22nd International Symposium on
Mathematical Theory of Networks and Systems
July 11-15, 2016. Minneapolis, MN, USA

Restatements of input-to-state stability in infinite dimensions: what goes wrong?

Andrii Mironchenko and Fabian Wirth

Abstract— **We show by means of counterexamples that many characterizations of input-to-state stability (ISS) known for ODE systems are not valid for general differential equations in Banach spaces. Moreover, these notions or combinations of notions are not equivalent to each other, and can be classified into several groups according to the type and grade of nonuniformity. We introduce the new notion of strong ISS which is equivalent to ISS in the ODE case, but which is strictly weaker than ISS in the infinite-dimensional setting. We characterize strong ISS as a strong asymptotic gain property plus global stability.**

Keywords: input-to-state stability, nonlinear systems, infinite-dimensional systems.

I. Introduction

For ordinary differential equations, the concept of input-to-state stability (ISS) was introduced in [1]. The corresponding theory is by now well developed with a firm theoretical basis. A variety of powerful tools for the investigation of ISS is available and a multitude of applications have been developed in nonlinear control theory, in particular, to robust stabilization of nonlinear systems [2], design of nonlinear observers [3], analysis of large-scale networks [4], [5], [6] etc.

Characterizations of ISS in terms of other stability properties [7], [8] are among the central theoretical results in ISS theory of finite-dimensional systems. In [7] Sontag and Wang have shown that ISS is equivalent to the existence of a smooth ISS Lyapunov function and in [8] the same authors proved a so-called ISS superposition theorem, saying that ISS is equivalent to the combination of stability of an undisturbed system together with an asymptotic gain property of the system with inputs. These theorems greatly simplify the proofs of other fundamental results, such as small-gain theorems [5], and are useful for analysis of other classes of systems, such as time-delay systems in the Lyapunov-Razumikhin framework [9], [10], hybrid systems [11] to name a few examples.

The success of ISS theory of ordinary differential equations and the need of proper tools for robust stability analysis of partial differential equations motivated the development of ISS theory in infinite-dimensional setting [12], [13], [14], [15], [16], [17].

A. Mironchenko is with Faculty of Computer Science and Mathematics, University of Passau, Innstraße 33, 94032 Passau, Germany. Corresponding author, andrii.mironchenko@uni-passau.de.

F. Wirth is with Faculty of Computer Science and Mathematics, University of Passau, Innstraße 33, 94032 Passau, Germany, fabian.(lastname)@uni-passau.de

In particular, in the recent paper [17] it was shown that uniform asymptotic stability at zero, local ISS and the existence of a LISS Lyapunov function are equivalent properties for a system of the form

$$\dot{x}(t) = Ax(t) + f(x(t), u(t)), \quad x(t) \in X,\ u(t) \in U, \qquad (1)$$

provided the right hand side has some sort of uniform continuity with respect to u. Here X is a Banach space, U is a linear normed space, A is the generator of a C_0-semigroup $\{T(t),\ t \geq 0\}$ and $f : X \times U \to X$ is sufficiently regular.

In addition, in [17] a system of the form (1) was constructed, which is locally ISS (LISS), uniformly globally asymptotically stable for a zero input (0-UGAS), globally stable (GS) and which has the asymptotic gain (AG) property, but which is not ISS. This is in contrast to ODE systems, for which we have the equivalences

$$\text{AG} \wedge \text{GS} \ \Leftrightarrow\ \text{AG} \wedge \text{0-UGAS} \ \Leftrightarrow\ \text{AG} \wedge \text{LISS} \ \Leftrightarrow\ \text{ISS},$$

see [7], [8]. This naturally leads to a set of challenging questions: which properties are still equivalent to ISS? Can one classify the non-ISS properties in a natural way? Is it possible to introduce a reasonable ISS-like property which will be equivalent to ISS in finite dimensions, but weaker than ISS for general systems (1)?

In this paper we make some steps towards the solution of these questions. As argued above, for (1) ISS is no longer equivalent to the combinations of not fully uniform notions (like AG ∧ GS or AG ∧ 0-UGAS). Moreover, these combinations of notions are no longer equivalent with each other, as we show by means of counterexamples. Instead, they can be classified into several groups, according to the type and grade of uniformity.

We start our analysis in Section V with a counterexample, showing that (nonuniform) global asymptotic stability at zero (0-GAS) together with local uniform asymptotic stability at zero (0-UAS) does not imply global stability (GS) of the system (1). Next in Section VI we show that the uniform asymptotic gain property (UAG) under minor requirements from the flow of the system is still equivalent to ISS. Using these results, in Section VII we classify the stability notions for undisturbed systems into four groups, which are all equivalent in the finite-dimensional case, but which are essentially different in infinite dimensions. Finally, we introduce a strong ISS (sISS) property, which for linear systems without inputs is resolved to strong stability of a semigroup T (ISS for linear systems without inputs corresponds to the exponential stability of T). In order to characterize strong

ISS we introduce a strong asymptotic gain (sAG) property, which is weaker than UAG, and prove that strong ISS is equivalent to global stability plus sAG property.

Although this paper does not solve the problem of characterizing the ISS property for system (1), we believe that it indicates a promising direction in order to achieve a complete solution of this fundamental problem.

II. Preliminaries

We study mild solutions of (1), i.e. solutions $x:[0,\tau]\to X$ of the integral equation

$$x(t)=T(t)x(0)+\int_0^t T(t-s)f(x(s),u(s))ds, \tag{2}$$

belonging to the space of continuous functions $C([0,\tau],X)$ for some $\tau>0$.

We assume that $f(0,0)=0$, i.e., $x^*=0$ is an equilibrium point of (1) when the input is $u\equiv 0$.

Let $\mathbb{R}_+ := [0,\infty)$. In the sequel we assume that the set of input values U is a normed linear space and that the input functions belong to the space $\mathscr{U} := PC(\mathbb{R}_+,U)$ of globally bounded, piecewise continuous functions $f:\mathbb{R}_+\to U$, which are right continuous. We denote the norm of $u\in\mathscr{U}$ by $\|u\|_{\mathscr{U}} := \sup_{t\geq 0}\|u(s)\|_U$. The closed ball of radius r around 0 in X is denoted by $B_r := \{x\in X : \|x\|_X\leq r\}$.

Throughout the paper, we use the following assumption concerning the nonlinearity f.

Assumption 1: The function $f: X\times U\to X$ is Lipschitz continuous on bounded subsets of X, uniformly with respect to the second argument, i.e. for all $C>0$, there exists a $L_f(C)>0$, such that for all $x,y\in B_C$ and for all $v\in U$, it holds that

$$\|f(y,v)-f(x,v)\|_X\leq L_f(C)\|y-x\|_X. \tag{3}$$

In addition, we assume that $f(x,\cdot)$ is continuous for all $x\in X$.

Since $\mathscr{U}=PC(\mathbb{R}_+,U)$, Assumption 1 ensures that the mild solution of (1) exists and is unique, according to a variation of a classical existence and uniqueness theorem [18, Proposition 4.3.3]. By $\phi(t,x,u)$ we denote the solution at time $t\in\mathbb{R}_+$ associated with the initial condition $x\in X$ at $t=0$ and the input $u\in\mathscr{U}$.

