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Construction of iSS Lyapunov functions for interconnected parabolic systems

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Abstract— **This paper develops tools to construct Lyapunov functions establishing integral input-to-state stability (iISS) and input-to-state stability (ISS) for several classes of nonlinear parabolic equations. Using these constructions and an infinite-dimensional iISS small-gain theorem we provide a stability criterion for interconnections of iISS parabolic systems. We show that for interconnections of partial differential equations it is essential to choose right state and input spaces, especially for iISS subsystems which are not ISS.**

I. Introduction

Input-to-state stability (ISS) unified into one framework two different types of stable behavior: asymptotic stability and input-output stability [22]. Within ISS theory an effective method for study of interconnected systems has been developed, on the basis of Lyapunov methods and small-gain arguments [14]. Nevertheless, saturation and limitations in actuation and processing rates, which are often encountered in the real-world systems, prevent these systems of being ISS, since the state of such systems stays bounded as long as the magnitude of the applied inputs remains below a specific threshold, but becomes unbounded when the input magnitude exceeds the threshold. However, these systems often possess a weaker stability property, called integral input-to-state stability (iISS) [21], [2]. Nowadays Lyapunov and small-gain theory for iISS and iISS-like systems are as powerful as its ISS counterparts [2], [8], [11], [1], [15], [9].

Recently a development of an ISS and iISS theory for partial differential equations (PDEs) has been started. In [13], [4], [5], [16], ISS of infinite-dimensional systems

$$\dot{x}(t) = Ax(t) + f(x(t),u(t)), \ x(t) \in X, u(t) \in U \qquad (1)$$

has been studied via methods of semigroup theory [12], [3]. Here the state space X and the space of input values U are Banach spaces, $A : D(A) \to X$ is the generator of a C_0-semigroup over X. Many classes of evolution PDEs, such as parabolic and hyperbolic equations are of this kind [7], [3]. As in the case of finite-dimensional systems [22], existence of an ISS Lyapunov function is sufficient for ISS of (1) (see [4]). This motivated the results in [4] on constructions of ISS Lyapunov functions for a class of parabolic systems belonging to (1). More direct approach to the construction of Lyapunov functions for some classes of nonlinear parabolic and linear time-varying hyperbolic systems has been proposed in [17], [20]. In [13] and [16], systems (1) with a sector-bounded nonlinearity f have been investigated via frequency-domain methods.

To the best of the authors' knowledge, except for time-delay systems, the study devoted to Lyapunov functions for iISS of infinite-dimensional systems has been started in [18], where equivalence of iISS and uniform global asymptotic stability for bilinear distributed parameter systems has been shown. The second result in [18] is an extension to bilinear systems over Hilbert spaces of a method from [21] for construction of iISS Lyapunov functions for bilinear ordinary differential equations (ODEs).

The abstract small gain type theorem proposed in [15] includes infinite-dimensional systems, but the applications of this framework to PDEs are not addressed. In [4], [5], ISS of large scale systems whose subsystems are in the form of (1) has been studied and the ISS small gain theorem, already available for finite-dimensional systems (see [14], [6]) has been extended to the infinite-dimensional systems. However, the method does not accommodate iISS subsystems which are not ISS.

This paper studies stability of interconnections of two parabolic systems, each of which is of the form

$$\frac{\partial x}{\partial t} = c\frac{\partial^2 x}{\partial l^2} + f(x(l,t), \tfrac{\partial x}{\partial l}(l,t), u(l,t)), \quad \forall t > 0, \qquad (2)$$

where $l \in (0,L)$, $x(l,t) \in \mathbb{R}$. This class of systems (2) allows more general functions f than the class considered in [4], [18], and possesses systems, which are not ISS. The primary goal of this paper is to accomplish an iISS small gain theorem [11], [9], originally proved for finite-dimensional systems, in the infinite-dimensional setting. In contrast to the small-gain theorem from [4], we require ISS property only from one subsystem and not from both of them; the other subsystem may be only iISS.

When working with PDEs, it is crucial to choose the state space in a right way. In particular, it is quite hard to find an iISS parabolic system whose state and input spaces are both L_p-spaces, while such difficulties do not arise in ISS systems. To address this issue, this paper reexamines constructions of iISS and ISS Lyapunov functions for parabolic systems developed in [4], [18], and actively exploits Sobolev spaces as state spaces. For interconnections of PDE systems additional difficulties arise since we need not only to choose right state and input spaces for every subsystem, but also to match them with the state and input spaces for other subsystems. Last but not least, incompatibility of spaces in the time domain,

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which is crucial for interconnections of ODE systems, is as important for PDE systems. All these issues make study of interconnections of iISS infinite-dimensional systems a challenging problem, which we solve here for some classes of parabolic systems in one-dimensional spatial domain.

This paper does not contain the proofs of the presented results and is a preliminary version of the paper [19]. For the proofs as well as for detailed discussions of results please consult [19].

