Global converse Lyapunov theorems for infinite-dimensional systems

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Abstract: We show that existence of a non-coercive Lyapunov function is sufficient for uniform global asymptotic stability (UGAS) of infinite-dimensional systems with external disturbances provided an additional mild assumption is fulfilled. For UGAS infinite-dimensional systems with external disturbances we derive a novel 'integral' construction of non-coercive Lipschitz continuous Lyapunov functions. Finally, converse Lyapunov theorems are used in order to prove Lyapunov characterizations of input-to-state stability of infinite-dimensional systems.

Keywords: nonlinear control systems, infinite-dimensional systems, input-to-state stability, Lyapunov methods

1. INTRODUCTION

We study converse Lyapunov theorems for robust stability of nonlinear infinite-dimensional systems with external inputs and we are interested both in uniform global asymptotic stability (UGAS) as well as input-to-state stability (ISS). Uniform global asymptotic stability is one of the central concepts in dynamical systems theory and dates back to the original works by Lyapunov. ISS was first introduced in the seminal paper Sontag (1989) and has since become an indispensable tool in robust nonlinear control with applications to robust stabilization of nonlinear systems Freeman and Kokotovic (2008), design of nonlinear observers Andrieu and Praly (2009), analysis of large-scale networks Jiang et al. (1994); Dashkovskiy et al. (2010) and numerous other branches of nonlinear control Kokotović and Arcak (2001). The success of ISS theory of ordinary differential equations and the need of proper tools for robust stability analysis of partial differential equations motivated the development of ISS theory in infinite-dimensional setting Dashkovskiy and Mironchenko (2013); Mironchenko and Ito (2016); Mazenc and Prieur (2011); Jayawardhana et al. (2008); Karafyllis and Krstic (2015); Karafyllis and Jiang (2011a); Mironchenko (2016).

Converse Lyapunov theorems for UGAS systems have many applications to other problems in stability theory and they have been proved in various contexts, Henry (1981); Deimling (1992); Karafyllis and Jiang (2011b); Clarke et al. (1998); Kellett (2015); Schönlein (2015); Teel and Praly (2000). For us the main motivation were the papers Lin et al. (1996); Sontag and Wang (1995), in which converse UGAS Lyapunov theorems have been applied in order to prove, in the case of ordinary differential equations (ODEs), the equivalence between ISS and the existence of a smooth ISS Lyapunov function. This result along with further restatements of ISS in terms of other stability notions Sontag and Wang (1995, 1996) and small-gain theorems Jiang et al. (1994); Dashkovskiy et al. (2010) is at the heart of ISS theory of systems of ordinary differential equations.

The paper can be divided into two parts. First, we study Lyapunov criteria for UGAS. In Section 2.1 we prove that existence of a *non-coercive* (positive definite, but not necessarily uniformly bounded from below) Lyapunov function ensures UGAS of a nonlinear infinite-dimensional system provided the system cannot grow arbitrarily fast on finite time-intervals. This result is obtained using uniform Barbalat-like estimates. Next, in Section 2.2 we give a novel "integral" construction of a non-coercive Lipschitz continuous Lyapunov function for a UGAS system with the Lipschitz continuous flow map. This is achieved using ideas from (Karafyllis and Jiang, 2011b, Section 3.4), which in turn are based on earlier local converse Lyapunov theorems, see e.g. (Yoshizawa, 1966, Theorem 19.3), (Henry, 1981, Theorem 4.2.1) and on using Sontags' $\mathcal{KL}$-Lemma (Sontag, 1998, Proposition 7).

