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Coercive quadratic ISS Lyapunov functions for analytic systems

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***Abstract*— We investigate the relationship between input-to-state stability (ISS) of linear infinite-dimensional systems and existence of coercive ISS Lyapunov functions. We show that input-to-state stability of a linear system does not imply existence of a coercive quadratic ISS Lyapunov function, even if the underlying semigroup is analytic, and the input operator is bounded. However, if in addition the semigroup is similar to a contraction semigroup on a Hilbert space, then a quadratic ISS Lyapunov function always exists.**

Next we consider analytic and similar to contraction semigroups in Hilbert spaces with unbounded input operator B. If B is slightly stronger than 2-admissible, we construct explicitly a coercive L^2-ISS Lyapunov function. If the generator of a semigroup is additionally self-adjoint, this Lyapunov function is precisely a square norm in the state space.

Keywords: infinite-dimensional systems, linear systems, input-to-state stability, Lyapunov methods, semigroup theory.

I. Introduction

Input-to-state stability (ISS) was introduced by E. Sontag in [2] and has rapidly become a foundational concept in robust nonlinear control with diverse applications to robust stabilization [3], nonlinear observer design [4], analysis of large-scale networks [5], [6], event-based control [7], networked control systems [8], ISS feedback redesign [2], quantized control [9], nonlinear detectability [10], etc. Recently, the ISS notion has been extended to wide classes of distributed parameter systems, including partial differential equations (PDEs) with distributed and boundary inputs, semilinear evolution equations in Banach spaces, delay systems, etc. [11], [12], [13], [14], [15], [16], [17], [18], [19]. We refer to [20] for a survey of the state of the art of infinite-dimensional ISS theory and its applications to robust control and observation of distributed parameter systems, as well as [21] for an overview of available results on ISS of linear boundary control systems and some semilinear extensions.

One of the cornerstone concepts within the ISS theory is a notion of ISS Lyapunov function. For ODEs with Lipschitz continuous right hand side, it is known that existence of a coercive ISS Lyapunov function is equivalent to ISS of a system [22], which was extended to a wide class of semilinear evolution equations with Lipschitz nonlinearities and distributed inputs in [23]. For PDEs with boundary inputs the situation is much more complex. For instance, the classic linear heat equation with Dirichlet boundary inputs is ISS, which has been verified by means of the notion of admissible operators [15], monotonicity approach [24], as well as using spectral analysis [14]. However, no coercive ISS Lyapunov function is known for this system, and an existence of such a function is neither proved nor disproved. To address this problem, in [11], [25] the concept of a non-coercive Lyapunov function has been proposed, and in [26] it was shown that existence of a non-coercive ISS Lyapunov function implies ISS, provided some further properties hold. In particular, a quadratic non-coercive ISS Lyapunov function has been constructed for the 1-D heat equation with a Dirichlet boundary input, see [26]. Yet, it remains an open problem, whether a coercive ISS Lyapunov function for such a simple boundary control system exists. The problem becomes even more intriguing as for linear parabolic systems with Neumann and Robin boundary inputs and for linear first order hyperbolic systems (systems of conservation laws) rather simple coercive quadratic ISS Lyapunov functions exist [17], [12]. All this makes developing systematic Lyapunov methods for analysis of linear and nonlinear boundary control systems a central problem in the infinite-dimensional ISS theory.

In this paper, we investigate the relationship between existence of coercive Lyapunov functions and the ISS property. In [23], it was shown that a linear system on a general Banach space with distributed control is ISS if and only if there exists a coercive ISS Lyapunov function. This Lyapunov function however is not necessarily quadratic, even if X is a Hilbert space. Moreover, on Hilbert spaces such a system is ISS if and only if there exists a non-coercive, quadratic ISS Lyapunov function. In this note, we show that one cannot expect to have a coercive quadratic Lyapunov function even if additionally the system is supposed to be analytic. However, we also show that a coercive, quadratic Lyapunov function exists if the underlying semigroup is similar to a Hilbert space contraction semigroup, which is not a strong limitation in practice.