Notation

We define $\mathbb{R}_+ := [0,\infty)$, and the symbol $\mathbb{N}$ denotes the set of natural numbers. By $C(\mathbb{R}_+,Y)$ we denote the space of continuous functions from $\mathbb{R}_+$ to Y, equipped with the standard sup-norm. We use the following function spaces:

- $C_0^k(0,L)$ is a space of k times continuously differentiable functions $f:(0,L)\to\mathbb{R}$ with a support compact in $(0,L)$.
- $L_p(0,L)$, $p\geq 1$ is a space of p-th power integrable functions $f:(0,L)\to\mathbb{R}$ with $\|f\|_{L_p(0,L)} = \left(\int_0^L |f(l)|^p dl\right)^{1/p}$.
- $W^{k,p}(0,L)$ is a Sobolev space of functions $f\in L_p(0,L)$, which have weak derivatives of order $\leq k$, all of which belong to $L_p(0,L)$. Norm in $W^{k,p}(0,L)$ is defined by $\|f\|_{W^{k,p}(0,L)} = \left(\int_0^L \sum_{1\leq s\leq k}\left|\frac{\partial^s f}{\partial l^s}(l)\right|^p dl\right)^{1/p}$.
- $W_0^{k,p}(0,L)$ is a closure of $C_0^k(0,L)$ in the norm of $W^{k,p}(0,L)$. We endow $W_0^{k,p}(0,L)$ with a norm $\|f\|_{W_0^{k,p}(0,L)} = \left(\int_0^L \left|\frac{\partial^k f}{\partial l^k}(l)\right|^p dl\right)^{1/p}$, equivalent to the norm $\|\cdot\|_{W^{k,p}(0,L)}$ on $W_0^{k,p}(0,L)$, see, [7, p.8].
- $H^k(0,L) = W^{k,2}(0,L)$, $H_0^k(0,L) = W_0^{k,2}(0,L)$.

To define and analyze stability properties we use so-called comparison functions

$$
\begin{aligned}
\mathcal{P} &:= \{\gamma\in C(\mathbb{R}_+,\mathbb{R}_+) \mid \gamma(0)=0,\ \gamma(r)>0 \text{ for } r>0\}\\
\mathcal{K} &:= \{\gamma\in\mathcal{P} \mid \gamma \text{ is strictly increasing}\}\\
\mathcal{K}_\infty &:= \{\gamma\in\mathcal{K} \mid \gamma \text{ is unbounded}\}\\
\mathcal{L} &:= \{\gamma:\mathbb{R}_+\to\mathbb{R}_+ \mid \gamma \text{ is continuous and strictly decreasing with } \lim_{t\to\infty}\gamma(t)=0\}\\
\mathcal{KL} &:= \{\beta:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+ \mid \beta \text{ is continuous, } \beta(\cdot,t)\in\mathcal{K},\ \forall t\geq 0,\ \beta(r,\cdot)\in\mathcal{L},\ \forall r>0\}
\end{aligned}
$$

II. Problem formulation

Consider the system (1) and assume throughout the paper that X and U are Banach spaces and that input functions belong to the space $C(\mathbb{R}_+,U)$. Let also T be a semigroup generated by A from (1). Under (weak) solutions of (1) we understand solutions of the integral equation

$$x(t) = T(t)x(0) + \int_0^t T(t-s)f(x(s),u(s))ds,\ \forall t\in[0,\tau] \quad (3)$$

belonging to $C([0,\tau],X)$ for some $\tau>0$.

We use the following assumption concerning nonlinearity f throughout the paper

Assumption 1: Let $f:X\times U\to X$ be Lipschitz continuous on bounded subsets of X, uniformly with respect to the second argument, i.e. $\forall C>0\ \exists K(C)>0$, such that $\forall x,y:$ $\|x\|_X\leq C,\ \|y\|_X\leq C,\ \forall v\in U$, it holds that

$$\|f(y,v)-f(x,v)\|_X \leq K(C)\|y-x\|_X. \quad (4)$$

Let also $f(x,\cdot)$ be continuous for all $x\in X$.

Since the inputs belong to $C(\mathbb{R}_+,U)$, Assumption 1 ensures that the weak solution of (1) exists and is unique, according to a variation of a classical existence and uniqueness theorem [3, Proposition 4.3.3].

Let $\phi(t,\phi_0,u)$ denote the state of a system (1), i.e. the solution to (1), at moment $t\in\mathbb{R}_+$ associated with an initial condition $\phi_0\in X$ at $t=0$, and input $u\in C(\mathbb{R}_+,U)$. Next we introduce stability properties for the system (1).