Next we state the other two assumptions which we require from the system (1) on the following pages. The first one is essentially a continuity property of the solution map in the equilibrium solution.

Assumption 2: We assume that $0\in X$ is a robust equilibrium point of (1), that is,

- $\phi(t,0,0)=0$ for all $t\geq 0$
- for every $\varepsilon>0$ and for any $h>0$ there exists $\delta=\delta(\varepsilon,h)>0$, so that

$$t\in[0,h]\wedge\|x\|_X\leq\delta\wedge\|u\|_{\mathscr{U}}\leq\delta\ \Rightarrow\ \|\phi(t,x,u)\|_X\leq\varepsilon. \tag{4}$$

Assumption 3: We assume that (1) is robustly forward complete (RFC), that is for any $C>0$ and any $\tau>0$ it holds that

$$\sup_{\|x\|_X\leq C,\ \|u\|_{\mathscr{U}}\leq C,\ t\in[0,\tau]}\|\phi(t,x,u)\|_X<\infty.$$

For the formulation of stability properties the following classes of comparison functions are useful:

$$\begin{aligned}
\mathscr{K} &:= \{\gamma:\mathbb{R}_+\to\mathbb{R}_+ \mid \gamma \text{ is continuous, strictly increasing and } \gamma(0)=0\},\\
\mathscr{K}_\infty &:= \{\gamma\in\mathscr{K} \mid \gamma \text{ is unbounded}\},\\
\mathscr{L} &:= \{\gamma:\mathbb{R}_+\to\mathbb{R}_+ \mid \gamma \text{ is continuous and strictly decreasing with } \lim_{t\to\infty}\gamma(t)=0\},\\
\mathscr{KL} &:= \{\beta:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+ \mid \beta \text{ is continuous, } \beta(\cdot,t)\in\mathscr{K},\ \beta(r,\cdot)\in\mathscr{L},\ \forall t\geq 0,\ \forall r>0\}.
\end{aligned}$$

Next we list the stability notions, which will be investigated on the following pages. We start with notions for systems without disturbances, followed by notions for systems with inputs.

Definition 1: System (1) is called

- uniformly stable at zero (0-ULS), if for all $\varepsilon>0$ there exists a $\delta>0$ so that
$$x\in B_\delta\quad\Rightarrow\quad\|\phi(t,x,0)\|_X<\varepsilon,\forall t\geq 0. \tag{5}$$
- *uniformly globally stable at zero (0-UGS)*, if there exists a $\sigma\in\mathscr{K}_\infty$ so that
$$\|\phi(t,x,0)\|_X\leq\sigma(\|x\|_X),\quad\forall x\in X,\forall t\geq 0. \tag{6}$$
- *practically uniformly globally stable at zero (0-pUGS)*, if there exist $\sigma\in\mathscr{K}_\infty$ and $c>0$ so that
$$\|\phi(t,x,0)\|_X\leq\sigma(\|x\|_X)+c,\quad\forall x\in X,\forall t\geq 0. \tag{7}$$
- *globally attractive at zero (0-GATT)*, if
$$\lim_{t\to\infty}\|\phi(t,x,0)\|_X=0,\quad\forall x\in X. \tag{8}$$
- a system with the *limit property at zero (0-LIM)*, if
$$\inf_{t\geq 0}\|\phi(t,x,0)\|_X=0,\quad\forall x\in X.$$
- *uniformly globally attractive at zero (0-UGATT)*, if for all $\varepsilon,\delta>0$ there exists a $\tau_a=\tau_a(\varepsilon,\delta)<\infty$ such that
$$t\geq\tau_a,\ x\in B_\delta\quad\Rightarrow\quad\|\phi(t,x,0)\|_X\leq\varepsilon. \tag{9}$$
- *globally asymptotically stable at zero* (0-GAS), if (1) is 0-ULS and 0-GATT.
- *asymptotically stable at zero uniformly with respect to the state* (0-UAS), if there exists a $\beta\in\mathscr{KL}$ and $r>0$, such that
$$\|\phi(t,x,0)\|_X\leq\beta(\|x\|_X,t),\quad\forall x\in B_r,\forall t\geq 0. \tag{10}$$
- *globally asymptotically stable at zero uniformly with respect to the state* (0-UGAS), if it is 0-UAS and (10) holds for all $x\in X$.

We stress the difference between the uniform notions 0-UGATT and 0-UGAS and the nonuniform notions 0-GATT and 0-GAS. For 0-GATT systems all trajectories converge to the origin, but their speed of convergence may differ drastically for initial values with the same norm, in contrast to 0-UGATT systems. The notions of 0-ULS, 0-UGS and 0-pUGS are uniform in the sense that there exists an upper bound of the norm of trajectories which is equal for initial states with the same norm.

Remark 1: 0-GAS is equivalent to 0-UGAS for ODE systems, but it is weaker than 0-UGAS for infinite-dimensional systems. For linear systems $\dot{x} = Ax$, where A generates a strongly continuous semigroup, the Banach-Steinhaus theorem implies that 0-GAS is equivalent to strong stability of the associated semigroup T and implies the UGS property.

The 0-LIM property describes systems whose trajectories approach the origin arbitrarily closely. Obviously, 0-GATT implies 0-LIM.

The notions of 0-ULS and 0-UGS lead to the following notions for the systems with inputs.

Definition 2: System (1) is

- *uniformly locally stable (ULS)*, if there exist $\sigma, \gamma \in \mathcal{K}_\infty$ and $r > 0$ such that for all $x \in B_r$ and all $u \in \mathcal{U}$ with $\|u\|_{\mathcal{U}} \le r$, it holds that

$$\|\phi(t,x,u)\|_X \le \sigma(\|x\|_X) + \gamma(\|u\|_{\mathcal{U}}), \quad \forall t \ge 0. \quad (11)$$

- *uniformly globally stable (UGS)*, if it is locally stable with $r = \infty$.
- *practically uniformly globally stable (pUGS)*, if there exist $\sigma, \gamma \in \mathcal{K}_\infty$ and $c > 0$ such that for all $x \in X$, and all $u \in \mathcal{U}$ it holds that

$$\|\phi(t,x,u)\|_X \le \sigma(\|x\|_X) + \gamma(\|u\|_{\mathcal{U}}) + c, \quad \forall t \ge 0. \quad (12)$$

Remark 2: It is easy to see that the notion of pUGS is equivalent to the boundedness property (BND), as defined in [8, p. 1285].

Next we list attractivity properties for systems with inputs.

