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A note on input-to-state stability of linear and bilinear infinite-dimensional systems

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Abstract— **We propose a construction of Lyapunov functions for linear and bilinear infinite-dimensional integral input-to-state stable systems. In contrast to other constructions available in the literature, we do not impose strong restrictions on the type of the bilinear systems. The coercivity of such Lyapunov functions as well as a possible extension to systems with unbounded input operators are discussed.**

Keywords: input-to-state stability, linear systems, bilinear systems, infinite-dimensional systems.

I. Introduction

Input-to-state stability (ISS) is widely recognized as a central framework for study of robust stability. It unifies the notions of internal and external stability [20], forms a basis for the robust stability and stabilizability theory of nonlinear systems [18], [7] and, by means of ISS small-gain theorems [12], [6], [5], provides a firm basis for the stability analysis of interconnected nonlinear control systems. However, in spite of all the advantages of ISS theory, input-to-state stable systems do not encompass all interesting systems with a certain kind of robustness. In particular, systems appearing in biochemical processes, population dynamics, traffic flows etc. often do not enjoy the ISS property due to saturation and limitations in actuators and processing rates. The states of such systems may grow to infinity, provided the magnitude of the applied input is large enough (but finite). Such a behavior is impossible for ISS systems, which have trajectories that stay bounded for inputs of a finite magnitude. To treat such systems a weaker robustness property called integral input-to-state stability (iISS) has been proposed [19]. Subsequently, the theory of iISS systems has been developed in the finite-dimensional setting, see [1], [9], [20] and references therein.

Success of ISS theory for ODE systems fostered development of infinite-dimensional ISS theory [4], [13], [11]. Recently, the first attempts to study infinite-dimensional iISS systems have been made [14], [15]. In particular, constructions of Lyapunov functions for nonlinear parabolic systems have been proposed in [4], [13] and interconnection of ISS and iISS systems have been studied in [15], [4].

In this paper we develop a Lyapunov theory for ISS of linear systems

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

and for iISS of bilinear systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t) + C(x(t), u(t)), \tag{2}$$

where A generates a strongly continuous semigroup T on a Banach space X, $B \in L(U,X)$ is a bounded linear operator, and $C : X \times U \to X$ is a nonlinear continuous operator, such that there exist $K > 0$ and $\xi \in \mathcal{K}$:

$$\|C(x,u)\|_X \le K\|x\|_X \xi(\|u\|_U). \tag{3}$$

for all $x \in X$ and all $u \in U$.

This class of systems is important due to several reasons. On the one hand, it includes bilinear systems with bounded operator C which arise in a number of applications such as biochemical reactions or quantum-mechanical processes [16]. On the other hand, bilinear systems is a basic class of systems which are iISS but which are not ISS (for linear systems with bounded input operators the notions of ISS and iISS coincide). Last but not least the Lyapunov functions for systems (2) play crucial role in the study of interconnections of systems (2) with nonlinear ISS systems by means of small-gain theorems, see [15].

The Lyapunov characterization of iISS for bilinear ODE systems (that is for systems (2) with $X = \mathbb{R}^n$) has been derived by Sontag in [19]. It states that a bilinear system is iISS iff it is 0-GAS and iff it possesses an iISS Lyapunov function of a certain form.

In the infinite-dimensional context, ISS and iISS of (1) and (2) have been studied in [14] for $B \in L(U,X)$ and C, satisfying (3). It has been shown that under these conditions (2) is iISS if and only if (2) is uniformly globally asymptotically stable for $u \equiv 0$. For the special case when A generates an analytic semigroup and X is a Hilbert space, iISS Lyapunov functions for the system (2) have been constructed. The restrictiveness of these requirements motivated us to seek for new constructions, which are valid for any Banach space X and for any strongly continuous semigroup T.

In this note we propose two constructions of ISS Lyapunov functions for linear ISS systems (1). The first of them results in a not necessarily coercive ISS Lyapunov function and another one provides a coercive ISS Lyapunov function. Next in Section IV we give a Lyapunov characterization of iISS for bilinear systems with bounded B and C. These constructions together with the results from [14] help to establish Lyapunov characterizations of ISS and iISS for systems (1) and (2) with bounded B and C.

Finally, in Sections V, VI we briefly describe difficulties, arising in ISS theory for linear systems (1) with unbounded

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admissible input operators. We show by means of examples that at least in some cases it is possible to construct ISS Lyapunov functions for such systems. At the same time, if V is an ISS Lyapunov function for (1) with a certain A and all bounded B, this does not mean that V will be an ISS-Lyapunov function for the system with the same A and any unbounded admissible input operator.