The second part of the paper is devoted to applications of converse Lyapunov theorems to the characterization of ISS for a class of infinite-dimensional systems. The main two lines of research within infinite-dimensional ISS theory are the development of a general ISS theory of evolution equations in Banach spaces and the application of ISS ideas to the stability analysis and control of specific important partial differential equations (PDEs). The results of the first line of research include small-gain theorems for interconnected infinite-dimensional systems and their applications to nonlinear interconnected parabolic PDEs over Sobolev spaces Dashkovskiy and Mironchenko (2013); Mironchenko and Ito (2015), characterizations of local and global ISS properties Mironchenko (2016); Mironchenko and Wirth (2016) etc. In the second line of research, for instance, constructions of ISS Lyapunov functions for nonlinear parabolic systems over L_p-spaces Mazenc and Prieur

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(2011), for linear time-variant systems of conservation laws Prieur and Mazenc (2012), for nonlinear Kuramoto-Sivashinsky equation Ahmadi et al. (2016) have been obtained. Non-Lyapunov methods were successfully applied to linear parabolic systems with boundary disturbances in Karafyllis and Krstic (2015).

In Section 3 we follow the first line of research and prove that ISS is equivalent to existence of a coercive Lipschitz continuous ISS Lyapunov function for infinite-dimensional systems over Banach spaces. To this end we exploit the method from Sontag and Wang (1995) and converse Lyapunov theorems for global asymptotic stability of systems with disturbances from Karafyllis and Jiang (2011b). We show that ISS is equivalent to the existence of a globally stabilizing feedback which is robust with respect to multiplicative actuator disturbances of bounded magnitude (weak uniform robust stability, WURS).

We conclude the paper in Section 4.

Next we introduce some notation. Let $\mathbb{R}_+ := [0, \infty)$.

Definition 1. For the formulation of stability properties the following classes of functions are useful:

$$\begin{aligned}
\mathcal{P} &:= \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous}, \gamma(r) = 0 \Leftrightarrow r = 0\} \\
\mathcal{K} &:= \{\gamma \in \mathcal{P} \mid \gamma \text{ is strictly increasing}\} \\
\mathcal{K}_\infty &:= \{\gamma \in \mathcal{K} \mid \gamma \text{ is unbounded}\} \\
\mathcal{L} &:= \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous and strictly decreasing with } \lim_{t\to\infty} \gamma(t) = 0\} \\
\mathcal{KL} &:= \{\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \mid \beta \text{ is continuous}, \beta(\cdot, t) \in \mathcal{K},\ \beta(r, \cdot) \in \mathcal{L},\ \forall t \geq 0,\ \forall r > 0\}
\end{aligned}$$

2. LYAPUNOV THEORY FOR SYSTEMS WITH TIME-VARYING PARAMETERS

In this paper we consider abstract axiomatically defined time-invariant and forward-complete control systems.

Definition 2. Consider the triple $\Sigma = (X, \mathcal{D}, \phi)$, consisting of

- A normed linear space $(X, \|\cdot\|_X)$, called the state space, endowed with the norm $\|\cdot\|_X$.
- A set of disturbance values D, which is a nonempty subset of a normed linear space S_d.
- A space of disturbances $\mathcal{D} \subset \{f : \mathbb{R}_+ \to D\}$ satisfying the following two axioms.

  The axiom of concatenation is defined by the requirement that for all $d_1, d_2 \in \mathcal{D}$ and for all $t > 0$ the concatenation of d_1 and d_2 at time t
  $$d(\tau) := \begin{cases} d_1(\tau), & \text{if } \tau \in [0, t], \\ d_2(\tau - t), & \text{otherwise}, \end{cases} \tag{1}$$
  belongs to $\mathcal{D}$.

  The axiom of shift invariance states that for all $d \in \mathcal{D}$ and all $\tau \geq 0$ the time shift $d(\cdot + \tau)$ is in $\mathcal{D}$.
- A transition map $\phi : \mathbb{R}_+ \times X \times \mathcal{D} \to X$.

The triple Σ is called a (forward-complete) control system, if the following properties hold:

(Σ1) Forward-completeness: for every $(x, d) \in X \times \mathcal{D}$ and for all $t \geq 0$ the value $\phi(t, x, d)$ is defined and finite.

(Σ2) The identity property: for every $(x, d) \in X \times \mathcal{D}$ it holds that $\phi(0, x, d) = x$.

(Σ3) Causality: for every $(t, x, d) \in \mathbb{R}_+ \times X \times \mathcal{D}$, for every $\tilde{d} \in \mathcal{D}$, such that $d(s) = \tilde{d}(s)$, $s \in [0, t]$ it holds that $\phi(t, x, d) = \phi(t, x, \tilde{d})$.