Definition 3: System (1) has the

- *limit property (LIM)*, if there exists a $\gamma \in \mathcal{K}_\infty \cup \{0\}$, such that

$$\inf_{t \ge 0} \|\phi(t,x,u)\|_X \le \gamma(\|u\|_{\mathcal{U}}) \qquad \forall x \in X,\ \forall u \in \mathcal{U}.$$

- *asymptotic gain (AG) property*, if there exists a $\gamma \in \mathcal{K}_\infty \cup \{0\}$ such that for all $\varepsilon > 0$, for all $x \in X$ and for all $u \in \mathcal{U}$ there exists a $\tau_a = \tau_a(\varepsilon, x, u) < \infty$ such that

$$t \ge \tau_a \quad \Rightarrow \quad \|\phi(t,x,u)\|_X \le \varepsilon + \gamma(\|u\|_{\mathcal{U}}). \quad (13)$$

- *strong asymptotic gain (sAG) property*, if there exists a $\gamma \in \mathcal{K}_\infty \cup \{0\}$ such that for all $x \in X$ and for all $\varepsilon > 0$ there exists a $\tau_a = \tau_a(\varepsilon, x) < \infty$ such that for all $u \in \mathcal{U}$

$$t \ge \tau_a \quad \Rightarrow \quad \|\phi(t,x,u)\|_X \le \varepsilon + \gamma(\|u\|_{\mathcal{U}}). \quad (14)$$

- *uniform asymptotic gain (UAG) property*, if there exists a $\gamma \in \mathcal{K}_\infty \cup \{0\}$ such that for all $\varepsilon, \delta > 0$ there exists a $\tau_a = \tau_a(\varepsilon, \delta) < \infty$ such that for all $u \in \mathcal{U}$ and all $x \in B_\delta$

$$t \ge \tau_a \quad \Rightarrow \quad \|\phi(t,x,u)\|_X \le \varepsilon + \gamma(\|u\|_{\mathcal{U}}). \quad (15)$$

All three properties AG, sAG and UAG imply that all trajectories converge to the ball of radius $\gamma(\|u\|_{\mathcal{U}})$ around the origin as $t \to \infty$. The difference between AG, sAG and UAG is in the kind of dependence of τ_a on the states and inputs. In UAG systems this time depends (besides ε) only on the norm of the state, in sAG systems it depends on the state x (and may vary for different states with the same norm), but it does not depend on u. In AG systems τ_a depends both on x and on u. For systems without inputs, the AG and sAG properties are reduced to 0-GATT, and the UAG property becomes 0-UGATT.

Now we proceed to the main notion of this paper:

Definition 4: System (1) is called *(uniformly) input-to-state stable (ISS)*, if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for all $x \in X$, $u \in \mathcal{U}$ and $t \ge 0$ the following holds

$$\|\phi(t,x,u)\|_X \le \beta(\|x\|_X, t) + \gamma(\|u\|_{\mathcal{U}}). \quad (16)$$

The local counterpart of the ISS property is

Definition 5: System (1) is called *(uniformly) locally input-to-state stable (LISS)*, if there exist $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$ and $r > 0$ such that the inequality (16) holds for all $x \in B_r$, $\|u\|_{\mathcal{U}} \le r$ and $t \ge 0$.

A powerful tool to investigate local ISS of control systems is a Lyapunov function.

Definition 6: Let $D \subset X$ with $0 \in \text{int}(D)$. A continuous function $V : D \to \mathbb{R}_+$ is called a *LISS Lyapunov function*, if there exist $r > 0$, $\psi_1, \psi_2 \in \mathcal{K}_\infty$, $\alpha \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{K}$ such that $B_r \subset D$,

$$\psi_1(\|x\|_X) \le V(x) \le \psi_2(\|x\|_X), \quad \forall x \in B_r \quad (17)$$

and the Lie derivative of V along the trajectories of the system (1) satisfies

$$\dot{V}_u(x) \le -\alpha(\|x\|_X) + \sigma(\|u(0)\|_U) \quad (18)$$

for all $x \in B_r$ and $u \in \mathcal{U}$ with $\|u\|_{\mathcal{U}} \le r$. Here the Lie derivative of V corresponding to the input u is defined by

$$\dot{V}_u(x) = \overline{\lim_{t \to +0}} \frac{1}{t}(V(\phi(t,x,u)) - V(x)). \quad (19)$$

The function V is called a 0-UAS Lyapunov function, if (17) is satisfied and if (18) holds for $u \equiv 0$.

III. Previous work

In this section we briefly describe results, related to the topic of this paper which are available in the literature.

The fundamental result due to Sontag and Wang [7], [8], which we informally described in the introduction, states that

Proposition 1: Let $X = \mathbb{R}^n$, $U = \mathbb{R}^m$. For an ODE system (1), satisfying Assumption 1 the combinations of properties depicted in Figure 1 are equivalent.

In particular, ISS $\Leftrightarrow$ UAG $\Leftrightarrow$ AG $\wedge$ UGS $\Leftrightarrow$ AG $\wedge$ 0-UGAS $\Leftrightarrow$ existence of a smooth ISS Lyapunov function.

One of the interesting features of Proposition 1 is that uniform properties like ISS or UAG are equivalent to combinations of nonuniform properties, like AG $\wedge$ ULS. We will see that this is no longer true in infinite dimensions.

Concerning infinite-dimensional systems, in [17, Theorem 4] the following characterization of LISS has been obtained:

Theorem 2: Let Assumption 1 hold and assume there exist $\sigma \in \mathcal{K}$ and $\rho > 0$ so that for all $v \in U$ such that $\|v\|_U \le \rho$ and all $x \in X$ with $\|x\|_X \le \rho$ we have

$$\|f(x,v) - f(x,0)\|_X \le \sigma(\|v\|_U). \quad (20)$$

Then for the system (1) the following properties are equivalent:

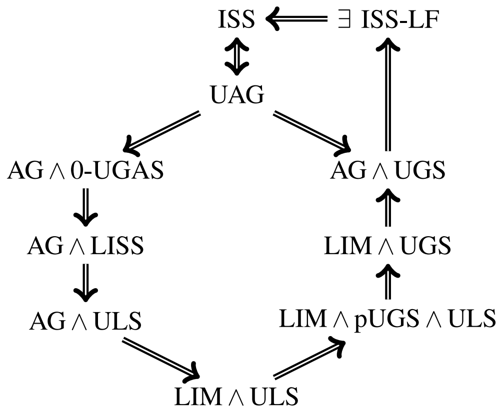

Fig. 1. Characterizations of ISS in finite dimensions

(i) 0-UAS.
(ii) Existence of a Lipschitz continous 0-UAS Lyapunov function.
(iii) Existence of a Lipschitz continuous LISS Lyapunov function.
(iv) LISS.

The situation concerning the global ISS property is more complicated, as indicated by the following example in [17].

Example 1: Consider a system Σ with state space $X = l_1 := \{(x_k)_{k=1}^{\infty} : \sum_{k=1}^{\infty} |x_k| < \infty\}$ and input space $\mathscr{U} := PC(\mathbb{R}_+, \mathbb{R})$.

Let the dynamics of the k-th mode of Σ be given by

$$\dot{x}_k(t) = -\frac{1}{1+|u(t)|^k} x_k(t). \tag{21}$$

This system is 0-UGAS, sAG, AG with zero gain, UGS with zero gain, and LISS with zero gain, but it is not ISS (and hence not UAG). □

Remark 3: In finite-dimensions AG ∧ 0-ULS ⇔ ISS ⇔ UAG which shows that for ODE systems the difference between AG and UAG is relatively small. The previous example shows that in infinite dimensions this difference is considerably bigger, since even sAG systems with as strong additional properties as UGS and 0-UGAS are still not ISS, and thus not UAG.