II. PRELIMINARIES

Consider the system (2) and assume throughout the paper that X and U are Banach spaces and that input functions belong to the space $\mathcal{U} := PC(\mathbb{R}_+, U)$ of piecewise continuous functions from $\mathbb{R}_+$ to U, which are right-continuous. Also, let A be the infinitesimal generator of a strongly continuous semigroup T. Under (weak) solutions of (2) we understand solutions of the integral equation

$$x(t) = T(t)x(0) + \int_0^t T(t-s)(Bu(s) + C(x(s), u(s)))ds, \quad (4)$$

belonging to $C([0,\tau], X)$ for some $\tau > 0$. We assume that for each $x(0) \in X$ such a solution exists and is unique and refer to [2, Proposition 4.3.3] for details on conditions guaranteeing existence and uniqueness.

Let $\phi(t, \phi_0, u)$ denote the state of the system (2), i.e. the solution to (2), at time $t \in \mathbb{R}_+$ associated with an initial condition $\phi_0 \in X$ at $t = 0$, and an input $u \in \mathcal{U}$.

We define the following classes of comparison functions:

$$\begin{aligned}
\mathcal{P} &:= \{\gamma \in C(\mathbb{R}_+, \mathbb{R}_+) \mid \gamma(0) = 0,\ \gamma(r) > 0 \text{ for } r > 0\} \\
\mathcal{K} &:= \{\gamma \in \mathcal{P} \mid \gamma \text{ is strictly increasing}\} \\
\mathcal{K}_\infty &:= \{\gamma \in \mathcal{K} \mid \gamma \text{ is unbounded}\} \\
\mathcal{L} &:= \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous and strictly decreasing with } \lim_{t\to\infty} \gamma(t) = 0\} \\
\mathcal{KL} &:= \{\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \mid \beta \text{ is continuous, } \beta(\cdot, t) \in \mathcal{K},\ \forall t \geq 0,\ \beta(r, \cdot) \in \mathcal{L},\ \forall r > 0\}
\end{aligned}$$

Next we introduce stability properties for the system (2).

Definition 1: System (2) is *globally asymptotically stable at zero uniformly with respect to the state* (0-UGASs), if there exists a $\beta \in \mathcal{KL}$, such that for all $\phi_0 \in X$, and all $t \geq 0$ it holds that

$$\|\phi(t, \phi_0, 0)\|_X \leq \beta(\|\phi_0\|_X, t). \quad (5)$$

To characterize stability properties of (2) with respect to external inputs, the notion of input-to-state stability [4] is of importance:

Definition 2: System (2) is called *input-to-state stable (ISS)*, if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for all $\phi_0 \in X$, all $u \in \mathcal{U}$ and all $t \geq 0$ it holds that

$$\|\phi(t, \phi_0, u)\|_X \leq \beta(\|\phi_0\|_X, t) + \gamma(\sup_{s\in[0,t]} \|u(s)\|_U). \quad (6)$$

Next we define another notion, which is strictly weaker than ISS in the finite-dimensional case:

Definition 3: System (2) is called *integral input-to-state stable (iISS)* if there exist $\alpha \in \mathcal{K}_\infty$, $\mu \in \mathcal{K}$ and $\beta \in \mathcal{KL}$ such that the inequality

$$\alpha(\|\phi(t, \phi_0, u)\|_X) \leq \beta(\|\phi_0\|_X, t) + \int_0^t \mu(\|u(s)\|_U) ds \quad (7)$$

holds for all $\phi_0 \in X$, $u \in \mathcal{U}$ and $t \geq 0$.

A crucial tool in study of ISS and iISS of control systems is that of a Lyapunov function.

Definition 4: A continuous function $V : X \to \mathbb{R}_+$ is called a non-coercive *iISS Lyapunov function*, if there exist $\psi_2 \in \mathcal{K}_\infty$, $\alpha \in \mathcal{P}$, $\sigma \in \mathcal{K}$ such that

$$0 < V(x) \leq \psi_2(\|x\|_X), \quad \forall x \neq 0 \quad (8)$$

and system (2) satisfies

$$\dot{V}_u(x) \leq -\alpha(\|x\|_X) + \sigma(\|u(0)\|_U) \quad (9)$$

for all $x \in X$ and $u \in \mathcal{U}$, where

$$\dot{V}_u(x) = \overline{\lim_{t\to+0}} \frac{1}{t}(V(\phi(t, x, u)) - V(x)). \quad (10)$$

Furthermore:

- if $\lim_{\tau\to\infty} \alpha(\tau) = \infty$ or $\liminf_{\tau\to\infty} \alpha(\tau) \geq \lim_{\tau\to\infty} \sigma(\tau)$ holds, V is called an *ISS Lyapunov function*.
- if in addition there exists $\psi_1 \in \mathcal{K}_\infty$ so that

$$\psi_1(\|x\|_X) \leq V(x) \leq \psi_2(\|x\|_X), \quad \forall x \in X \quad (11)$$

holds, then V is called a coercive iISS/ISS Lyapunov function for (2).