If B is unbounded operator, it is well-known, that L^2-ISS of $\dot{x} = Ax + Bu$ is equivalent to 2-admissibility of B and exponential stability of the underlying semigroup. Next we argue that for analytic and similar to contraction semigroups the condition $B \in L(U, X_{-p})$, $p \in (0, \frac{1}{2})$ (which is somewhat stronger condition than 2-admissibility) implies the existence of a coercive L^2-ISS Lyapunov function.

This paper is a conference version of [1].

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If A is additionally self-adjoint, then a weaker condition $B \in L(U, X_{-\frac{1}{2}})$ (still slightly stronger than 2-admissibility) already suffices for the same claim.

As a rule, Lyapunov methods are the most realistic way to study stability and ISS of nonlinear systems. The results that we obtain here are a part of our long-term aim to rigorously settle the applicability of Lyapunov methods to linear systems with unbounded input operators and then to extend the methods to treat the nonlinear PDEs with boundary controls and more generally nonlinear boundary control systems.

Notation. Throughout this note X and Y will refer to Banach spaces which may, at instances, be specified to be a Hilbert space with an inner product $\langle \cdot, \cdot \rangle$.

For a Banach space $(U, \|\cdot\|_U)$, we denote by $L^\infty(0,t;U)$ the space of Bochner measurable functions $u:(0,t) \to U$ with finite essential supremum norm $\|u\|_{L^\infty(0,t)} := \operatorname{ess\,sup}_{s\in(0,t)} \|u(s)\|_U$ and similarly we define the common Lebesgue Bochner spaces $L^p(\mathbb{R}_0^+, U)$, $p \in [1,\infty]$. We denote the space of bounded operators acting from X to a Banach space Y by $L(X,Y)$. For short, $L(X) := L(X,X)$.

Denote $\mathbb{R}_0^+ := [0,+\infty)$, $(\mathbb{R}_0^+)^2 := \mathbb{R}_0^+ \times \mathbb{R}_0^+$ and recall the following well-known classes of comparison functions.

$$\mathcal{K} = \{\mu \in C(\mathbb{R}_0^+, \mathbb{R}_0^+) \mid \mu(0) = 0, \mu \text{ strictly increasing}\},$$
$$\mathcal{K}_\infty = \{\theta \in \mathcal{K} \mid \lim_{x\to\infty} \theta(x) = \infty\},$$
$$\mathcal{L} = \{\gamma \in C(\mathbb{R}_0^+, \mathbb{R}_0^+) \mid \gamma \text{ str. decreas.}, \lim_{t\to\infty} \gamma(t) = 0\},$$
$$\mathcal{KL} = \{\beta : (\mathbb{R}_0^+)^2 \to \mathbb{R}_0^+ \mid \beta(\cdot,t) \in \mathcal{K}\ \forall t, \beta(s,\cdot) \in \mathcal{L}\ \forall s \neq 0\}.$$

II. Linear systems and their stability

In the following, let $A : D(A) \subset X \to X$ always be an infinitesimal generator of a strongly continuous semigroup $T := (T(t))_{t\geq 0}$ on X with a nonempty resolvent set $\rho(A)$. Recall that a semigroup T is called analytic if T extends to an analytic mapping $z \mapsto T(z)$ on a sector

$$S_\zeta = \{z \in \mathbb{C} \setminus \{0\} \colon \arg(z) < \zeta\}$$

for some $\zeta \in (0, \pi/2]$ and $\lim_{z\to 0, z\in S_\theta} T(z)x = x$ for all $x \in X$ and some $\theta \in (0,\zeta)$.

For the rest of the paper, we will be interested in systems $\Sigma(A,B)$ given by abstract Cauchy problems of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad t > 0,\ x(0) = x_0, \qquad (1)$$

where $A : D(A) \to X$ generates a C_0-semigroup $(T(t))_{t\geq 0}$ and B is an operator, which is possibly unbounded, acting on the input space U. The reason for allowing unbounded operators B stems from the study of systems with boundary or point controls; see, e.g., [27], [28]. Note that in contrast to the (in general unbounded) operator A, B will always be defined on the "full space" U, and the "unboundedness" is only reflected in the norm on the image space. To clarify the precise assumptions on B, let us recall a solution concept for (1): Consider the function x (formally) given by