IV. Main result and structure of the paper

As indicated in the title of this paper, our aim is to show what 'goes wrong' in the characterizations of ISS in infinite dimensions. We describe this in Figure 2: the black arrows shows the implications or equivalences which hold in infinite dimensions and the red arrows (with the negation sign) are the implications which do not hold, due to the counterexamples presented in this paper.

In Section VI we show the equivalence between the properties of ISS and UAG. This generalizes the well-known fact that the undisturbed system (1) is 0-UGAS iff it is uniformly globally attractive (0-UGATT).

Next, in Section VIII we introduce the new concept of strong input-to-state stability (sISS). For linear systems without inputs sISS is equivalent to strong stability of the associated semigroup T. We show that sISS is equivalent to sAG ∧ UGS.

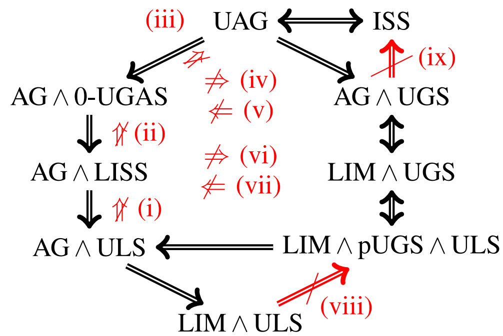

Fig. 2. Characterizations of ISS in infinite dimensions

V. Counterexamples

In this section we provide counterexamples which show that several implications which are true in finite dimensions fail to hold in infinite dimensions. These examples show that all the crossed out implication arrows (marked in red) in Figure 2 indeed correspond to implications which do not hold.

Already looking at linear systems without inputs we see that several implications do not hold.

Lemma 1: The following implications do not hold (in general) for infinite-dimensional systems (1), even for linear systems:

1) AG ∧ UGS does not imply ISS.
2) AG ∧ ULS does not imply AG ∧ LISS.

Proof: Consider the linear system $\dot{x} = Ax$, where A is the generator of a C_0-semigroup $T(\cdot)$. For this system it is observed in [19] that ISS is equivalent to 0-UGAS which is in turn equivalent to exponential stability of the semigroup $T(\cdot)$. By linearity and as there is no input these properties are also equivalent to LISS and thus also to AG ∧ LISS.

On the other hand, using linearity we have the equivalences AG ∧ UGS ⇔ AG ∧ ULS ⇔ 0-GATT ∧ 0-ULS ⇔ 0-GAS ⇔ strong stability of $T(\cdot)$ (for the last equivalence see Remark 1). Since strong stability of a semigroup does not imply exponential stability in general, the claim of the lemma follows. ■

Remark 4: The previous Lemma 1 allows us to dispose of several of the arrows in Figure 2. The arrows marked with (i) and (ix) are immediate from the statement. On the other hand from the proof we also see that AG ∧ UGS does not imply 0-UGAS, which yields the claim associated to the arrow (v). Finally, the claim of arrow (vii) follows as AG ∧ UGS is easily seen to be equivalent to LIM ∧ UGS and by the proof of the lemma this property does not imply AG ∧ LISS.

Example 2: According to Remark 1 for *linear* infinite-dimensional systems 0-GAS implies 0-UGS. We now show that this is false for nonlinear systems. Consider the nonlinear infinite-dimensional system Σ defined by

$$\Sigma : \begin{cases} \Sigma_k : \begin{cases} \dot{x}_k = -x_k + x_k^2 y_k - \frac{1}{k^2} x_k^3, \\ \dot{y}_k = -y_k. \end{cases} \\ k = 1, \ldots, \infty \end{cases} \tag{22}$$

with state space

$$X := l_2 = \left\{ (z_k)_{k=1}^{\infty} : \sum_{k=1}^{\infty} |z_k|^2 < \infty, \quad z_k = (x_k, y_k) \in \mathbb{R}^2 \right\}.$$

We show that Σ is 0-GAS and 0-UAS but is not 0-pUGS and thus is not 0-UGS.

First we argue that Σ is 0-UAS. Indeed, for $r < 1$ the Lyapunov function $V(z) = \|z\|_{l_2}^2 = \sum_{k=1}^{\infty} (x_k^2 + y_k^2)$ satisfies for all z_k with $|x_k| \le r$ and $|y_k| \le r$ ($k \in \mathbb{N}$) the estimate

$$\begin{aligned} \dot{V}(z) &= 2\sum_{k=1}^{\infty} (-x_k^2 + x_k^3 y_k - \frac{1}{k^2} x_k^4 - y_k^2) \\ &\le 2\sum_{k=1}^{\infty} (-x_k^2 + |x_k|^3 |y_k| - \frac{1}{k^2} x_k^4 - y_k^2) \\ &\le 2\sum_{k=1}^{\infty} ((r^2 - 1) x_k^2 - y_k^2) \\ &\le 2(r^2 - 1) V(z). \end{aligned}$$

We see that V is an exponential local Lyapunov function for the system (22) and thus (22) is locally uniformly asymptotically stable. Indeed it it not hard to show that the domain of attraction contains $\{z \in l_2 : |x_k| < r, |y_k| < r, \forall k\}$.

To show the global attractivity of Σ we first point out that every Σ_k is 0-GAS. This follows from the fact that any Σ_k is a cascade interconnection of an ISS x_k-system (with y_k as an input) and a globally asymptotically stable y_k-system, see [1].

Furthermore, for any $z(0) \in l_2$ there exists a finite $N > 0$ such that $|z_k(0)| \le \frac{1}{2}$ for all $k \ge N$. Decompose the norm of $z(t)$ as follows

$$\|z(t)\|_{l_2} = \sum_{k=1}^{N-1} |z_k(t)|^2 + \sum_{k=N}^{\infty} |z_k(t)|^2.$$

According to the previous arguments, $\sum_{k=1}^{N-1} |z_k(t)|^2 \to 0$ as $t \to 0$ since all Σ_k are GAS for $k = 1, \ldots, N-1$. Also $\sum_{k=N}^{\infty} |z_k(t)|^2$ decays monotonically and exponentially to 0 as $t \to \infty$. Overall, $\|z(t)\|_{l_2} \to 0$ as $t \to \infty$ which shows that Σ is 0-GAS and 0-UAS.

Finally we show that Σ is not 0-pUGS. To prove this, it is enough to show that there exists an $r > 0$ so that for any $M > 0$ there exist $z \in l_2$ and $t \ge 0$ so that $\|z\|_{l_2} = r$ and $\|\phi(t, z, 0)\|_X > M$.

Let us consider Σ_k. For $y_k \ge 1$ and for $x_k \in [0, k]$ it holds that

$$\dot{x}_k \ge -2x_k + x_k^2. \tag{23}$$

Pick an initial state $x_k(0) = c > 0$ (which is independent of k) so that the solution of $\dot{x}_k = -2x_k + x_k^2$ blows up to infinity in time $t^* = 1$. Now pick $y_k(0) = e$ (Euler's constant) for all $k = 1, \ldots, \infty$. For this initial condition we obtain $y_k(t) = e^{1-t} \ge 1$ for $t \in [0, 1]$. And consequently for $z_k(0) = (c, e)^T$ there exists a time $\tau_k \in (0, 1)$ such that $x_k(\tau_k) = k$ for the solution of Σ_k.