Definition 1: System (1) is *globally asymptotically stable at zero uniformly with respect to state* (0-UGASs), if $\exists\beta\in\mathcal{KL}$, such that $\forall\phi_0\in X$, $\forall t\geq 0$ it holds

$$\|\phi(t,\phi_0,0)\|_X \leq \beta(\|\phi_0\|_X,t). \quad (5)$$

To study stability properties of (1) with respect to external inputs, we use the notion of input-to-state stability [4]:

Definition 2: System (1) is called *input-to-state stable (ISS)*, if there exist $\beta\in\mathcal{KL}$ and $\gamma\in\mathcal{K}$: $\forall\phi_0\in X$, $\forall u\in C(\mathbb{R}_+,U)$ and $\forall t\geq 0$ it holds that

$$\|\phi(t,\phi_0,u)\|_X \leq \beta(\|\phi_0\|_X,t) + \gamma(\sup_{s\in[0,t]}\|u(s)\|_U). \quad (6)$$

If the system is not ISS, it may still have some sort of robustness. Thus we introduce another stability property

Definition 3: System (1) is called *integral input-to-state stable (iISS)* if there exist $\alpha\in\mathcal{K}_\infty$, $\mu\in\mathcal{K}$ and $\beta\in\mathcal{KL}$ such that the inequality

$$\alpha(\|\phi(t,\phi_0,u)\|_X) \leq \beta(\|\phi_0\|_X,t) + \int_0^t \mu(\|u(s)\|_U)ds \quad (7)$$

holds $\forall\phi_0\in X$, $\forall u\in C(\mathbb{R}_+,U)$ and $\forall t\geq 0$.

The following defines a useful notion for studying iISS.

Definition 4: A continuous function $V:X\to\mathbb{R}_+$ is called an *iISS Lyapunov function*, if $\exists\psi_1,\psi_2\in\mathcal{K}_\infty$, $\alpha\in\mathcal{P}$, $\sigma\in\mathcal{K}$:

$$\psi_1(\|x\|_X)\leq V(x)\leq\psi_2(\|x\|_X),\quad \forall x\in X \quad (8)$$

and system (1) satisfies

$$\dot{V}_u(x) \leq -\alpha(\|x\|_X) + \sigma(\|u(0)\|_U) \quad (9)$$

for all $x\in X$ and $u\in C(\mathbb{R}_+,U)$, where

$$\dot{V}_u(x) = \overline{\lim_{t\to+0}}\frac{1}{t}(V(\phi(t,x,u))-V(x)). \quad (10)$$

Furthermore, if $\lim_{\tau\to\infty}\alpha(\tau)=\infty$ or $\liminf_{\tau\to\infty}\alpha(\tau)\geq\lim_{\tau\to\infty}\sigma(\tau)$ holds, V is called an *ISS Lyapunov function*.

We write $\dot{V}$ instead of $\dot{V}_u(x)$ when it is clear along which solutions the derivative is taken.

Proposition 1 (Prop. 1, [18]): If there exist an iISS (resp. ISS) Lyapunov function for (1), then (1) is iISS (resp. ISS).

As a rule a construction of a Lyapunov function is the only realistic way to prove ISS/iISS of finite-dimensional systems. In the next sections we propose a method for constructing iISS and ISS Lyapunov functions for subclasses of a parabolic equation (2). Then we show how to construct the Lyapunov functions for systems of PDEs from the information about Lyapunov functions of subsystems by means of an small-gain approach.

III. ISS Lyapunov functions for a class of nonlinear parabolic systems: Sobolev state space

The purpose of this section is to develop a Lyapunov-type characterization of ISS for PDEs in (2). In several papers such characterizations for parabolic systems whose state space is an L_p space have been provided [17], [4]. However, as we will see in Section VII the iISS systems in many cases cannot have the L_p space both as an input and state space. Since our final goal is to consider interconnections of iISS and ISS systems, we need to have the constructions of ISS Lyapunov functions with Sobolev state spaces. This section provides one of such constructions.

Consider a system

$$\frac{\partial x}{\partial t} = c\frac{\partial^2 x}{\partial l^2} + f\big(x(l,t), \tfrac{\partial x}{\partial l}(l,t)\big) + u(l,t), \quad \forall t > 0 \qquad (11)$$

defined on the spatial domain $(0,L)$ with the Dirichlet boundary conditions

$$x(0,t) = x(L,t) = 0, \quad \forall t \geq 0. \qquad (12)$$

The next theorem gives a sufficient condition for ISS of (11) w.r.t. the state space $X = W_0^{1,2q}(0,L)$, $q \in \mathbb{N}$ and certain types of spaces U of input values.

Theorem 2: Suppose

$$\int_0^L \left(\frac{\partial x}{\partial l}\right)^{2q-2} \frac{\partial^2 x}{\partial l^2} f\left(x, \tfrac{\partial x}{\partial l}\right) dl \geq \int_0^L \eta\left(\left(\tfrac{\partial x}{\partial l}\right)^{2q}\right) dl \qquad (13)$$

holds for all twice continuously differentiable $x \in W_0^{1,2q}(0,L)$ with some convex continuous function $\eta : \mathbb{R}_+ \to \mathbb{R}$ and some $\varepsilon > 0$ such that

$$\hat{\alpha}(s) := \frac{\pi^2}{q^2 L^2}(c-\varepsilon)s + L\eta\left(\frac{s}{L}\right) \geq 0, \ \forall s \in \mathbb{R}_+. \qquad (14)$$

Then

$$V(x) = \int_0^L \left(\frac{\partial x}{\partial l}(l)\right)^{2q} dl = \|x\|^{2q}_{W_0^{1,2q}(0,L)} \qquad (15)$$

is an ISS Lyapunov function of (11)-(12) with respect to the space $U = L_{2q}(0,L)$ of input values and $U = W_0^{1,2q}(0,L) \cap W^{2,2q}(0,L)$ as well.