We write $\dot{V}$ instead of $\dot{V}_u(x)$ when it is clear along which solution the derivative is taken. Lyapunov functions help to prove iISS and ISS of control systems:

Proposition 1 (Prop. 2, [14]): If there exists a coercive iISS (resp. ISS) Lyapunov function for (2), then (2) is iISS (resp. ISS).

III. LINEAR SYSTEMS

The aim of this section is to derive a converse Lyapunov theorem for linear systems with a bounded input operator B of the form

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ x(0) &= x_0. \end{aligned} \quad (12)$$

To this end we will need an auxiliary lemma.

Lemma 1: Let $B \in L(U, X)$ and let T be a C_0-semigroup. Then for any $u \in \mathcal{U}$ it holds that

$$\lim_{h\to+0} \frac{1}{h} \int_0^h T(h-r)Bu(r)dr = Bu(0). \quad (13)$$

Proof: The proof is straightforward and is omitted. ■

The main technical result of this section is as follows:

Proposition 2: If (12) is 0-UGASs, then $V : X \to \mathbb{R}_+$, defined as

$$V(x) = \int_0^\infty \|T(t)x\|_X^2 dt \quad (14)$$

is a non-coercive ISS Lyapunov function for (12). Moreover, $\forall x \in X$, $\forall u \in \mathcal{U}$ and $\forall \varepsilon > 0$ it holds that

$$\dot{V}_u(x) \leq -\|x\|_X^2 + \frac{\varepsilon M^2}{2\lambda} \|x\|_X^2 + \frac{M^2}{2\lambda\varepsilon} \|B\|^2 \|u(0)\|_U^2, \quad (15)$$

where $M, \lambda > 0$ are so that

$$\|T(t)\| \leq Me^{-\lambda t}. \quad (16)$$

Proof: Let (12) be 0-UGASs and pick $u \equiv 0$. Then (5) implies $\|T(t)x\|_X \leq \beta(1, t)$ for all $t \geq 0$ and for all x

with $\|x\|_X = 1$. Since $\beta \in \mathcal{KL}$, there exists t^* such that $\|T(t^*)x\|_X < 1$ for all x, $\|x\|_X = 1$. Thus, $\|T(t^*)\| < 1$ and consequently T is an exponentially stable semigroup [3, Theorem 2.1.6], i.e. there exist $M, \lambda > 0$ such that (16) holds.

Consider $V : X \to \mathbb{R}_+$ as defined in (14). We have

$$V(x) \le \int_0^\infty \|T(t)\|^2 \|x\|_X^2 dt \le \frac{M^2}{2\lambda}\|x\|_X^2. \tag{17}$$

Let $V(x) = 0$. Then $\|T(\cdot)x\| \equiv 0$ a.e. on $[0,\infty)$. Strong continuity of T implies that $x = 0$, and thus (8) holds.

Next we estimate the Lie derivative of V:

$$\begin{aligned}
\dot{V}_u(x) &= \overline{\lim_{h\to+0}} \frac{1}{h}\big(V(\phi(h,x,u)) - V(x)\big)\\
&= \overline{\lim_{h\to+0}} \frac{1}{h}\Big(\int_0^\infty \|T(t)\phi(h,x,u)\|_X^2 dt - \int_0^\infty \|T(t)x\|_X^2 dt\Big)\\
&= \overline{\lim_{h\to+0}} \frac{1}{h}\Big(\int_0^\infty \Big\|T(t)\Big(T(h)x + \int_0^h T(h-r)Bu(r)dr\Big)\Big\|_X^2 dt\\
&\qquad - \int_0^\infty \|T(t)x\|_X^2 dt\Big)\\
&= \overline{\lim_{h\to+0}} \frac{1}{h}\Big(\int_0^\infty \Big\|T(t+h)x + T(t)\int_0^h T(h-r)Bu(r)dr\Big\|_X^2 dt\\
&\qquad - \int_0^\infty \|T(t)x\|_X^2 dt\Big)\\
&\le \overline{\lim_{h\to+0}} \frac{1}{h}\Big(\int_0^\infty \Big(\Big\|T(t+h)x\Big\|_X\\
&\qquad + \Big\|T(t)\int_0^h T(h-r)Bu(r)dr\Big\|_X\Big)^2 dt - \int_0^\infty \|T(t)x\|_X^2 dt\Big)\\
&= I_1 + I_2,
\end{aligned}$$

where

$$I_1 := \overline{\lim_{h\to+0}} \frac{1}{h}\Big(\int_0^\infty \|T(t+h)x\|_X^2 dt - \int_0^\infty \|T(t)x\|_X^2 dt\Big)$$

and

$$\begin{aligned}
I_2 := \overline{\lim_{h\to+0}} \frac{1}{h}\int_0^\infty \Big(&2\|T(t+h)x\|_X \Big\|T(t)\int_0^h T(h-r)Bu(r)dr\Big\|_X\\
&+ \Big\|T(t)\int_0^h T(h-r)Bu(r)dr\Big\|_X^2\Big)dt.
\end{aligned}$$