$$x(t) = T(t)x_0 + \int_0^t T(t-s)Bu(s)\mathrm{d}s, \qquad t \geq 0, \qquad (2)$$

for any $x_0 \in X$ and $u \in L^1_{\mathrm{loc}}(\mathbb{R}_{\geq 0}, U)$. If x, which we denote also $\phi(\cdot, x_0, u)$, maps $[0,\infty)$ to X, then we call x the *mild solution* of (1). If B is a bounded operator in the sense that $B \in L(U,X)$, then (2) indeed defines such mild solution. For more general operators B, suitable properties are required, in particular such that the integral in (2) is well-defined in X for all $t > 0$ and inputs u from a space of U-valued (equivalence classes of) functions such as $L^q(0,\infty;U)$

To introduce these properties, we will view the convergence of the integral in a weaker norm on X as follows. Define the extrapolation space X_{-1} as the completion of X with respect to the norm $\|x\|_{X_{-1}} := \|(aI - A)^{-1}x\|_X$ for some $a \in \rho(A)$. X_{-1} is a Banach space (see [29, Theorem 5.5, p. 126]) and different choices of $a \in \rho(A)$ generate equivalent norms on X_{-1}, see [29, p. 129]. As we know from the representation theorem [30, Theorem 3.9], the input operator B must satisfy the condition $B \in L(U, X_{-1})$ in order to give rise to a well-defined control system. Lifting of the state space X to a larger space X_{-1} is natural because the semigroup $(T(t))_{t\geq 0}$ extends uniquely to a strongly continuous semigroup $(T_{-1}(t))_{t\geq 0}$ on X_{-1} whose generator $A_{-1} : X_{-1} \to X_{-1}$ is an extension of A with $D(A_{-1}) = X$, see, e.g., [29, Section II.5]. If clear from the context, we may drop the subscript "-1" in our notation. Thus we may consider Equation (1) on the Banach space X_{-1} by replacing A by A_{-1} and henceforth interpret (2) in X_{-1} as the integral exists in X_{-1} when the extension of the semigroup is considered. The standing assumption for systems $\Sigma(A,B)$ is thus that $B \in L(U, X_{-1})$ and where X and U are general Banach spaces. The lifting comes however at a price that x has values in X_{-1} in general. This motivates the following definition:

Definition 1 *The operator $B \in L(U, X_{-1})$ is called a q-*admissible control operator *for $(T(t))_{t\geq 0}$, where $1 \leq q \leq \infty$, if for all $t \geq 0$ and $u \in L^q([0,t], U)$, it holds that*

$$\int_0^t T_{-1}(t-s)Bu(s)ds \in X. \qquad (3)$$

If the analogous property holds for continuous functions $u \in C([0,t], U)$, we say that B is C-admissible.

If B is q-admissible, then x defined by (2) is indeed a mild solution of (1). Note that q-admissibility means precisely the forward-completeness of control systems for all inputs from L^q. It is easy to see that any mild solution is continuous if B is q-admissible for $q < \infty$, see [30, Proposition 2.3]. In the critical case $q = \infty$, this is also known in many practically relevant situations, see e.g. [15], [16], but not entirely clear in general.

For linear systems with admissible B, we study the following stability notions.

Definition 2 *System $\Sigma(A,B)$ is called L^p-*input-to-state stable (*L^p*-ISS)*, if there exist functions $\beta \in \mathcal{KL}$ and $\mu \in \mathcal{K}_\infty$ such that for every $x_0 \in X$, every $t \geq 0$ and every*

$u \in L^p(0,t;U)$*, the mild solution* x *of* (1) *satisfies* $x(t) \in X$ *and*

$$\|x(t)\| \le \beta(\|x_0\|, t) + \mu(\|u\|_{L^p(0,t)}). \tag{4}$$

ISS of (1) can be characterized as follows, see [15, Proposition 2.10]:

Theorem 3 *Let* $p \in [1, +\infty]$*. The following properties are equivalent for a system* $\Sigma(A,B)$