According to (9), iISS allows the decay rate of V to be much slower for large magnitude of state variables (since $\alpha \in \mathscr{P}$ can be bounded) than ISS can allow. This indicates that significantly different constructions for iISS Lyapunov functions are needed. Next section is devoted to this question.

IV. iISS of a class of nonlinear parabolic systems: L_p state space

Consider a system

$$\frac{\partial x}{\partial t} = c\frac{\partial^2 x}{\partial l^2} + f(x(l,t), u(l,t)), \quad \forall t > 0 \qquad (16)$$

defined on the spatial domain $(0,L)$ with

$$x(0,t)\frac{\partial x}{\partial l}(0,t) = x(L,t)\frac{\partial x}{\partial l}(L,t) = 0, \quad \forall t \geq 0 \qquad (17)$$

which represents boundary conditions of Dirichlet, Neumann or mixed type. The state space for (16) we choose as $X = L_{2q}(0,L)$ for some $q \in \mathbb{N}$ and input space we take as $U = L_\infty(0,L)$ and $H_0^1(0,L)$.

Define the following ODE associated with (16) given by

$$\dot{y}(t) = f(y(t), u(t)), \quad y(t), u(t) \in \mathbb{R}. \qquad (18)$$

The next theorem provides a construction of an iISS Lyapunov function for systems of the form (16).

Theorem 3: Suppose that $W : y \mapsto y^{2q}$ satisfies

$$\dot{W}(y) := 2qy^{2q-1} f(y,u) \leq -\alpha(W(y)) + W(y)\sigma(|u|) \qquad (19)$$

for some $\alpha \in \mathscr{K}_\infty \cup \{0\}$, $\sigma \in \mathscr{K}$. Let any of the following conditions hold:

1) $x(0,t) = 0$ for all $t \geq 0$ or $x(L,t) = 0$ for all $t \geq 0$.
2) α is convex and $\mathscr{K}_\infty$.

Then an iISS Lyapunov function of (16) with (17) with respect to the spaces of input values $U = L_\infty(0,L)$ as well as $U = H_0^1(0,L)$ is given by

$$V(x) = \ln(1 + Z(x)), \qquad (20)$$

where Z is defined as

$$Z(x) = \int_0^L W(x(l))dl = \|x\|^{2q}_{L_{2q}(0,L)}. \qquad (21)$$

Furthermore, if α is convex and satisfies

$$\liminf_{s \to \infty} \frac{\alpha(s)}{s} = \infty, \qquad (22)$$

then V given above is an ISS Lyapunov system of (16) with (17) with respect to $U = L_\infty(0,L)$ as well as $U = H_0^1(0,L)$. The term $W(y)\sigma(|u|)$ in (19) allows to analyze PDEs (16) with bilinear or generalized bilinear terms which do not possess ISS property.

Remark 4: One should choose an input space carefully. First we verified iISS of the system (23) for the input space $L_\infty(0,L)$. For many applications this choice of input space is reasonable and sufficient. However, in interconnected systems the input to one system is a state of another system. Thus, having $L_\infty(0,L)$ as an input space of the first subsystem automatically means that it is a state space of another subsystem, which complicates the proof of its ISS, since the constructions of Lyapunov functions for this choice of state space are hard to find (e.g. how to differentiate such Lyapunov functions?), if possible. As we have seen in Section III, the choice of $H_0^1(0,L)$ instead of $L_\infty(0,L)$ resolves the problem.

V. iISS of a class of nonlinear parabolic systems: Sobolev state space

Instead of the L_{2q} state space we used for characterizing iISS in Section IV, in this section Sobolev state spaces are used to establish iISS. We consider

$$\frac{\partial x}{\partial t} = c\frac{\partial^2 x}{\partial l^2} + f\big(x(l,t), \tfrac{\partial x}{\partial l}(l,t)\big) + \frac{\partial x}{\partial l}(l,t)u(l,t) \qquad (23)$$

defined for $(l,t) \in (0,L) \times (0,\infty)$ with the Dirichlet boundary conditions

$$x(0,t) = x(L,t) = 0, \quad \forall t \geq 0. \qquad (24)$$

Taking $X = W_0^{1,2q}(0,L)$, $q \in \mathbb{N}$ we can verify the following.

Theorem 5: Suppose that (13) holds for all $x \in X$ with some convex continuous function $\eta : \mathbb{R}_+ \to \mathbb{R}$ and some $\varepsilon \in \mathbb{R}_+$ s. t. (14) holds. If $\varepsilon > 0$, then the function V given by

$$V(x) = \ln(1 + Z(x)), \tag{25}$$

$$Z(x) = \int_0^L \left(\frac{\partial x}{\partial l}\right)^{2q} dl = \|x\|^{2q}_{W_0^{1,2q}(0,L)} \tag{26}$$

is an iISS Lyapunov function of (23)-(24) w.r.t. the space $U = L_\infty(0,L)$ of input values and $U = H_0^1(0,L)$ as well.