Let us compute I_1:

$$\begin{aligned}
I_1 &= \overline{\lim_{h\to+0}} \frac{1}{h}\Big(\int_h^\infty \|T(t)x\|_X^2 dt - \int_0^\infty \|T(t)x\|_X^2 dt\Big)\\
&= \overline{\lim_{h\to+0}} -\frac{1}{h}\int_0^h \|T(t)x\|_X^2 dt\\
&= -\|x\|_X^2.
\end{aligned}$$

Now we proceed with I_2:

$$\begin{aligned}
I_2 = \overline{\lim_{h\to+0}} \int_0^\infty 2\Big\|T(t+h)x\Big\|_X \Big\|T(t)\frac{1}{h}\int_0^h T(h-r)Bu(r)dr\Big\|_X dt\\
+ \overline{\lim_{h\to+0}} \int_0^\infty \frac{1}{h}\Big\|T(t)\int_0^h T(h-r)Bu(r)dr\Big\|_X^2 dt.
\end{aligned}$$

The limit of the second term equals zero since

$$\begin{aligned}
\overline{\lim_{h\to+0}} \int_0^\infty \frac{1}{h}\Big\|T(t)\int_0^h T(h-r)Bu(r)dr\Big\|_X^2 dt\\
\le \overline{\lim_{h\to+0}} \int_0^\infty \frac{1}{h} M^4 e^{-2\lambda t}\|B\|\|u\|_{\mathcal{U}} h^2 dt\\
= 0.
\end{aligned}$$

To compute the limit of the first term, note that

$$\begin{aligned}
2\Big\|T(t+h)x\Big\|_X \Big\|T(t)\frac{1}{h}\int_0^h T(h-r)Bu(r)dr\Big\|_X\\
\le 2M\|x\|_X\|T(t)\|M\|B\| \sup_{r\in[0,h]}\|u(r)\|_U\\
\le 2M^3\|x\|_X\|B\|\|u\|_{\mathcal{U}} e^{-\lambda t}.
\end{aligned}$$

and thus we can apply the dominated convergence theorem. Together with Lemma 1 and Youngs' inequality this leads to

$$\begin{aligned}
I_2 &= \int_0^\infty 2\|T(t)x\|_X \|T(t)Bu(0)\|_X dt\\
&\le \int_0^\infty \varepsilon\|T(t)x\|_X^2 + \frac{1}{\varepsilon}\|T(t)Bu(0)\|_X^2 dt\\
&\le \int_0^\infty \varepsilon\|T(t)\|^2 dt\|x\|_X^2 + \frac{1}{\varepsilon}\int_0^\infty \|T(t)\|^2\|Bu(0)\|_X^2 dt\\
&\le \frac{\varepsilon M^2}{2\lambda}\|x\|_X^2 + \frac{M^2}{2\lambda\varepsilon}\|B\|^2\|u(0)\|_U^2,
\end{aligned}$$

for any $\varepsilon > 0$.

Overall, we obtain that $\forall x \in X$, $\forall u \in \mathcal{U}$ and for all $\varepsilon > 0$ the inequality (15) holds. Considering $\varepsilon < \frac{2\lambda}{M^2}$ this shows that V is a non-coercive ISS Lyapunov function for (12). ■

Remark 2: The ISS Lyapunov function V defined in (14) is not coercive in general. Noncoercivity of V defined by (14) implies that the system

$$\dot{x} = Ax, \quad y = x$$

is not exactly observable on $[0,+\infty)$ (although we can measure the full state!), see [3, Corollary 4.1.14]. The reason for this is that for any given exponential decay rate there are states that decay faster that this given rate, and thus we lose a part of the information about the state "infinitely fast".

Under the additional assumption that

$$\|T(t)x\|_X \ge M_2 e^{-\lambda_2 t}\|x\|_X \tag{18}$$

for some $M_2, \lambda_2 > 0$ and for all $x \in X$, V defined in (14) is a coercive ISS-Lyapunov function for (12), since

$$V(x) \ge \alpha_1\|x\|_X^2 \tag{19}$$

holds for some $\alpha_1 > 0$. ■

Below we provide another construction of ISS Lyapunov functions for the system (12) with bounded input operators. This is based on a standard construction in the analysis of C_0-semigroups, see e.g. [17, Eq. (5.14)].