(i) (1) *is* L^p*-ISS*

(ii) T *is exponentially stable and* B *is* p*-admissible*

Definition 4 (ISS Lyapunov function) *Consider* $\Sigma(A,B)$ *and suppose that* B *is* ∞*-admissible. A continuous function* $V : X \to \mathbb{R}_0^+$ *is called a* (non-coercive) ISS Lyapunov function *for* $\Sigma(A,B)$ *if there exist* $\alpha_2 \in \mathcal{K}$ *and* $\sigma \in \mathcal{K}_\infty$ *such that*

$$0 < V(x) \le \alpha_2(\|x\|), \quad x \neq 0,$$

and for all $x \in X$ *and all* $u \in L^\infty_{loc}(0,\infty;U)$*,*

$$\dot{V}_u(x) \le \; -\alpha_3(\|x\|) + \sigma(\limsup_{t\to 0^+} \|u\|_{L^\infty(0,t)}), \tag{5}$$

where $\dot{V}_u(x)$ *is the right-hand Dini derivative of* $V(x(\cdot))$ *at* $t = 0$*:*

$$\dot{V}_u(x) := \limsup_{h\to 0^+} \frac{1}{h}\big(V(\phi(t,x,u)) - V(x)\big)$$

and $\phi(\cdot, x, u)$ *is the mild solution* (2) *of* (1) *with initial condition* x *and input* u*.*

An ISS Lyapunov function is called coercive *if there exists another* $\mathcal{K}_\infty$*-function* α_1 *such that*

$$\alpha_1(\|x\|) \le V(x), \qquad x \in X.$$

The map V *is called* Lyapunov function *(for the uncontrolled system* (1)*) if* V *is an ISS Lyapunov function for* $\Sigma(A,0)$*.*

We emphasize that any ISS Lyapunov function is a Lyapunov function for the uncontrolled system.

Definition 5 (Quadratic Lyapunov function) *Let* X *be a Hilbert space. An ISS Lyapunov function* $V : X \to \mathbb{R}_0^+$ *is called a* quadratic ISS Lyapunov function *if there exists a self-adjoint operator* $P \in L(X)$ *such that* $V(x) = \langle Px, x\rangle$ *for all* $x \in X$*. In this case, we also say that* V is quadratic.

We call a bounded, self-adjoint operator P positive if $\langle Px, x\rangle > 0$ for all $x \in X \setminus \{0\}$. Clearly, if V is a quadratic Lyapunov function, then the corresponding operator P such that $V(x) = \langle Px, x\rangle$, $x \in X$, is positive in the sense that $\langle Px, x\rangle > 0$ for all $x \in X \setminus \{0\}$. Furthermore, P is invertible (with a bounded inverse) if and only if V is coercive.

Classical constructions of Lyapunov functions via solution of the Lyapunov operator equation, see [31, Theorem 4.1.3], lead to quadratic Lyapunov functions. Furthermore, quadratic Lyapunov functions can be easily differentiated, and there are efficient numerical schemes for construction of quadratic Lyapunov functions, such as sum of squares (SoS).

The following characterization is elementary, but motivates how to define quadratic Lyapunov functions for general Banach spaces.

Proposition 6 *Let* X *be a Hilbert space. An ISS Lyapunov function* $V : X \to \mathbb{R}_0^+$ *is a quadratic ISS Lyapunov function if and only if there exists* $F \in L(X)$ *such that*

$$V(x) = \|Fx\|^2, \qquad x \in X. \tag{6}$$

In view of Proposition 6, in the Banach space setting, we will call functions V as in (6) quadratic ISS Lyapunov functions.

The use of $\mathcal{K}$-functions in the definition of ISS Lyapunov functions is natural as one is interested in general nonlinear Lyapunov functions. It is not surprising that for quadratic Lyapunov functions there is no need to consider general comparison functions.

Lemma 7 *Let* X *be a Hilbert space. Let* $V : X \to \mathbb{R}_0^+$ *be a coercive, quadratic ISS Lyapunov function for* $\Sigma(A,B)$ *with* ∞*-admissible* B *and with* $\sigma(r) = a_4 r^2$ *in* (5)*. Then there exist constants* $a_1, a_2, a_3 > 0$ *such that*

$$a_1\|x\|^2 \le V(x) \le a_2\|x\|^2, \qquad x \in X, \tag{7}$$

and for all $u \in L^\infty(\mathbb{R}_0^+, U)$ *we have*

$$\dot{V}_u(x) \le -a_3\|x\|^2 + a_4(\limsup_{t\to 0^+} \|u\|_{L^\infty(0,t)})^2, \quad x \in X. \tag{8}$$

ISS Lyapunov functions as defined above, are of virtue to study L^∞-ISS. For the analysis of L^p-ISS, another type of ISS Lyapunov functions is needed.