VI. Interconnections of iISS systems

Consider the following interconnected system:

$$\begin{aligned} \dot{x}_i(t) &= A_i x_i(t) + f_i(x_1, x_2, u), \quad i = 1,2 \\ x_i(t) &\in X_i, \quad u \in C(\mathbb{R}_+, U), \end{aligned} \tag{27}$$

where X_i is a state space of the i-th subsystem, $A_i : D(A_i) \to X_i$ is a generator of a strongly continuous semigroup over X_i. Let $X = X_1 \times X_2$ which is the space of $x = (x_1, x_2)$, and the norm in X is defined as $\|\cdot\|_X = \|\cdot\|_{X_1} + \|\cdot\|_{X_2}$. In this section, we assume that there exist continuous functions $V_i : X_i \to \mathbb{R}_+$, $\psi_{i1}, \psi_{i2} \in \mathscr{K}_\infty$, $\alpha_i \in \mathscr{P}$, $\sigma_i \in \mathscr{K}$ and $\kappa_i \in \mathscr{K} \cup \{0\}$ for $i = 1,2$ such that

$$\psi_{i1}(\|x_i\|_{X_i}) \le V_i(x_i) \le \psi_{i2}(\|x_i\|_{X_i}), \quad \forall x_i \in X_i \tag{28}$$

and system (27) satisfies

$$\dot{V}_i(x_i) \le -\alpha_i(\|x_i\|_{X_i}) + \sigma_i(\|x_{3-i}\|_{X_{3-i}}) + \kappa_i(\|u(0)\|_U) \tag{29}$$

for all $x_i \in X_i$, $x_{3-i} \in X_{3-i}$ and $u \in C(\mathbb{R}_+, U)$, where the Lie derivative of V_i corresponding to the inputs $u \in C(\mathbb{R}_+, U)$ and $v \in PC(\mathbb{R}_+, X_{3-i})$ with $v(0) = x_{3-i}$ is defined by

$$\dot{V}_i(x_i) = \overline{\lim_{t \to +0}} \frac{1}{t}(V_i(\phi_i(t, x_i, v, u)) - V_i(x_i)). \tag{30}$$

To present a small-gain criterion for the interconnected system (27) whose components are not necessarily ISS, we make use of a generalized expression of inverse mappings on the set of extended non-negative numbers $\overline{\mathbb{R}}_+ = [0, \infty]$. For $\omega \in \mathscr{K}$, define the function $\omega^\ominus$: $\overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ as $\omega^\ominus(s) = \sup\{v \in \mathbb{R}_+ : s \ge \omega(v)\}$. Notice that $\omega^\ominus(s) = \infty$ holds for $s \ge \lim_{\tau\to\infty} \omega(\tau)$, and $\omega^\ominus(s) = \omega^{-1}(s)$ holds elsewhere. A function $\omega \in \mathscr{K}$ is extended to ω: $\overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ as $\omega(s) := \sup_{v \in \{y \in \mathbb{R}_+ : y \le s\}} \omega(v)$. These notations are useful for presenting the following result succinctly.

Theorem 6: Suppose that

$$\lim_{s\to\infty} \alpha_i(s) = \infty \text{ or } \lim_{s\to\infty} \sigma_{3-i}(s)\kappa_i(1) < \infty \tag{31}$$

is satisfied for $i = 1,2$. If there exists $c > 1$ such that

$$\psi_{11}^{-1} \circ \psi_{12} \circ \alpha_1^\ominus \circ c\sigma_1 \circ \psi_{21}^{-1} \circ \psi_{22} \circ \alpha_2^\ominus \circ c\sigma_2(s) \le s \tag{32}$$

holds for all $s \in \mathbb{R}_+$, then system (27) is iISS. Moreover, if additionally $\alpha_i \in \mathscr{K}_\infty$ for $i = 1,2$, then system (27) is ISS. Furthermore,

$$V(x) = \int_0^{V_1(x_1)} \lambda_1(s) ds + \int_0^{V_2(x_2)} \lambda_2(s) ds \tag{33}$$

is an iISS (ISS) Lyapunov function for (27), where $\lambda_i \in \mathscr{K}$ is given for $i = 1,2$ by

$$\lambda_i(s) = [\alpha_i(\psi_{i2}^{-1}(s))]^\psi [\sigma_{3-i}(\psi_{3-i\,1}^{-1}(s))]^{\psi+1}, \forall s \in \mathbb{R}_+ \tag{34}$$

with an arbitrary $\psi \ge 0$ satisfying

$$\begin{aligned} &\psi = 0 && ,\text{if } c > 2 \\ &\psi^{-\frac{\psi}{\psi+1}} < \frac{c}{\psi+1} \le 1 && ,\text{otherwise.} \end{aligned} \tag{35}$$

It is straightforward to see that there always exists $\psi \ge 0$ satisfying (35). It is also worth mentioning that the Lyapunov function (33) is not in the maximization form, employed in [4] for establishing ISS. The use of the summation form (33) for systems which are not necessarily ISS is motivated by the limitation of the maximization form and clarified in [10] for finite-dimensional systems.

VII. Examples

This section exploits obtained results to analyze two reaction-diffusion systems.