For exponentially stable C_0-semigroup T there exist $M, \lambda > 0$ such that the estimate (16) holds. Choose $\gamma > 0$ such that $\gamma - \lambda < 0$. Then

$$V^\gamma(x) := \max_{s\ge 0}\|e^{\gamma s}T(s)x\|_X \tag{20}$$

defines an equivalent norm on X, for which we have

$$\begin{aligned}
V^\gamma(T(t)x) &= \max_{s\ge 0}\|e^{\gamma s}T(s)T(t)x\|_X\\
&= e^{-\gamma t}\max_{s\ge 0}\|e^{\gamma(s+t)}T(s+t)x\|_X \le e^{-\gamma t}V^\gamma(x).
\end{aligned} \tag{21}$$

Based on this inequality we obtain the following statement for ISS Lyapunov functions.

Proposition 3: Let (12) be 0-UGASs. Let $M,\lambda > 0$ be such that (16) holds and let $0 < \gamma < \lambda$. Then $V^\gamma : X \to \mathbb{R}_+$, defined by (20) is a coercive ISS Lyapunov function for (12). In particular, for any $u \in \mathscr{U}$, $x \in X$, we have the dissipation inequality

$$\dot{V}_u^\gamma(x) \le -\gamma\, V^\gamma(x) + V^\gamma(Bu(0))\,. \tag{22}$$

Proof: In order to obtain the infinitesimal estimate, we compute, using the triangle inequality as V^γ is a norm, the estimate (21) and Lemma 1,

$$\begin{aligned}
\dot{V}_u^\gamma(x) &= \overline{\lim_{h\to+0}} \frac{1}{h}\left(V^\gamma(\phi(h,x,u)) - V^\gamma(x)\right)\\
&= \overline{\lim_{h\to+0}} \frac{1}{h}\Big(V^\gamma\Big(T(h)x + \int_0^h T(h-r)Bu(r)dr\Big) - V^\gamma(x)\Big)\\
&\le \overline{\lim_{h\to+0}} \frac{1}{h}\Big(V^\gamma\Big(T(h)x\Big) + V^\gamma\Big(\int_0^h T(h-r)Bu(r)dr\Big)\\
&\qquad - V^\gamma(x)\Big)\\
&\le \overline{\lim_{h\to+0}} \frac{1}{h}\Big((e^{-\gamma h}-1)V^\gamma(x) + V^\gamma\Big(\int_0^h T(h-r)Bu(r)dr\Big)\Big)\\
&\le -\gamma\, V^\gamma(x) + V^\gamma(Bu(0))\,.
\end{aligned}$$

This shows V^γ is an ISS-Lyapunov function and that (22) holds. Coercivity is clear by construction. ■

Finally we can state the main result of this section:

Theorem 4: Let $B : U \to X$ be an arbitrary bounded linear operator. The following statements are equivalent:

(i) (12) is ISS
(ii) (12) is 0-UGAS
(iii) $T(\cdot)$ is an exponentially stable semigroup
(iv) V defined in (14) is a (not necessarily coercive) ISS Lyapunov function for (12).
(v) V^γ defined in (20) is a coercive ISS Lyapunov function for (12).

Proof: Equivalence between items (i) and (ii) can be easily derived from the variation of constants formula. The implications (ii) ⇒ (iii) ⇒ (iv) follow from Proposition 2. Item (iv) implies (iii) due to Datko's Lemma, see [3, Lemma 5.1.2, Theorem 5.1.3, p. 215]. Implication (iii) ⇒ (ii) is clear. (ii) implies (v) due to Proposition 3 and (v) implies (i) by Proposition 1. ■

IV. Bilinear systems

Now we turn our attention to bilinear systems of the form (2). Note that this class includes systems with C linear in both variables and bounded in the sense that $\|C\| := \sup_{\|x\|_X=1,\|u\|_U=1} \|C(x,u)\|_X < \infty$ (then $K = \|C\|$ and $\xi(r) = r$ for all $r \in \mathbb{R}_+$ in (3)).

We have the following bilinear counterpart to Proposition 2:

Proposition 5: Let (2) be 0-UGASs and let V be defined as in (14). Then $W : X \to \mathbb{R}_+$, defined as

$$W(x) = \ln(1+V(x)), \tag{23}$$

is an (in general non-coercive) iISS Lyapunov function for (12).

If T satisfies (18) for some $M_2,\lambda_2 > 0$ and for all $x \in X$, then W is a coercive iISS Lyapunov function for (2).

Proof: Let (12) be 0-UGASs. Then $T(\cdot)$ is an exponentially stable semigroup and thus there exist $M,\lambda > 0$ so that (16) holds.