Now consider an initial state $z(0)$ for Σ, where $z_k(0) = (c, e)^T$ and $z_j(0) = (0, 0)^T$ for $j \ne k$. For this initial state we have that $\|z(t)\|_{l_2} = |z_k(t)|$ and

$$\sup_{t \ge 0} \|z(t)\|_{l_2} = \sup_{t \ge 0} |z_k(t)| \ge |x_k(\tau_k)| \ge k.$$

As $k \in \mathbb{N}$ was arbitrary, this shows that the system Σ is not 0-pUGS. ■

Corollary 3: The following implications do not hold (in general) for an infinite-dimensional nonlinear system of the form (1):

(i) AG $\wedge$ ULS $\not\Rightarrow$ AG $\wedge$ UGS.
(ii) LIM $\wedge$ ULS $\not\Rightarrow$ LIM $\wedge$ UGS.
(iii) AG $\wedge$ LISS $\not\Rightarrow$ AG $\wedge$ 0-UGAS.

Remark 5: We note that item (iii) of Corollary 3 immediately justifies the arrow marked by (ii) in Figure 2.

Proof: (i) For a nonlinear system without inputs it holds trivially that

$$\text{AG} \wedge \text{ULS} \Leftrightarrow \text{0-GATT} \wedge \text{0-ULS} \Leftrightarrow \text{0-GAS}.$$

Again by direct inspection of the definitions we have

$$\text{AG} \wedge \text{UGS} \Leftrightarrow \text{0-GATT} \wedge \text{0-UGS} \Leftrightarrow \text{0-GAS} \wedge \text{0-UGS}.$$

Now Example 2 presents a system that is 0-GAS but is not 0-UGS. This shows the claim.

The items (ii) and (iii) are proved by a similar argument. Again for a system without inputs we have the equivalences

1) LIM $\wedge$ ULS $\Leftrightarrow$ 0-LIM $\wedge$ 0-ULS $\Leftrightarrow$ 0-GAS,
2) AG $\wedge$ LISS $\Leftrightarrow$ 0-GATT $\wedge$ 0-UAS $\Leftrightarrow$ 0-GAS $\wedge$ 0-UAS,

where the last equivalence of 1) is shown in Lemma 6. On the other hand we have

3) LIM $\wedge$ UGS $\Leftrightarrow$ 0-LIM $\wedge$ 0-UGS $\Leftrightarrow$ 0-GAS $\wedge$ 0-UGS,
4) AG $\wedge$ 0-UGAS $\Leftrightarrow$ 0-GATT $\wedge$ 0-UGAS $\Leftrightarrow$ 0-UGAS.

The equivalences 1) and 3) together with Example 2 show (ii). For (iii) we note that the last item of 4) implies 0-UGS. ■

Remark 6: In [8] it was proved that a finite-dimensional forward complete system which is LIM is necessarily pUGS, see [8, Lemma I.4, p.1915, for the proof see p.1920]. Example 2 shows that this is no longer true in infinite dimensions. Indeed, the system in Example 2 is forward-complete and is 0-GAS. In particular, it is ULS and LIM, but the system is not practically globally stable. This justifies the claim associated with arrow (viii) in Figure 2.

Example 3: In this modification of Example 2 it is demonstrated that 0-UGAS $\wedge$ AG $\wedge$ LISS does not imply UGS. Let Σ be defined by

$$\Sigma : \begin{cases} \Sigma_k : \begin{cases} \dot{x}_k = -x_k + x_k^2 y_k |u| - \frac{1}{k^2} x_k^3, \\ \dot{y}_k = -y_k. \end{cases} \\ k = 1, \ldots, \infty \end{cases}$$

And let a state space of Σ be

$$X := l_2 = \{(z_k)_{k=1}^{\infty} : \sum_{k=1}^{\infty} |z_k|^2 < \infty\}, \quad z_k = (x_k, y_k) \in \mathbb{R}^2.$$

and its input space be $\mathcal{U} := PC(\mathbb{R}_+, \mathbb{R})$.

Evidently, this system is 0-UGAS. Also it is clear that Σ is not UGS, since for $u \equiv 1$ we obtain exactly the system from Example 2, which is not practically globally stable. The proof that this system is LISS and AG with zero gain mimics the argument we exploited to show 0-GATT of Example 2 and thus we skip it.

In view of Remark 4 we note the following immediate consequence.

Corollary 4: AG $\wedge$ UGS and AG $\wedge$ 0-UGAS are different notions, in sense that neither combination implies the other one.

Remark 7: With Corollary 4 we have treated the arrows marked (iv) and (v) in Figure 2. Furthermore, the system in Example 3 is AG and 0-UGAS but is not ISS, as it is not even UGS. In view of the next section where we show that ISS is equivalent to UAG, this disposes of the arrow (iii). Finally, the same system is AG and LISS, but not UGS, which shows arrow (vi).

VI. ISS $\Leftrightarrow$ UAG

In this section we prove that ISS is equivalent to UAG. We start with a simple lemma:

Lemma 2: If (1) is ISS, then it is UAG.

Proof: Let (1) be ISS with the corresponding $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$. Take arbitrary $\varepsilon, \delta > 0$. Define $\tau_a = \tau_a(\varepsilon, \delta)$ as the solution of the equation $\beta(\delta, \tau_a) = \varepsilon$ (if it exists, then it is unique, because of monotonicity of β in the second argument, if it does not exist, we set $\tau_a(\varepsilon, \delta) := 0$). Then for all $t \geq \tau_a$, all $x \in X$ with $\|x\|_X \leq \delta$ and all $u \in \mathcal{U}$ we have

$$\begin{aligned} \|\phi(t,x,u)\|_X &\leq \beta(\|x\|_X, t) + \gamma(\|u\|_{\mathcal{U}}) \\ &\leq \beta(\|x\|_X, \tau_a) + \gamma(\|u\|_{\mathcal{U}}) \\ &\leq \varepsilon + \gamma(\|u\|_{\mathcal{U}}), \end{aligned}$$

and the estimate (15) holds. ■

To prove the converse claim, we need an auxiliary lemma.

Lemma 3: System (1) is ULS if and only if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\|x\|_X \leq \delta \ \wedge \ \|u\|_{\mathcal{U}} \leq \delta \ \Rightarrow \ \|\phi(t,x,u)\|_X \leq \varepsilon \ \forall t \geq 0. \quad (24)$$

This proof is a standard reformulation of ε-δ formulations of stability in terms of $\mathcal{K}$-functions. We include it for the convenience of the reader.

Proof: "$\Rightarrow$" Let (1) be ULS. Let $\sigma, \gamma \in \mathcal{K}_\infty$ and $r > 0$ be such that (11) holds for these functions and the neighborhood specified by r. Let $\varepsilon > 0$ be arbitrary and choose

$$\delta = \delta(\varepsilon) := \min\left\{\sigma^{-1}\left(\frac{\varepsilon}{2}\right), \gamma^{-1}\left(\frac{\varepsilon}{2}\right), r\right\}.$$

With this choice (24) follows from (11).