Definition 8 (Quadratic L^p**-ISS Lyapunov function)** *Consider a system* $\Sigma(A,B)$ *with* C*-admissible* $B \in L(U, X_{-1})$*. A continuous function* $V : X \to \mathbb{R}_0^+$ *is called a* (non-coercive) quadratic L^p-ISS Lyapunov function *for* $\Sigma(A,B)$ *if there is an invertible operator* $F \in L(X)$ *and constants* $a_3, a_4 > 0$ *such that* (6) *holds and*

$$\dot{V}_u(x) \le -a_3\|x\|^2 + a_4\|u(0)\|_U^p, \tag{9}$$

for all $x \in X$ *and all* $u \in C(\mathbb{R}_0^+, U)$*. If in addition* (7) *holds for some* $a_1, a_2 > 0$*, then* V *is a called* coercive quadratic L^2-ISS Lyapunov function.

As shown in [32, Theorem 1], the following holds:

Proposition 9 *If* B *is* 2*-admissible and there is a quadratic coercive* L^2*-ISS Lyapunov function for* $\Sigma(A,B)$*, then* $\Sigma(A,B)$ *is* L^p*-ISS for all* $p \in [2, +\infty)$*.*

Proof. As the flow ϕ of $\Sigma(A,B)$ depends continuously on inputs, the claim follows from [32, Theorem 1], where using nonlinear rescaling of V, an explicit construction of the (non-quadratic) L^q-ISS Lyapunov functions was provided, for all $q \in (2, +\infty)$. □

Next we show that L^p-ISS Lyapunov functions for linear systems cannot be quadratic unless $p = 2$.

Proposition 10 *Let V be a (coercive or non-coercive) quadratic L^p-ISS Lyapunov function for $\Sigma(A,B)$ with $B \neq 0$ and $p \in [1,+\infty)$. Then $p=2$.*

Proof. By definition, there is $a \geq 0$ such that

$$\dot{V}_u(0) \leq a\|u(0)\|_U^p, \quad u \in C(\mathbb{R}_0^+, U).$$

Take a_m as the infimum of $a > 0$ satisfying the previous property. As $B \neq 0$, $a_m > 0$.

By the linearity of ϕ in u, and as V is quadratic, we see that for any $u \in C(\mathbb{R}_0^+, U)$, and any $c > 0$

$$\dot{V}_{cu}(0) = \limsup_{h\to+0} \frac{V(\phi(h,0,cu))}{h} = c^2 \limsup_{h\to+0} \frac{V(\phi(h,0,u))}{h} \leq c^2 a_m \|u(0)\|_U^p = c^{2-p} a_m \|cu(0)\|_U^p,$$

and taking $c > 1$ for $p > 2$ and $c < 1$ for $p < 2$, we come to a contradiction to the choice of a_m. □

III. Coercive quadratic Lyapunov functions for linear systems with bounded input operators

Importance of Lyapunov functions is due to the fact, that they are certificates for important stability properties. As we are interested in this note in ISS, we recall the following result from [23, Theorem 8]:

Proposition 11 *Let X be a Hilbert space, A be a generator of a strongly continuous semigroup and $B \in L(U,X)$. The following statements are equivalent:*

- *(i)* (1) *is L^p-ISS for some $p \in [1,+\infty]$.*
- *(ii)* (1) *is L^p-ISS for all $p \in [1,+\infty]$.*
- *(iii) There is a coercive L^1-ISS Lyapunov function for* (1)*, which is an equivalent norm on X.*
- *(iv) There is a non-coercive quadratic L^2-ISS Lyapunov for* (1).