The chain rule implies

$$\dot{W}_u(x) = \frac{1}{1+V(x)}\dot{V}_u(x). \tag{24}$$

In order to compute $\dot{V}_u(x)$ we perform the same derivations as in the previous theorem with the difference, that instead of $Bu(t)$ we need to consider $Bu(t) + C(x(t),u(t))$. Then we obtain for any $\varepsilon > 0$ the estimate:

$$\begin{aligned}
\dot{V}_u(x) \;&\le\; -\|x\|_X^2 + \int_0^\infty \varepsilon\|T(t)\|^2 dt\|x\|_X^2\\
&\quad + \frac{1}{\varepsilon}\int_0^\infty \|T(t)\|^2\|Bu(0)+C(x,u(0))\|_X^2 dt\\
&\le\; \Big(\frac{\varepsilon M^2}{2\lambda} - 1\Big)\|x\|_X^2\\
&\quad + \frac{M^2}{2\lambda\varepsilon}\Big(\|B\|^2\|u(0)\|_U^2 + \|C(x,u(0))\|_X^2\Big)\\
&\le\; \Big(\frac{\varepsilon M^2}{2\lambda} - 1\Big)\|x\|_X^2\\
&\quad + \frac{M^2}{2\lambda\varepsilon}\Big(\|B\|^2\|u(0)\|_U^2 + K^2\|x\|_X^2\xi^2(\|u(0)\|_U)\Big)
\end{aligned}$$

Due to (24) we obtain

$$\begin{aligned}
\dot{W}_u(x) \;&\le\; \Big(\frac{\varepsilon M^2}{2\lambda} - 1\Big)\frac{\|x\|_X^2}{1+V(x)} + \frac{M^2}{2\lambda\varepsilon}\frac{\|B\|^2\|u(0)\|_U^2}{1+V(x)}\\
&\quad + \frac{M^2}{2\lambda\varepsilon}\frac{1}{1+V(x)}K^2\|x\|_X^2\xi^2(\|u(0)\|_U)\\
&\le\; \Big(\frac{\varepsilon M^2}{2\lambda} - 1\Big)\frac{\|x\|_X^2}{1+\frac{M^2}{2\lambda}\|x\|_X^2} + \frac{M^2}{2\lambda\varepsilon}\|B\|^2\|u(0)\|_U^2\\
&\quad + \frac{M^2K^2}{2\lambda\varepsilon}\frac{\|x\|_X^2}{1+\alpha_1\|x\|_X^2}\xi^2(\|u(0)\|_U)\\
&\le\; \Big(\frac{\varepsilon M^2}{2\lambda} - 1\Big)\frac{\|x\|_X^2}{1+\frac{M^2}{2\lambda}\|x\|_X^2} + \frac{M^2}{2\lambda\varepsilon}\|B\|^2\|u(0)\|_U^2\\
&\quad + \frac{M^2K^2}{2\lambda\varepsilon\alpha_1}\xi^2(\|u(0)\|_U),
\end{aligned}$$

where we have estimated V from below by (19).

Considering $\varepsilon \in (0, \frac{2\lambda}{M^2})$ we see that W is an iISS Lyapunov function for (2). ■

Finally, we state a converse iISS Lyapunov theorem for bilinear systems.

Theorem 6: Let $B : U \to X$ be a bounded linear operator and let C satisfy (3). The following statements are equivalent:

(i) (2) is iISS
(ii) (2) is 0-UGAS
(iii) $T(\cdot)$ is an exponentially stable semigroup
(iv) W defined in (23) is a (not necessarily coercive) iISS Lyapunov function for (12).

Proof: Equivalence between items (i) and (ii) has been proved in [14, Theorem 7]. The implications (ii) ⇒ (iii) ⇒ (iv) follow from Proposition 5. Item (iv) implies (iii) due to Datko's Lemma, see [3, Lemma 5.1.2, Theorem 5.1.3, p. 215]. Implication (iii) $\Rightarrow$ (ii) is clear.

If W is a coercive iISS Lyapunov function, then Proposition 1 already implies iISS of (2). ■

V. Unbounded input operators

Having obtained Lyapunov characterizations for linear and bilinear systems with bounded input operators and bilinearities satisfying (3), we consider again a linear system (12), but now we do not assume anymore that B is a bounded operator.

Pick any $z \in \rho(A)$ (the resolvent set of A) and define X_{-1} to be the completion of X w.r.t. the norm

$$\|x\|_{-1} = \|(zI - A)^{-1}x\|_X.$$

Note that X_{-1} is a Banach space and X is dense in X_{-1}. It is known that the space X_{-1} does not depend on the choice of $z \in \rho(A)$ and different choices of z give rise to equivalent norms on X_{-1}.