"$\Leftarrow$" Let (24) hold. For $\varepsilon \geq 0$ define

$$\delta(\varepsilon) := \sup\{s \geq 0 : \|x\|_X \leq s \ \wedge \ \|u\|_{\mathcal{U}} \leq s \Rightarrow \sup_{t\geq 0}\|\phi(t,x,u)\|_X \leq \varepsilon\}.$$

Clearly (24) implies that $\delta(\cdot)$ is well defined, increasing and continuous in 0. Let $\hat{\delta} \in \mathcal{K}$ be any function with $\hat{\delta} \leq \delta$ and set $r := \sup_{s\geq 0} \hat{\delta}(s) \in \mathbb{R}_+ \cup \infty$. Define $\gamma = \hat{\delta}^{-1} : [0, r) \to \mathbb{R}$. Then for $\|x\|_X < r$ and $\|u\|_{\mathcal{U}} < r$ we have

$$\|\phi(t,x,u)\|_X \leq \gamma(\max\{\|x\|_X, \|u\|_{\mathcal{U}}\}) \leq \gamma(\|x\|_X) + \gamma(\|u\|_{\mathcal{U}}),$$

which shows ULS. ■

Lemma 4: If system (1) is UAG, then it is ULS.

Proof: We will show that (24) holds so that the claim follows from the previous lemma. Let τ_a and γ be the functions given by (15). Let $\varepsilon > 0$ and $\tau := \tau_a(\varepsilon/2, 1)$. Pick any $\delta_1 > 0$ so that $\gamma(\delta_1) < \varepsilon/2$. Then for all $x \in X$ with $\|x\|_X \leq 1$ and all $u \in \mathcal{U}$ with $\|u\|_{\mathcal{U}} \leq \delta_1$ we have

$$\sup_{t\geq\tau} \|\phi(t,x,u)\|_X \leq \frac{\varepsilon}{2} + \gamma(\|u\|_{\mathcal{U}}) < \varepsilon. \quad (25)$$

Due to Assumption 2, there is some $\delta_2 = \delta_2(\varepsilon, \tau) > 0$ so that

$$\|\eta\|_X \leq \delta_2 \ \wedge \ \|u\|_{\mathcal{U}} \leq \delta_2 \quad \Rightarrow \quad \sup_{t\in[0,\tau]} \|\phi(t,\eta,u)\|_X \leq \varepsilon.$$

Together with (25), this proves (24) with $\delta := \min\{1, \delta_1, \delta_2\}$. ■

Lemma 5: If (1) is UAG, then it is UGS.

Proof: Assume (1) is UAG and let $\varepsilon := 1$ in the definition of UAG. Then for any $\delta > 0$ there exists a $\tau := \tau_a(1, \delta)$ such that

$$\|x\|_X < \delta \ \wedge \ \|u\|_{\mathcal{U}} < \delta \ \wedge \ t \geq \tau_a \ \Rightarrow \ \|\phi(t,x,u)\|_X < 1 + \gamma(\delta).$$

Now Assumption 3 implies that for this δ there exists a $K(\delta)$ such that for all $t \leq \tau_a$, all $x \in X$ with $\|x\|_X < \delta$, and all $u \in \mathcal{U}$ with $\|u\|_{\mathcal{U}} < \delta$ we have

$$\|\phi(t,x,u)\|_X \leq K(\delta).$$

Without loss of generality we can assume that K is an increasing function of δ.

Then for all $\delta > 0$, all $t \geq 0$ and for all $\|x\|_X < \delta$ and $\|u\|_{\mathcal{U}} < \delta$ we have

$$\|\phi(t,x,u)\|_X \leq 1 + K(\delta) + \gamma(\delta).$$

Since (1) is ULS, there exist $\sigma \in \mathcal{K}_\infty$ and $r > 0$ such that for all $x \in X$ and all $u \in \mathcal{U}$ with $\|x\|_X \leq r$ and $\|u\|_{\mathcal{U}} \leq r$ it holds that

$$\|\phi(t,x,u)\|_X \leq \sigma(\|x\|_X) + \sigma(\|u\|_{\mathcal{U}}), \quad \forall t \geq 0.$$

Now define

$$w(s) := \begin{cases} \sigma(s) & , \text{if } s \in [0, r] \\ 1 + K(s) & , \text{otherwise} \end{cases}$$

Then $w(s) = 0$ whenever $s = 0$ and $w(s) > 0$ when $s > 0$.

Pick any $\kappa \in \mathcal{K}_\infty$ such that $w(s) \leq \kappa(s)$ for all $s \geq 0$. Then

$$\|\phi(t,x,u)\|_X \leq \kappa(\|x\|_X) + \kappa(\|u\|_{\mathcal{U}})$$

for all $x \in X$, $u \in \mathcal{U}$ and all $t \geq 0$, which shows UGS of (1). ■

The final result of this section is:

Proposition 5: (1) is UAG if and only if (1) is ISS.

Proof: '$\Leftarrow$' This is proved in Lemma 2.

'⇒' Fix arbitrary $\delta \in \mathbb{R}_+$. We are going to construct a function $\beta \in \mathcal{KL}$ so that (16) holds.

From global stability it follows that for all $t \geq 0$, all $x \in X$ with $\|x\|_X \leq \delta$ and $\forall u \in \mathcal{U}$ we have

$$\|\phi(t,x,u)\|_X \leq \sigma(\delta) + \gamma(\|u\|_{\mathcal{U}}).$$

Define $\varepsilon_n := \frac{1}{2^n}\sigma(\delta)$, for $n \in \mathbb{N}$. The UAG property implies that there exists a sequence of times $\tau_n := T(\varepsilon_n, \delta)$ that we may without loss of generality assume to be strictly increasing, such that for all $x \in X$ with $\|x\|_X \leq \delta$ and all $u \in \mathcal{U}$

$$\|\phi(t,x,u)\|_X \leq \varepsilon_n + \gamma(\|u\|_{\mathcal{U}}), \quad \forall t \geq \tau_n.$$

Define $\omega(\delta, \tau_n) := \varepsilon_{n-1}$, for $n \in \mathbb{N}$, $n \neq 0$.

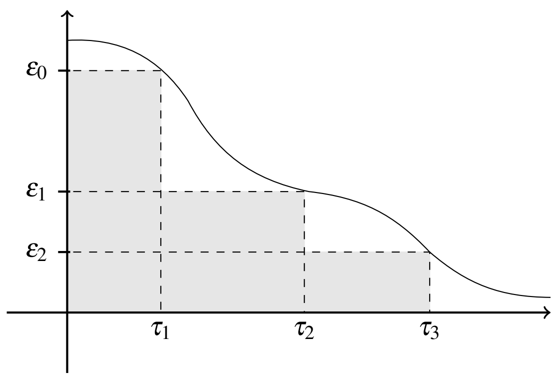

Fig. 3. Construction of the function ω

Now extend the function $\omega(\delta, \cdot)$ for $t \in \mathbb{R}_+ \backslash \{\tau_n, n \in \mathbb{N}\}$ so that $\omega(\delta, \cdot) \in \mathcal{L}$. All such functions satisfy the estimate (16), because for all $t \in (\tau_n, \tau_{n+1})$ it holds that $\|\phi(t,x,u)\|_X \leq \varepsilon_n + \gamma(\|u\|_{\mathcal{U}}) < \omega(\delta,t) + \gamma(\|u\|_{\mathcal{U}})$. Doing this for all $\delta \in \mathbb{R}_+$ we obtain the definition of the function ω.