Coercive L^1-ISS Lyapunov functions constructed in the proof of Proposition 11 in [23] to show the equivalence between (i) and (iii) are never quadratic. In fact, they are norms on X, equivalent to $\|\cdot\|$, and thus they are homogeneous of degree one. In this section, we show a criterion for existence of a quadratic coercive L^2-ISS Lyapunov function for linear systems with bounded input operators.

We say that a *semigroup T is similar to a contraction semigroup*, if there is a boundedly invertible operator $S : X \to X$ so that $(ST(t)S^{-1})_{t\geq 0}$ is a contraction semigroup[1].

We call $\langle\cdot,\cdot\rangle_{new}$ an *equivalent scalar product* in X, if $\langle\cdot,\cdot\rangle_{new}$ is a scalar product in X that induces a norm $\|\cdot\|_{new} = \sqrt{\langle\cdot,\cdot\rangle_{new}}$ that is equivalent to the norm $\|\cdot\|$ in X.

The following result settles existence of coercive, quadratic Lyapunov functions for systems with bounded B.

Theorem 12 *Let X be a Hilbert space, $B \in L(U,X)$ and let A generate an exponentially stable semigroup T on X. The following statements are equivalent:*

- *(i) There is a coercive, quadratic L^2-ISS Lyapunov function for* (1)*;*
- *(ii) There is a coercive, quadratic Lyapunov function for* (1) *with $B=0$;*
- *(iii) There exists an equivalent scalar product $\langle\cdot,\cdot\rangle_{new}$ such that A is dissipative, i.e.*
$$\Re\langle Ax,x\rangle_{new} \leq 0, \quad x \in D(A) \qquad (10)$$
and $x \mapsto \langle x,x\rangle_{new} = \|x\|_{new}^2$ is a quadratic ISS Lyapunov function for (1).
- *(iv) There exists an equivalent scalar product $\langle\cdot,\cdot\rangle_{new}$ such that A is dissipative.*
- *(v) T is similar to a contraction semigroup;*
- *(vi) There is an equivalent norm in X of the form $W(x) = \|Fx\|$, for $F \in L(X)$, such that $\dot{W}(x) \leq -W(x)$ for $B=0$.*

Supported by Theorem 12, we have the following negative result on existence of coercive quadratic Lyapunov functions for exponentially stable systems.

Proposition 13 *For any infinite-dimensional Hilbert space X there exists a generator A of an exponentially stable, analytic semigroup T on X such that the system $\Sigma(A,0)$ has no coercive, quadratic Lyapunov function.*

Proof. Assume that there exists a coercive, quadratic Lyapunov function $V : X \to \mathbb{R}_0^+$. By Theorem 12, it follows that A generates a semigroup, similar to a contraction semigroup. However, for any infinite-dimensional Hilbert space, it is possible to construct analytic, exponentially stable semigroups, which are not similar to a contraction semigroup, see [33] and also [34, Chapter 9]. □

Remark 14 *Let $B \in L(U,X)$. Combining Theorem 12 with Proposition 11, we obtain another negative result: Existence of a coercive ISS Lyapunov function for $\Sigma(A,B)$ does not imply existence of a coercive quadratic ISS Lyapunov function for $\Sigma(A,B)$ (in contrast to finite-dimensional linear case).*

Recall that for generator A of an exponentially stable analytic semigroup, the operator $-A$ is sectorial (of angle less than $\pi/2$), see [34]. Thus the fractional power $(-A)^{-\alpha}$, $\alpha \in (0,1)$ can be defined as bounded operator via a contour integral of an operator-valued analytic function; this is an instance of the Riesz–Dunford functional calculus for sectorial operators. It can be shown that this operator is injective since A is injective and hence $(-A)^{\alpha}$ can be defined as the inverse. The operator $(-A)^{\alpha}$ is closed and densely defined. For more information on this construction see the books [34] and [35, Section 1.4]. For a very brief description of the essentials required here, [21].

The following result provides an alternative construction of a coercive quadratic Lyapunov function for exponentially stable *analytic* systems with bounded B.

[1] Following standard conventions, we tacitly assume here that S^{-1} is defined on the whole space X.