The operator $B : U \to X$ can now be viewed as a bounded operator $B \in L(U, X_{-1})$ and the semigroup $T(\cdot)$ can be extended to the C_0-semigroup $T_{-1}(\cdot)$: $T_{-1}(t) \in L(X_{-1})$. In particular, for all $x \in X$ we have $T(t)x = T_{-1}(t)x$ for all $t \geq 0$. The generator of T_{-1} is A_{-1} (which is the extension of A).

Thus instead of (12), we can study the system

$$\dot{x} = A_{-1}x + Bu,$$

the solution of which is given by

$$\phi(t,x,u) = T_{-1}(t)x + \int_0^t T_{-1}(t-s)Bu(s)ds. \qquad (25)$$

For any $t > 0$ and $u \in \mathscr{U}$ it holds that

$$\int_0^t T_{-1}(t-s)Bu(s)ds \in X_{-1}.$$

In order to ensure that the solution, corresponding to $x \in X$ will stay in X for all inputs $u \in \mathscr{U}$ it is natural to assume

$$\int_0^t T_{-1}(t-s)Bu(s)ds \in X$$

for all u and all t. If B satisfies this condition, B is called an ∞-admissible control (input) operator [21], [10]. In other words, the control operator B is ∞-admissible, if

$$\left\| \int_0^t T(s)Bu(s)\,ds \right\|_X \leq h_t \sup_{s \in [0,t]} \|u(s)\|_U \qquad (26)$$

holds for all $u \in \mathscr{U}$, all $t \geq 0$ and some constant $h_t > 0$ dependent on t. If h does not depend on t, then B is called infinite-time ∞-admissible operator.

The basic characterization of ISS for (12) with unbounded input operators is given by

Proposition 7: The following notions are equivalent for systems (12):

1) ISS
2) 0-UGASs + infinite-time ∞-admissibility of B
3) 0-UGASs + ∞-admissibility of B

Proof: The equivalence between 1) and 2) follows by considering $u \equiv 0$ and $\phi_0 \equiv 0$. The equivalence between 2) and 3) holds since for 0-UGASs systems (i.e. for exponentially stable semigroups T) infinite-time ∞-admissibility of B and ∞-admissibility of B are equivalent notions, see [8, Lemma 1.1] (in a dual form). ■

It seems that the technique of proof used in Proposition 2 is not sharp enough to treat the case of linear systems with admissible input operators. However, we may hope to achieve some results also in the latter case, as the following considerations show.

Let X be a Hilbert space with a scalar product $\langle \cdot, \cdot \rangle$. Then V can be equivalently written [3, Theorem 5.1.3, p. 217] as

$$V(x) = \langle Px, x \rangle, \qquad (27)$$

where $P \in L(X)$ is a positive operator (i.e. P is self-adjoint and $\langle Px, x \rangle > 0$ for all $x \neq 0$), which is a solution of the Lyapunov equation

$$\langle Ax, Px \rangle + \langle Px, Ax \rangle = -\langle x, x \rangle, \quad x \in D(A). \qquad (28)$$

Next we formally differentiate V for $x \in D(A)$ and $u \in U$ (the proof is not strict, since Bu exists only for $u \in D(B)$; however, it is possible to give a strict (but longer) argument using the definition (10)):

$$\begin{aligned} \dot{V}_u(x) &= \langle P(Ax+Bu), x \rangle + \langle Px, Ax+Bu \rangle \\ &= -\|x\|_X^2 + \langle PBu, x \rangle + \langle Px, Bu \rangle \\ &= -\|x\|_X^2 + 2\langle PBu, x \rangle. \end{aligned} \qquad (29)$$

In the above estimates we used the facts that P is an self-adjoint operator and solves the Lyapunov equation (28). The last transition is important since although $Bu \notin X$ for $u \notin D(B)$, but if $PB \in L(U,X)$, then $PBu \in X$ for all $u \in U$. Consequently, if $PB \in L(U,X)$ we can proceed with

$$\dot{V}_u(x) \leq -(1-\varepsilon)\|x\|_X^2 + \frac{1}{\varepsilon}\|PB\|^2\|u\|_U^2, \qquad (30)$$

which holds for all $x \in D(A)$, all $u \in U$ and all $\varepsilon > 0$. Using density argument one can prove that (30) holds for all $x \in X$.

VI. Example

Consider the system (12) with the state space

$$X = l_2(\mathbb{N}) := \left\{ x = \{x_k\}_{k=1}^{\infty} : \|x\|_X = \Big(\sum_{k=1}^{\infty} x_k^2 \Big)^{1/2} < \infty \right\}.$$

endowed in the usual way with the scalar product $\langle \cdot, \cdot \rangle$. Let $U := \mathbb{R}$ (the field of scalars).

