{\pi}{L}\right)^2\|x\|_{L_2(0,L)}^2 \\ &+ \frac{1}{\sqrt{1-L}}\|x\|_{L_2(0,L)}\|u\|_{L_2(0,L)} \\ \le& -\left(2c\left(\frac{\pi}{L}\right)^2 - w\right)\|x\|_{L_2(0,L)}^2 \\ &+ \frac{1}{4(1-L)w}\|u\|_{L_2(0,L)}^2\end{aligned} \tag{23}$$

for $0 < w < 2c(\pi/L)^2$. Recall that $\|u\|_{L_2(0,L)}^2 \le L\|u\|_{C(0,L)}^2$. Thus, by virtue of Proposiiton 2, system (18) is ISS whenever $L < 1$. It is stressed that the coefficient of $\|u\|_{L_2(0,L)}^2$ in (23) goes to ∞ as L tends to 1 from below. Hence, the ISS estimate (23) is valid only if $L < 1$.

For the choice of input space $U = L_p(0, L)$ with $p \ge 1$, the case of $L \ge 1$ does not allow us to have an ISS estimate like (23). In fact, if $L \ge 1$ and $U = L_p(0, L)$ for any $p \ge 1$, the right hand side of (18) system is undefined. To see this take $u : l \mapsto |l-1|^{-\frac{1}{2p}} \in L_p(0, L)$ and $x : l \mapsto |l-1|^{-\frac{1}{2}+\frac{1}{2p}} \in L_2(0, L)$. Then $f(x, u) : l \mapsto$

$\frac{x(l,t)}{1+|l-1|x(l,t)^2}u(l,t) \notin L_2(0,L)$. Thus, according to our formulation of (1), the system (18) is not well-defined for $U = L_p(0,L)$ for any real $p \geq 1$.

For the choice of input space $U = C(0,L)$, we expect that the system (18) is not ISS for $L \geq 1$, but we have not proved it at this time. The blow-up of the σ-term in (23) corresponding to the dissipation inequality (8) for $V = W$ suggests the absence of ISS for the system (18) in the case of $L \geq 1$. It is worth noticing that the iISS estimate (20) is valid for all $L > 0$, that is for all $L > 0$ we do no have in (21) a blowup of σ, and α does not become a zero function. In fact, one can recall the idea demonstrated by Proposition 2 with Definition 5. An iISS Lyapunov function characterizes the absence of ISS by only allowing the decay rate α in the dissipation inequality (8) to satisfy $\liminf_{s\to\infty} \alpha(s) < \lim_{s\to\infty} \sigma(s)$. The iISS Lyapunov function yields ISS when $\liminf_{s\to\infty} \alpha(s) \geq \lim_{s\to\infty} \sigma(s)$, which is not the case in (21). Being able to uniformly characterize iISS irrespectively of whether systems are ISS or not should be advantageous in many applications. For instance, ISS of subsystems is not necessary for stability of their interconnections, and there are examples of UGAS interconnections involving iISS systems which are not ISS [11], [13], [1], [18].

V. CONCLUSION

We have proved that infinite-dimensional bilinear systems described by differential equations in Banach spaces are integral input-to-state stable provided they are uniformly globally asymptotically stable. For systems whose state space is Hilbert we have obtained under some additional restrictions, another proof of this result, which leads to a construction of an iISS Lyapunov function for the system.

Among possible directions for future research are investigation of iISS of more general nonlinear control systems and development of novel methods for construction of iISS Lyapunov functions for such systems. Another challenging problem is a study of interconnected infinite-dimensional systems, whose subsystems are iISS or ISS.

REFERENCES

[1] D. Angeli and A. Astolfi. A tight small gain theorem for not necessarily ISS systems. *Syst. Control Lett.*, 56:87–91, 2007.

[2] D. Angeli, E. D. Sontag, and Y. Wang. A characterization of integral input-to-state stability. *IEEE Transactions on Automatic Control*, 45(6):1082–1097, 2000.

[3] T. Cazenave and A. Haraux. *An Introduction To Semilinear Evolution Equations*, volume 13 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, New York, 1998.

[4] R. F. Curtain and H. Zwart. *An Introduction to Infinite-dimensional Linear Systems Theory*, volume 21 of *Texts in Applied Mathematics*. Springer-Verlag, New York, 1995.

[5] S. Dashkovskiy and A. Mironchenko. Dwell-time conditions for robust stability of impulsive systems. In *Proceedings of the 20th International Symposium on Mathematical Theory of Systems and Networks (MTNS 2012)*, 2012.

[6] S. Dashkovskiy and A. Mironchenko. Input-to-state stability of infinite-dimensional control systems. *Mathematics of Control, Signals, and Systems*, 25(1):1–35, 2013.