Theorem 15 *Let X be a Hilbert space, $B \in L(U, X)$, and let A generate an exponentially stable analytic semigroup T on X. The conditions (i)–(vi) in Theorem 12 are equivalent to*

(vii) The function

$$V : D(A) \to \mathbb{R}_0^+, x \mapsto \int_0^\infty \|(-A)^{\frac{1}{2}} T(t) x\|^2 \, \mathrm{d}t \quad (11)$$

extends to a coercive, quadratic ISS Lyapunov function from X to $\mathbb{R}_0^+$ for the system $\Sigma(A, B)$. We denote this extension again by V.

Remark 16 *Lyapunov function in* (11) *takes a particularly simple form in case if A is a self-adjoint operator.*

Let P be a solution of the Lyapunov equation

$$\langle Px, Ax\rangle + \langle Ax, Px\rangle = -\|x\|^2, \quad x \in D(A). \quad (12)$$

Pick any $x \in D(A^{1/2}) \supset D(A)$. It holds that

$$\begin{aligned} V(x) &= \int_0^{+\infty} \left\langle (-A)^{\frac{1}{2}} T(t)x, (-A)^{\frac{1}{2}} T(t)x \right\rangle dt \\ &= \int_0^{+\infty} \left\langle T(t)(-A)^{\frac{1}{2}} x, T(t)(-A)^{\frac{1}{2}} x \right\rangle dt \\ &= \left\langle P(-A)^{\frac{1}{2}} x, (-A)^{\frac{1}{2}} x \right\rangle \\ &= \left\langle \left((-A)^{\frac{1}{2}}\right)^* P(-A)^{\frac{1}{2}} x, x \right\rangle. \end{aligned}$$

If A is self-adjoint, then $(-A)^{\frac{1}{2}}$ is again self-adjoint. Furthermore, $P := -\frac{1}{2}A^{-1}$ solves the equation (12) *and V takes the form*

$$V(x) = \frac{1}{2} \left\langle \left((-A)^{\frac{1}{2}}\right) A^{-1} (-A)^{\frac{1}{2}} x, x \right\rangle = \frac{1}{2} \langle x, x\rangle \quad (13)$$

As $D(A^{1/2})$ is dense in X, V clearly extends by formula (13) *to the whole X.*

IV. Coercive L^2-ISS Lyapunov functions for linear L^2-ISS systems

In view of Proposition 9 and Theorem 3, the existence of a coercive L^2-ISS Lyapunov function implies 2-admissibility of B, for any well-posed linear system $\Sigma(A, B)$.

In this section, we establish the converse results for analytic semigroups that are similar to contraction semigroups. The following lemma on sufficient conditions for admissibility of B will be helpful on this way:

Lemma 17 *Let A be an analytic semigroup over a Hilbert space X. Then*

(i) If $B \in L(U, X_{-\frac{1}{2}+p})$ for some $p \geq 0$, then B is q-admissible for all $q \in (\frac{2}{1+2p}, +\infty]$. In particular, if $p > 0$, then B is 2-admissible.

(ii) If A is self-adjoint, and $B \in L(U, X_{-\frac{1}{2}})$, then B is 2-admissible.

(iii) If A is self-adjoint, and B is 2-admissible, then B does not necessarily satisfy the condition $B \in L(U, X_{-\frac{1}{2}})$.

Proof. (i) follows from [21, Proposition 2.13].

(ii) follows directly from [28, Proposition 5.1.3] by duality of admissible control and observation operators.

(iii) The claim follows from the following counterexample, which is an adaptation of [28, Example 5.3.11] by duality.

Consider the state space $X = \ell_2$, a diagonal operator $A = (-2^n)_{n\in\mathbb{Z}_+}$, and an input operator $B : \mathbb{C} \to X$, given by $Bc = (2^{n/2})_{n\in\mathbb{Z}_+} c$. Then

$$(-A)^{-\frac{1}{2}} B = (1, 1, 1, \ldots),$$

which is not a well-defined operator from $\mathbb{C}$ to X.

At the same time, A is 2-admissible by Carleson criterion [28, Theorem 5.3.2].