Now choose $\beta(r,t) = \sup_{0 \leq s \leq r} \omega(s,t) \geq \omega(r,t)$. From this definition it follows that β is continuous and $\beta(\cdot, t) \in \mathcal{K}_\infty$. Also $\beta(r, \cdot) \in \mathcal{L}$, because $\omega(r, \cdot) \in \mathcal{L}$. Thus, $\beta \in \mathcal{KL}$ and the estimate (16) is satisfied with such a β. ∎

VII. Systems without inputs

In this section we classify the stability notions for systems without inputs. We feel that this simplified picture can be helpful in understanding the general case and at the same time it is interesting in its own right.

We start with a lemma.

Lemma 6: (1) is 0-LIM and 0-ULS if and only if (1) is 0-GAS.

Proof: It is clear that 0-GAS implies 0-LIM and 0-ULS. So we only prove the converse direction.

Pick any $x \in X$. Since (1) is 0-ULS, for any $\varepsilon > 0$ there exist $\delta = \delta(\varepsilon) > 0$ such that $\|x\|_X \leq \delta$ implies $\|\phi(t,x,0)\|_X \leq \varepsilon$ for all $t \geq 0$. Pick any such ε, and denote it by ε_1 with the corresponding δ_1. Since (1) is 0-LIM, for this $\delta_1 > 0$ there exists a $T_1 = T_1(x) > 0$ such that $\|\phi(T_1,x,0)\|_X \leq \delta_1$. Due to the semigroup property, $\phi(t+T_1,x,0) = \phi(t, \phi(T_1,x,0),0)$ and consequently $\|\phi(t+T_1,x,0)\|_X \leq \varepsilon_1$ for all $t \geq 0$.

Pick a sequence $\{\varepsilon_i\}_{i=1}^\infty$ with $\varepsilon_i \to 0$ as $i \to \infty$. According to the above argument, there exists a sequence of times $T_i = T_i(x)$ such that $\|\phi(t,x,0)\|_X \leq \varepsilon_i$ for all $t \geq T_i$, and thus $\|\phi(t,x,0)\|_X \to 0$ as $t \to \infty$. This shows that (1) is 0-GATT, and since we assumed that (1) is 0-ULS, (1) is also 0-GAS. ∎

Now we are in a position to state the main result of this section.

Proposition 6: For the system (1) without inputs the relations depicted in Figure 4 hold.

Proof: The equivalences on the uniform level follow directly from the equivalence between UAG and ISS. 0-GAS is equivalent to 0-GATT ∧ 0-ULS according to the definition of 0-GAS, and it is equivalent to 0-LIM ∧ 0-ULS according to Lemma 6.

The implications (2) follow since 0-UAS ⇔ 0-UGAS and 0-ULS ⇔ 0-UGS for linear systems. The implications (1) are well-known.

The observation that 0-UAS ∧ 0-GATT is not implied by and does not imply 0-GAS ∧ 0-UGS follows from Example 2 and since strong stability of strongly continuous semigroups is weaker than the exponential stability. ∎

We see that there are two main groups of stability notions: uniform and nonuniform ones. Between these two levels there are two other combinations: 0-UAS ∧ 0-GATT and 0-GATT ∧ 0-UGS ⇔ 0-GAS ∧ 0-UGS. These notions neither belong to the fully uniform nor to the fully nonuniform level. Rather, they possess different types of uniformity: 0-UAS ∧ 0-GATT has uniform attraction times near the origin and the pair 0-GATT ∧ 0-UGS has the uniform global bound for solutions of (1) with $u \equiv 0$.

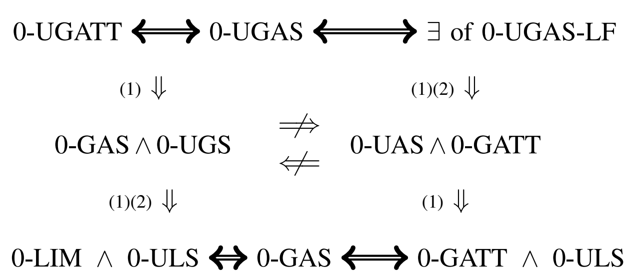

Fig. 4. Characterizations of 0-UGAS. Implications marked by (1) resp. (2) become equivalences for (1) ODE systems, see e.g. [20, Proposition 2.5] and (2) linear systems (as a consequence of the Banach-Steinhaus theorem).

VIII. Strong ISS

As has been shown in Lemma 1, the combination of the properties AG and UGS is weaker than ISS. Therefore it is natural to ask for a weaker property than ISS which would be equivalent to the combination AG ∧ UGS. In this section we prove a partial result of this kind.

Definition 7: System (1) is called *strongly input-to-state stable (sISS)*, if there exist $\gamma \in \mathcal{K}$, $\sigma \in \mathcal{K}_\infty$ and $\beta : X \times \mathbb{R}_+ \to \mathbb{R}_+$, so that

1) $\beta(x, \cdot) \in \mathcal{L}$ for all $x \in X$, $x \neq 0$

2) $\beta(x,t) \leq \sigma(\|x\|_X)$ for all $x \in X$ and all $t \geq 0$
3) for all $x \in X$, all $u \in \mathscr{U}$ and all $t \geq 0$ it holds that

$$\|\phi(t,x,u)\|_X \leq \beta(x,t) + \gamma(\|u\|_{\mathscr{U}}). \tag{26}$$

Remark 8: Clearly, ISS implies sISS, but not vice versa.

By Proposition 5 we know that ISS is equivalent to UAG. Next we prove the corresponding characterization for sISS.

Theorem 7: Consider system (1) and let Assumption 1 hold. The following statements are equivalent.

(i) System (1) is sISS.
(ii) System (1) is sAG and UGS.

Proof: Let (1) be sISS with corresponding $\beta : X \times \mathbb{R}_+ \to \mathbb{R}_+$ and $\gamma \in \mathscr{K}_\infty$. Then (1) is automatically UGS.

Fix any $x \in X$ and any $\varepsilon > 0$. Define $\tau = \tau(\varepsilon, x)$ as the solution of the equation $\beta(x,\tau) = \varepsilon$ (if this solution exists, then it is unique, because of the monotonicity of β in the second argument, if it does not exist, we set $\tau(\varepsilon,x) = 0$). Then for all $t \geq \tau$ and all $u \in \mathscr{U}$

$$\begin{aligned} \|\phi(t,x,u)\|_X &\leq \beta(x,t) + \gamma(\|u\|_{\mathscr{U}}) \\ &\leq \beta(x,\tau) + \gamma(\|u\|_{\mathscr{U}}) \\ &\leq \varepsilon + \gamma(\|u\|_{\mathscr{U}}), \end{aligned}$$

and the estimate (14) holds. Thus, sISS implies sAG.