A. Example 1

Consider a nonlinear interconnected parabolic system

$$\begin{cases} \dfrac{\partial x_1}{\partial t}(l,t) = \dfrac{\partial^2 x_1}{\partial l^2}(l,t) + x_1(l,t) x_2^4(l,t), \\ x_1(0,t) = x_1(\pi,t) = 0; \\ \dfrac{\partial x_2}{\partial t} = \dfrac{\partial^2 x_2}{\partial l^2} + a x_2 - b x_2 \left(\dfrac{\partial x_2}{\partial l}\right)^2 + \left(\frac{x_1^2}{1+x_1^2}\right)^{\frac{1}{2}}; \\ x_2(0,t) = x_2(\pi,t) = 0. \end{cases} \tag{36}$$

defined on the region $(l,t) \in (0,\pi) \times (0,\infty)$. To fully define the system we should choose the state spaces of subsystems. We take $X_1 := L_2(0,\pi)$ for $x_1(\cdot,t)$ and $X_2 := H_0^1(0,\pi)$ for $x_2(\cdot,t)$ as in Table I (a). We are going to analyze UGASs of (36). We divide the analysis into three parts. We omit technical details of the argument. They can be found in [18].

1) The first subsystem is iISS: First we invoke Item 1 of Theorem 3 with $q = 1$ to see that V_1 defined by

$$V_1(x_1) := \ln\left(1 + \|x_1\|^2_{L_2(0,\pi)}\right) \tag{37}$$

is an iISS Lyapunov function for x_1-subsystem. According to the proof of Theorem 3 (see [19]), it holds that

$$\dot{V}_1 \le -\frac{2\|x_1\|^2_{L_2(0,\pi)}}{1 + \|x_1\|^2_{L_2(0,\pi)}} + 2\|x_2\|^4_{L_\infty(0,\pi)}. \tag{38}$$

To replace $L_\infty(0,\pi)$ with $X_2 = H_0^1(0,\pi)$ for the input space used in (38), we recall Agmon's inequality which results in

$$\dot{V}_1(x_1) \le -\frac{2\|x_1\|^2_{L_2(0,\pi)}}{1 + \|x_1\|^2_{L_2(0,\pi)}} + 8\|x_2\|^4_{H_0^1(0,\pi)}, \tag{39}$$

and x_1-subsystem is iISS w.r.t. the state space $X_1 = L_2(0,\pi)$ and the input space $X_2 = H_0^1(0,\pi)$.

2) The second subsystem is ISS provided $a<0$ and $b\geq 0$: We invoke Theorem 2 with $q=1$. As in (15), let

$$V_2(x_2) := \int_0^\pi \left(\frac{\partial x_2}{\partial l}\right)^2 dl = \|x_2\|^2_{H_0^1(0,\pi)}. \tag{40}$$

Notice that x_2-subsystem is of the form (11) with $c=1$, $f(x_2, \frac{\partial x_2}{\partial l}) = ax_2 - bx_2(\frac{\partial x_2}{\partial l})^2$. To arrive at (13), we obtain by integration by parts

$$\int_0^L \frac{\partial^2 x_2}{\partial l^2}\Big(ax_2 - bx_2\Big(\frac{\partial x_2}{\partial l}\Big)^2\Big)dl = -aV_2(x_2) - b\int_0^L \frac{\partial^2 x_2}{\partial l^2} x_2 \Big(\frac{\partial x_2}{\partial l}\Big)^2 dl$$

Due to $x_2(0,t) = x_2(\pi,t) = 0$ for all $t\in\mathbb{R}_+$, we have

$$\int_0^\pi \frac{\partial^2 x_2}{\partial l^2} x_2 \Big(\frac{\partial x_2}{\partial l}\Big)^2 dl = -\int_0^\pi \frac{\partial x_2}{\partial l}\left(2x_2\frac{\partial x_2}{\partial l}\frac{\partial^2 x_2}{\partial l^2} + \Big(\frac{\partial x_2}{\partial l}\Big)^3\right)dl$$

which implies that

$$\int_0^\pi \frac{\partial^2 x_2}{\partial l^2} x_2 \Big(\frac{\partial x_2}{\partial l}\Big)^2 dl = -\frac{1}{3}\int_0^\pi \Big(\frac{\partial x_2}{\partial l}\Big)^4 dl. \tag{41}$$

Define $\forall s\in\mathbb{R}_+$

$$\eta(s) = -as + \frac{b}{3}s^2, \quad \hat{\alpha}(s) = (1-a-\varepsilon)s + \frac{b}{3\pi}s^2.$$

For $b\geq 0$ η is convex and satisfies (13). The inequality in (14) is achieved for $\varepsilon = 1-a>0$ provided $a<1$.

Hence, if $a<1$ and $b\geq 0$ hold, Theorem 2 with $q=1$ shows that for $\omega\in(0,2(1-a)]$, the function V_2 satisfies (with $\varepsilon = 1-a$, see [19])

$$\dot{V}_2 \leq -2(1-a-\frac{\omega}{2})V_2(x_2) - \frac{2b}{3\pi}V_2(x_2)^2 + \frac{1}{\omega}\|u_2\|^2_{L_2(0,\pi)}, \tag{42}$$

where we denote $u_2 := (x_1^2/(1+x_1^2))^{1/2}$.