However, $(-A)^{-p}B = (2^{(\frac{1}{2}-p)n})_{n\in\mathbb{Z}_+}$ is a bounded operator from $\mathbb{C}$ to X for $p > \frac{1}{2}$. □

We start with a converse coercive Lyapunov result for systems with self-adjoint A:

Proposition 18 *Let $A = A^*$ be a self-adjoint operator, generating an exponentially stable semigroup T. Let further $B \in L(U, X_{-\frac{1}{2}})$. Then*

$$V : x \mapsto \frac{1}{2}\|x\|^2, \quad x \in X \quad (14)$$

is a (coercive) L^2-ISS Lyapunov function for $\Sigma(A, B)$.

Proof. By Lemma 17(ii), B is 2-admissible, and thus in particular C-admissible. In view of Theorem 15 and Remark 16, V is a coercive quadratic Lyapunov function for the system $\Sigma(A, 0)$.

Now take any $x_0 \in D(A)$ and any $u \in C^1_{loc}(\mathbb{R}_0^+, U)$ with $u(0) = 0$. Then the corresponding mild solution x is even classical and we can differentiate the Lyapunov function along trajectories, to obtain for all $t > 0$

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) = \langle \dot{x}(t), x(t)\rangle = \langle Ax(t), x(t)\rangle + \langle Bu(t), x(t)\rangle,$$

where the last term has to be understood as duality bracket between $X_{-\frac{1}{2}}$ and $X_{\frac{1}{2}} \subset X_1$. Hence, the modules of that term can be bound from above by

$$\frac{1}{\varepsilon}\|B\|^2_{L(U,X_{-1/2})}\|u(t)\|^2 + \varepsilon^2\|(-A)^{\frac{1}{2}} x(t)\|^2, \quad \varepsilon > 0.$$

By assumption A is strictly dissipative in the sense that there exists a constant $a > 0$ such that

$$-\|(-A)^{\frac{1}{2}} x(t)\|^2 = \langle Ax(t), x(t)\rangle \leq -a\|x(t)\|^2.$$

Thus, choosing ε sufficiently small yields

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) \leq -\kappa V(x(t)) + c\|u(t)\|^2, \quad t > 0,$$

for constants $\kappa, c > 0$. Integrating the equation gives

$$\|x(t)\|^2 - \|x_0\|^2 \leq -\kappa \int_0^t \|x(s)\|^2 \, \mathrm{d}s + c \int_0^t \|u(s)\|^2 \, \mathrm{d}s.$$

By the continuous dependence of x on x_0 and u in X and C_{loc}, respectively, the last inequality extends to all $x_0 \in X$ and $u \in C(\mathbb{R}_0^+, U)$. By [31, Cor. A.5.45], we conclude that the Dini derivative satisfies

$$\dot{V}_u(x) \leq -\kappa\|x\|^2 + c\|u(0)\|^2, \qquad x \in X, u \in C(\mathbb{R}_0^+, U).$$

□

Next we formulate the converse L^2-ISS Lyapunov theorem for linear systems governed by general analytic semigroups under a slightly stronger assumption on B.

Theorem 19 *Let X be a Hilbert space and let A generate an exponentially stable analytic semigroup T on X, which is similar to a contraction semigroup. Furthermore, let $B \in L(U, X_{-p})$ for some $p < \frac{1}{2}$. Then*

$$V : D(A) \to \mathbb{R}_0^+, \quad x \mapsto \int_0^\infty \|(-A)^{\frac{1}{2}} T(t)x\|^2 \, \mathrm{d}t \qquad (15)$$

(extended to X) is a coercive quadratic L^2-ISS Lyapunov function for (1).

V. Discussion

In this work, we have shown that quadratic Lyapunov functions are a natural Lyapunov function concept to study the L^2-ISS of linear analytic systems in Hilbert spaces. Under assumption that A generates an analytic semigroup that is similar to a contraction, and if $B \in L(U, X_{-\frac{1}{2}+p})$ for $p > 0$, we give an explicit construction of a coercive L^2-ISS Lyapunov function for such a system. If A is self-adjoint, then this function is just $V(x) = \|x\|^2$, and is a coercive L^2-ISS Lyapunov function under a milder assumption that $B \in L(U, X_{-\frac{1}{2}})$. Note that if B is not 2-admissible, then no coercive quadratic L^p-ISS Lyapunov function can be constructed for this system, and non-quadratic coercive Lyapunov functions should be considered.

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