Conversely, assume that (1) is UGS and sAG. Fix an arbitrary $\delta \in \mathbb{R}_+$ and any $x \in X$ with $\|x\|_X \leq \delta$. We are going to construct $\beta : X \times \mathbb{R}_+ \to \mathbb{R}_+$ with the properties as in Definition 7, so that (26) holds.

Uniform global stability of (1) implies that there exist $\sigma, \gamma \in \mathscr{K}_\infty$ so that for all $t \geq 0$ and for all $u \in \mathscr{U}$ it holds that

$$\|\phi(t,x,u)\|_X \leq \sigma(\delta) + \gamma(\|u\|_{\mathscr{U}}).$$

Define $\varepsilon_n := \frac{1}{2^n}\sigma(\delta)$, for all $n \in \mathbb{N}$. Due to sAG there exists a sequence of times $\tau_n := \tau(\varepsilon_n, x)$, which we assume without loss of generality to be strictly increasing in n, such that

$$\|\phi(t,x,u)\|_X \leq \varepsilon_n + \gamma(\|u\|_{\mathscr{U}}) \quad \forall u \in \mathscr{U}, \ \forall t \geq \tau_n.$$

Define $\beta(x,\tau_n) := \varepsilon_{n-1}$, for $n \in \mathbb{N}$, $n \neq 0$.

Now extend the function $\beta(x,\cdot)$ for $t \in \mathbb{R}_+ \backslash \{\tau_n, n \in \mathbb{N}\}$ so that $\beta(x,\cdot) \in \mathscr{L}$ and $\beta(x,t) \leq 2\sigma(\|x\|_X)$ for all $t \geq 0$ (this can be done by choosing the values of $\beta(x,t)$ sufficiently small for $t \in [0,\tau_1)$).

The function β satisfies the estimate (26), because for all $t \in (\tau_n, \tau_{n+1})$ it holds that $\|\phi(t,x,u)\|_X \leq \varepsilon_n + \gamma(\|u\|_{\mathscr{U}}) < \beta(x,t) + \gamma(\|u\|_{\mathscr{U}})$. Performing this procedure for all $x \in X$ we obtain the definition of the function β. This shows sISS of (1). ■

IX. Conclusions

We have demonstrated by means of counterexamples that many conditions which are equivalent to ISS in finite dimensions, no longer provide equivalent characterizations in the infinite-dimensional setting. As a positive result, it has been shown that one of the finite-dimensional equivalences, namely the equivalence between ISS and the uniform asymptotic gain property, still holds in infinite dimensions. Finally, we have introduced a strong ISS property which is equivalent to ISS in the ODE case, but which is strictly weaker than ISS for differential equations in Banach spaces. Strong ISS has been characterized as as the strong asymptotic gain property together with global stability.

References

[1] E. D. Sontag, “Smooth stabilization implies coprime factorization,” *IEEE Transactions on Automatic Control*, vol. 34, no. 4, pp. 435–443, 1989.
[2] R. A. Freeman and P. V. Kokotovic, *Robust Nonlinear Control Design: State-Space and Lyapunov Techniques*. Boston, MA: Birkhäuser, 2008.
[3] M. Arcak and P. Kokotović, “Nonlinear observers: a circle criterion design and robustness analysis,” *Automatica*, vol. 37, no. 12, pp. 1923–1930, 2001.
[4] Z.-P. Jiang, A. R. Teel, and L. Praly, “Small-gain theorem for ISS systems and applications,” *Mathematics of Control, Signals, and Systems*, vol. 7, no. 2, pp. 95–120, 1994.
[5] S. Dashkovskiy, B. S. Rüffer, and F. R. Wirth, “An ISS small gain theorem for general networks,” *Mathematics of Control, Signals, and Systems*, vol. 19, no. 2, pp. 93–122, 2007.
[6] S. Dashkovskiy, B. Rüffer, and F. Wirth, “Small gain theorems for large scale systems and construction of ISS Lyapunov functions,” *SIAM Journal on Control and Optimization*, vol. 48, no. 6, pp. 4089–4118, 2010.
[7] E. D. Sontag and Y. Wang, “On characterizations of the input-to-state stability property,” *Systems & Control Letters*, vol. 24, no. 5, pp. 351–359, 1995.
[8] ——, “New characterizations of input-to-state stability,” *IEEE Transactions on Automatic Control*, vol. 41, no. 9, pp. 1283–1294, 1996.
[9] A. R. Teel, “Connections between Razumikhin-type theorems and the ISS nonlinear small-gain theorem,” *IEEE Transactions on Automatic Control*, vol. 43, no. 7, pp. 960–964, 1998.
[10] S. Dashkovskiy, M. Kosmykov, A. Mironchenko, and L. Naujok, “Stability of interconnected impulsive systems with and without time delays, using Lyapunov methods,” *Nonlinear Analysis: Hybrid Systems*, vol. 6, no. 3, pp. 899–915, 2012.
[11] S. Dashkovskiy and M. Kosmykov, “Input-to-state stability of interconnected hybrid systems,” *Automatica*, vol. 49, no. 4, pp. 1068–1074, 2013.
[12] S. Dashkovskiy and A. Mironchenko, “Input-to-state stability of infinite-dimensional control systems,” *Mathematics of Control, Signals, and Systems*, vol. 25, no. 1, pp. 1–35, 2013.
[13] A. Mironchenko and H. Ito, “Integral input-to-state stability of bilinear infinite-dimensional systems,” in *Proc. of the 53th IEEE Conference on Decision and Control*, 2014, pp. 3155–3160.
[14] F. Mazenc and C. Prieur, “Strict Lyapunov functions for semilinear parabolic partial differential equations,” *Mathematical Control and Related Fields*, vol. 1, no. 2, pp. 231–250, 2011.
[15] B. Jayawardhana, H. Logemann, and E. P. Ryan, “Infinite-dimensional feedback systems: the circle criterion and input-to-state stability,” *Communications in Information and Systems*, vol. 8, no. 4, pp. 413–444, 2008.
[16] I. Karafyllis and M. Krstic, “ISS With Respect to Boundary Disturbances for 1-D Parabolic PDEs,” *Submitted to IEEE Transactions on Automatic Control. arXiv preprint arXiv:1505.06457*, 2015.
[17] A. Mironchenko, “Local input-to-state stability: Characterizations and counterexamples,” *Systems & Control Letters*, vol. 87, pp. 23–28, 2016.
[18] T. Cazenave and A. Haraux, *An Introduction To Semilinear Evolution Equations*. New York: Oxford University Press, 1998.
[19] A. Mironchenko and F. Wirth, “A note on input-to-state stability of linear and bilinear infinite-dimensional systems,” in *Proc. 54th IEEE Conference on Decision and Control, CDC2015*, 2015.
[20] Y. Lin, E. D. Sontag, and Y. Wang, “A Smooth Converse Lyapunov Theorem for Robust Stability,” *SIAM Journal on Control and Optimization*, vol. 34, no. 1, pp. 124–160, 1996.