[7] S. Dashkovskiy and A. Mironchenko. Input-to-State Stability of Nonlinear Impulsive Systems. *SIAM Journal on Control and Optimization*, 51(3):1962–1987, 2013.

[8] S. Dashkovskiy, B. S. Rüffer, and F. R. Wirth. An ISS small gain theorem for general networks. *Mathematics of Control, Signals, and Systems*, 19(2):93–122, 2007.

[9] R. A. Freeman and P. V. Kokotovic. *Robust Nonlinear Control Design: State-Space and Lyapunov Techniques*. Modern Birkhäuser Classics. Birkhäuser Boston, Boston, MA, 2008.

[10] D. Henry. *Geometric Theory of Semilinear Parabolic Equations*, volume 840 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1981.

[11] H. Ito. State-dependent scaling problems and stability of interconnected iISS and ISS systems. *IEEE Trans. Automat. Contr.*, 51:1626–1643, 2006.

[12] H. Ito. Utility of iISS in composing lyapunov functions. In *9th IFAC Symposium on Nonlinear Control Systems*, pages 723–730, 2013.

[13] H. Ito and Z.-P. Jiang. Necessary and Sufficient Small Gain Conditions for Integral Input-to-State Stable Systems: A Lyapunov Perspective. *IEEE Transactions on Automatic Control*, 54(10):2389–2404, 2009.

[14] H. Ito, Z. P. Jiang, S. Dashkovskiy, and B. Rüffer. Robust Stability of Networks of iISS Systems: Construction of Sum-Type Lyapunov Functions. *IEEE Transactions on Automatic Control*, 58(5):1192–1207, 2013.

[15] B. Jacob and H. J. Zwart. *Linear Port-Hamiltonian Systems on Infinite-dimensional Spaces*. Springer Basel, Basel, 2012.

[16] B. Jayawardhana, H. Logemann, and E. P. Ryan. Infinite-dimensional feedback systems: the circle criterion and input-to-state stability. *Communications in Information and Systems*, 8(4):413–444, 2008.

[17] Z.-P. Jiang, I. M. Y. Mareels, and Y. Wang. A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems. *Automatica*, 32(8):1211–1215, 1996.

[18] I. Karafyllis and Z.-P. Jiang. A new small-gain theorem with an application to the stabilization of the chemostat. *Int. J. Robust. Nonlinear Control*, 22(14):1602–1630, 2012.

[19] P. Kokotović and M. Arcak. Constructive nonlinear control: a historical perspective. *Automatica*, 37(5):637–662, 2001.

[20] D. Liberzon. Observer-based quantized output feedback control of nonlinear systems. In *15th Mediterranean Conference on Control and Automation*, pages 1–5, June 2007.

[21] H. Logemann. Stabilization of Well-Posed Infinite-Dimensional Systems by Dynamic Sampled-Data Feedback. *SIAM Journal on Control and Optimization*, 51(2):1203–1231, 2013.

[22] F. Mazenc and C. Prieur. Strict Lyapunov functions for semilinear parabolic partial differential equations. *Mathematical Control and Related Fields*, 1(2):231–250, 2011.

[23] A. Mironchenko and H. Ito. Characterizations of integral input-to-state stability for bilinear systems in infinite dimensions. http://arxiv.org/abs/1406.2458, 2014.

[24] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink. *Inequalities involving functions and their integrals and derivatives*, volume 53 of *Mathematics and its Applications (East European Series)*. Kluwer Academic Publishers Group, Dordrecht, 1991.

[25] C. Prieur and F. Mazenc. ISS-Lyapunov functions for time-varying hyperbolic systems of balance laws. *Mathematics of Control, Signals, and Systems*, 24(1-2):111–134, 2012.

[26] E. D. Sontag. Smooth stabilization implies coprime factorization. *IEEE Transactions on Automatic Control*, 34(4):435–443, 1989.

[27] E. D. Sontag. Comments on integral variants of ISS. *Systems & Control Letters*, 34(1-2):93–100, 1998.

[28] E. D. Sontag. Input to State Stability: Basic Concepts and Results. In P. Nistri and G. Stefani, editors, *Nonlinear and Optimal Control Theory*, volume 1932 of *Lecture Notes in Mathematics*, chapter 3, pages 163–220. Springer, Berlin, Heidelberg, 2008.

[29] E. D. Sontag and Y. Wang. New characterizations of input-to-state stability. *IEEE Transactions on Automatic Control*, 41(9):1283–1294, 1996.

[30] G. Teschl. *Ordinary Differential Equations and Dynamical Systems*. Graduate studies in mathematics. American Mathematical Society, 2012.