Let (12) be diagonal, i.e. let $Ae_k = -\lambda_k e_k$, where e_k is the k-th unity vector of $l_2(\mathbb{N})$ and $\lambda_k \in \mathbb{R}$. For simplicity we assume that $\lambda_k < \lambda_{k+1}$ for all k and $\lambda_k \to \infty$ as $k \to \infty$. Also assume that A generates an exponentially stable semigroup, i.e. there exists $\varepsilon > 0$: $\lambda_k > \varepsilon$ for all $k > 0$[1].

We are going to find an ISS Lyapunov function for this system. We consider two cases: bounded and unbounded input operators. First let B be an arbitrary operator in $L(U,X)$.

[1]The idea to use this system arose after a personal communication with Birgit Jacob from the University of Wuppertal, who considered this system in order to show certain other properties of infinite-dimensional ISS systems.

The generator $A: x \mapsto \sum_{k=1}^{\infty} -\lambda_k \langle x, e_k \rangle e_k$ is self-adjoint as well as its inverse $A^{-1}: x \mapsto \sum_{k=1}^{\infty} -\frac{1}{\lambda_k} \langle x, e_k \rangle e_k$. Since $\lambda_k > \varepsilon$ for some $\varepsilon > 0$, A^{-1} is bounded. Thus the solution of (28) is given by $P = -\frac{1}{2}A^{-1} > 0$, which is a (bounded) positive operator.

The Lyapunov function V given by (27) is equal to

$$V(x) = \langle Px, x \rangle = \sum_{k=1}^{\infty} \frac{1}{2\lambda_k} \langle x, e_k \rangle^2. \quad (31)$$

It is easy to see that P (as well as V) is not coercive since $\lambda_k \to \infty$ as $k \to \infty$.

However, for this example it is easy to find a coercive ISS Lyapunov function V_2:

$$V_2 : x \mapsto \langle x, x \rangle = \|x\|_X^2.$$

Using the definition of A we see that the Lyapunov inequality

$$\langle Ax, x \rangle + \langle x, Ax \rangle \le -\lambda \langle x, x \rangle, \quad x \in D(A) \quad (32)$$

holds, where $\lambda := \inf_{k \ge 1} \lambda_k$. In turn, this implies the following estimates for $x \in D(A)$ and any $\varepsilon > 0$:

$$\begin{aligned} \dot{V}_{2,u}(x) &\le -\lambda \|x\|_X^2 + 2\langle x, Bu \rangle \\ &\le -\lambda \|x\|_X^2 + 2\|x\|_X \|B\| \|u\|_U \\ &\le -(\lambda - \varepsilon)\|x\|_X^2 + \frac{1}{\varepsilon}\|B\|^2 \|u\|_U^2. \end{aligned}$$

Using density arguments one can show that the last inequality is valid for all $x \in X$, which implies for $\varepsilon < \lambda$ that V_2 is a coercive ISS Lyapunov function for (12) with our operator A and for any $B \in L(U, X)$.

Now let us assume that B is merely an admissible operator. Since $0 \in \rho(A)$, X_{-1} is equal to

$$X_{-1} := \Big\{ \{x_k\}_{k=1}^{\infty} : \sum_{k=1}^{\infty} \frac{|x_k|^2}{\lambda_k^2} < \infty \Big\}.$$

Since $U = \mathbb{R}$, every operator $B \in L(U, X_{-1})$ can be identified with a sequence $\{b_k\}$ in X_{-1}. It can be shown that

Lemma 3: B is ∞-admissible for $T \Leftrightarrow \sum_{k=1}^{\infty} \frac{|b_k|^2}{\lambda_k^2} < \infty$.

Now let us consider the product PB, where P is a solution (31) of the Lyapunov equation (28). Clearly,

$$PBu = \frac{1}{2} \sum_{k=1}^{\infty} \frac{b_k}{\lambda_k} u.$$

Thus,

$$\|PBu\|_X = \frac{1}{2} \Big(\sum_{k=1}^{\infty} \frac{|b_k|^2}{\lambda_k^2} \Big)^{1/2} |u| = M|u|,$$

with a finite M, due to admissibility of the operator B and Lemma 3. Thus, $PB \in L(U, X)$ and the estimate (29) shows that V defined by (27) is a non-coercive ISS Lyapunov function for our system for any admissible operator B.

Remark 4: Interestingly, $V_2 : x \mapsto \|x\|_X^2$, which is a coercive ISS Lyapunov function for the above example for any bounded B is no longer an ISS Lyapunov function for the same system with any admissible unbounded B.

This Lemma has been told to authors by Birgit Jacob, see the footnote on the page 499

VII. Conclusions

We have derived converse ISS/iISS Lyapunov theorems for linear and bilinear systems over Banach spaces with bounded input operators. In Sections V, VI we have shown by means of an example that the situation is getting more complex if systems with unbounded admissible operators are considered. A Lyapunov characterization of ISS for systems (1) with unbounded admissible B remains an open problem.

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