Since $s\mapsto s/(1+s)$ is a concave function of $s\in\mathbb{R}_+$, Jensen's inequality yields

$$\int_0^\pi \frac{x_1^2}{1+x_1^2}dl \leq \pi\frac{(1/\pi)\|x_1\|^2_{L_2(0,\pi)}}{1+(1/\pi)\|x_1\|^2_{L_2(0,\pi)}} \leq \frac{\pi\|x_1\|^2_{L_2(0,\pi)}}{1+\|x_1\|^2_{L_2(0,\pi)}}.$$

Using this property in (42) we have

$$\begin{aligned}\dot{V}_2 \leq -2(1-a-\frac{\omega}{2})\|x_2\|^2_{H_0^1(0,\pi)} - \frac{2b}{3\pi}\|x_2\|^4_{H_0^1(0,\pi)} \\ + \frac{\pi}{\omega}\frac{\|x_1\|^2_{L_2(0,\pi)}}{1+\|x_1\|^2_{L_2(0,\pi)}}.\end{aligned} \tag{43}$$

Therefore, V_2 is an ISS Lyapunov function of x_2-subsystem with respect to the state space $X_2 = H_0^1(0,\pi)$ for $x_2(\cdot,t)$ and the input space $X_1 = L_2(0,\pi)$ for $x_1(\cdot,t)$.

Although property (43) is satisfactory for establishing UGASs of the overall system (36), one can obtain another estimate for $\dot{V}_2$, see [19].

3) Interconnection is UGASs: Now we collect the findings of two previous subsections. Assume that $a<1$ and $b\geq 0$. For the space $X = L_2(0,\pi)\times H_0^1(0,\pi)$, the Lyapunov functions defined as (37) and (40) for both subsystems satisfy (28) with the $\mathcal{K}_\infty$-functions $\psi_{11} = \psi_{12} : s\mapsto \ln(1+s^2)$ and $\psi_{21} = \psi_{22} : s\mapsto s^2$. Due to (39) and (43), we have (29) for

$$\alpha_1(s) = \frac{2s^2}{1+s^2}, \quad \sigma_1(s) = 8s^4, \quad \kappa_1(s) = 0$$

$$\alpha_2(s) = 2\Big(1-a-\frac{\omega}{2}\Big)s^2 + \frac{2b}{3\pi}s^4, \ \sigma_2(s) = \frac{\pi}{\omega}\frac{s^2}{1+s^2}, \ \kappa_2(s) = 0$$

TABLE I

USEFUL SPACES FOR INTERCONNECTIONS WITH AN iISS SUBSYSTEM.

(a) Choice #1

	State values X_i	Input values U_i
iISS subsystem ($i=1$)	$L_2(0,L)$	$H_0^1(0,L)$
ISS subsystem ($i=2$)	$H_0^1(0,L)$	$L_2(0,L)$

(b) Choice #2

	State values X_i	Input values U_i
iISS subsystem ($i=1$)	$H_0^1(0,L)$	$H_0^1(0,L)$
ISS subsystem ($i=2$)	$H_0^1(0,L)$	$H_0^1(0,L)$

TABLE II

STANDARD SPACES FOR INTERCONNECTIONS OF ISS SUBSYSTEMS.

	State values X_i	Input values U_i
ISS subsystem ($i=1$)	$L_p(0,L)$	$L_q(0,L)$
ISS subsystem ($i=2$)	$L_q(0,L)$	$L_p(0,L)$

defined with $\omega\in(0,2(1-a)]$. For these functions, condition (32) holds for all $s\in\mathbb{R}_+$ if and only if

$$\frac{12c^2\pi^2}{b\omega}\left(\frac{s^2}{1+s^2}\right) \leq \frac{2s^2}{1+s^2}, \quad \forall s\in\mathbb{R}_+ \tag{44}$$

is satisfied. Thus, there exists $c>1$ such that (32) holds if and only if $6\pi^2/b<\omega$ holds. Combining this with $\omega\in(0,2(1-a)]$, $a<1$ and $b\geq 0$, Theorem 6 establishes UGASs of $x=0$ for the whole system (36) when

$$a + \frac{3\pi^2}{b} < 1, \quad b\geq 0. \tag{45}$$

Note that (31) is satisfied. Due to the boundary conditions of x_2, Friedrichs' inequality ensures $\|x_2(\cdot,t)\|_{L_2(0,\pi)} \leq \|x_2(\cdot,t)\|_{H_0^1(0,\pi)}$. Thus, the UGASs guarantees the existence of $\beta\in\mathcal{KL}$ s.t. for all $\phi_0\in X = L_2(0,\pi)\times H_0^1(0,\pi)$ and all $t\in\mathbb{R}_+$ it holds that

$$\|\phi(t,\phi_0,0)\|_{L_2(0,\pi)\times L_2(0,\pi)} \leq \|\phi(t,\phi_0,0)\|_X \leq \beta(\|\phi_0\|_X, t). \tag{46}$$

B. Example 2

Consider

$$\begin{cases} \frac{\partial x_1}{\partial t}(l,t) = \frac{\partial^2 x_1}{\partial l^2}(l,t) + \frac{\partial x_1}{\partial t}(l,t)x_2^4(l,t), \\ x_1(0,t) = x_1(\pi,t) = 0; \\ \frac{\partial x_2}{\partial t} = \frac{\partial^2 x_2}{\partial l^2} + ax_2 - bx_2\Big(\frac{\partial x_2}{\partial l}\Big)^2 + \Big(\frac{x_1^2}{1+x_1^2}\Big)^{\frac{1}{2}}, \\ x_2(0,t) = x_2(\pi,t) = 0. \end{cases} \tag{47}$$

defined on the region $(l,t)\in(0,\pi)\times(0,\infty)$. For (47), we take $X_1 := H_0^1(0,\pi)$ and $X_2 := H_0^1(0,\pi)$ as in Table I (b).

1) The first subsystem is iISS: apply Theorem 5 to x_1-subsystem on $X_1 = H_0^1(0,\pi)$ by taking $q=1$. Let $V_1(x_1)$ be

$$V_1(x_1) := \ln\Big(1+\|x_1\|^2_{H_0^1(0,\pi)}\Big) \tag{48}$$

We can use $\eta=0$, which is convex on $\mathbb{R}_+$. Let $\varepsilon = c = 1$. Then Property (14) holds with $\hat{\alpha}=0$.

The proof of Theorem 5 and Agmon's inequality lead to

$$\dot{V}_1 \leq -\frac{\|x_1\|^2_{H_0^1(0,\pi)}}{1+\|x_1\|^2_{H_0^1(0,\pi)}} + 4\|x_2\|^4_{H_0^1(0,\pi)}. \tag{49}$$

Hence, inequality (29) is obtained for $i=1$, and the V_1 in (48) is an iISS Lyapunov function of x_1-subsystem with respect to the input space $X_2 = H_0^1(0,\pi)$.

2) The second subsystem is ISS: Since x_2-subsystem of (47) is identical with that of (36), if $a<1$ and $b \geq 0$ holds, the function V_2 defined by (40) is an ISS Lyapunov function for x_2-subsystem of (47) and satisfies

$$\begin{aligned}\dot{V}_2 \leq -2(1-a-\frac{\omega}{2})\|x_2\|^2_{H_0^1(0,\pi)} - \frac{2b}{3\pi}\|x_2\|^4_{H_0^1(0,\pi)} \\ + \frac{\pi}{\omega}\frac{\|x_1\|^2_{H_0^1(0,\pi)}}{1+\|x_1\|^2_{H_0^1(0,\pi)}}.\end{aligned} \tag{50}$$

This can be obtained from (43) since $\|x_1\|_{H_0^1(0,\pi)} \geq \|x_1\|_{L_2(0,\pi)}$ due to Friedrich's inequality and $s \mapsto \frac{s}{1+s}$ is strictly increasing for $s>0$. This shows ISS of x_2-subsystem of (47) with respect to the state space $X_2 = H_0^1(0,\pi)$ and the input space $X_1 = H_0^1(0,\pi)$.

3) Interconnection is UGASs: The above analysis for system (47) yields (28) and (29) for $i=1,2$, with functions which are the same as those for (36) except for

$$\alpha_1(s) = \frac{s^2}{1+s^2}, \quad \sigma_1(s) = 4s^4. \tag{51}$$

Again, condition (32) holds for all $s \in \mathbb{R}_+$ if and only if (44) is satisfied. Hence, Theorem 6 establishes UGASs of $x=0$ for the whole system (47) if (45) holds. The UGASs ensures the existence of $\beta \in \mathcal{KL}$ satisfying (46) for all $\phi_0 \in X$ and all $t \in \mathbb{R}_+$ in terms of $X = H_0^1(0,\pi) \times H_0^1(0,\pi)$.

VIII. Conclusion

For interconnected nonlinear parabolic systems a small-gain criterion has been proposed together with a method to construct Lyapunov functions of interconnected systems. We emphasized the importance of a correct choice of state spaces for iISS subsystems which are not ISS. In ISS literature about parabolic systems [4], [17] the systems over L_p-spaces (which is the simplest possible case) have been studied most extensively. However, as indicated in [18], the presence of a bilinear term in a PDE makes the L_p setting break down. Indeed, pointwise multiplication of state and input variables in PDEs defined on L_2 state space cannot be bounded by the product of the spatial L_2-norm of the state and the spatial L_2-norm of the input, while this is true for norms in Euclidean space in case of ODEs. Importantly, when two systems are connected to each other, a choice of state and input and spaces of one system affects the pair of the other systems. Thus, the bilinearity makes the choice in Table II useless, while the choice is often satisfactory for interconnections of ISS subsystems. To be able to formulate interconnections involving iISS subsystems as in Table I, this paper proposed constructions of Lyapunov functions characterizing iISS of parabolic systems with Sobolev state spaces, which are not covered by ISS Lyapunov functions. It is worth mentioning that different choices of input spaces leading to different properties of a single system are demonstrated through an example in [19].

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