(Σ4) Continuity: for each $(x, d) \in X \times \mathcal{D}$ the map $t \mapsto \phi(t, x, d)$ is continuous.

(Σ5) The cocycle property: for all $t, h \geq 0$, for all $x \in X$, $d \in \mathcal{D}$ we have $\phi(h, \phi(t, x, d), d(t + \cdot)) = \phi(t + h, x, d)$.

Here $\phi(t, x, d)$ denotes the state of a system at the moment $t \in \mathbb{R}_+$ corresponding to initial condition $x \in X$ and the disturbance $d \in \mathcal{D}$.

Definition 3. We say that the flow of $\Sigma = (X, \mathcal{D}, \phi)$ is Lipschitz continuous on compact intervals, if for any $\tau > 0$ and any $R > 0$ there exists $L > 0$ so that for any $x, y \in X$: $\|x\|_X \leq R$, $\|y\|_X \leq R$, for all $t \in [0, \tau]$ and for all $d \in \mathcal{D}$ it holds that
$$\|\phi(t, x, d) - \phi(t, y, d)\|_X \leq L\|x - y\|_X. \tag{2}$$

We exploit the following stronger version of forward completeness:

Definition 4. The system $\Sigma = (X, \mathcal{D}, \phi)$ is called robustly forward complete (RFC) if for any $C > 0$ and any $\tau > 0$ it holds that
$$\sup_{\|x\|_X \leq C,\ d \in \mathcal{D},\ t \in [0, \tau]} \|\phi(t, x, d)\|_X < \infty.$$

The condition of robust forward completeness is satisfied by large classes of infinite-dimensional systems. In Definition 14 below we introduce a sufficient condition.

Definition 5. We call $0 \in X$ an equilibrium point of the control system $\Sigma = (X, \mathcal{D}, \phi)$, if $\phi(t, 0, d) = 0$ for all $t \geq 0$ and all $d \in \mathcal{D}$.

Definition 6. We call $0 \in X$ a robust equilibrium point of the control system $\Sigma = (X, \mathcal{D}, \phi)$, if

(i) 0 is an equilibrium point of Σ

(ii) for every $\varepsilon > 0$ and for any $h > 0$ there exists $\delta = \delta(\varepsilon, h) > 0$, so that

$$t \in [0, h],\ \|x\|_X \leq \delta,\ d \in \mathcal{D} \quad \Rightarrow \quad \|\phi(t, x, d)\|_X \leq \varepsilon. \tag{3}$$

Example 7. Let $X = D = \mathbb{R}$ and let $\mathcal{D} = L^\infty(\mathbb{R}_+, D)$. The following examples show the relations between forward-completeness, robust forward-completeness and robustness of the equilibrium point.

(1) Σ is RFC, but 0 is not REP of Σ:
$$\dot{x}(t) = |d(t)|(x(t) - x^3(t)).$$

(2) Σ is forward-complete, but not RFC and 0 is not REP:
$$\dot{x} = dx.$$

(3) 0 is REP of Σ, Σ is forward-complete, but not RFC:
$$\dot{x} = \frac{1}{|d| + 1}x + d \max\{|x| - 1, 0\}.$$

(4) 0 is REP of Σ and Σ is RFC:
$$\dot{x} = \frac{1}{|d| + 1}x.$$

For systems with a Lipschitz continuous flow the equilibrium point is necessarily robust, which is clear as the robustness property is essentially a continuity property of the flow at 0.

In this paper we investigate the following stability property of abstract control systems:

Definition 8. The system $\Sigma = (X, \mathcal{D}, \phi)$ is called uniformly globally asymptotically stable (UGAS) if there exists a $\beta \in \mathcal{KL}$ such that

$$\|\phi(t,x,d)\|_X \le \beta(\|x\|_X, t) \quad \forall d \in \mathcal{D},\ \forall x \in X,\ \forall t \ge 0. \quad (4)$$

As we will see a concept of uniform global attractivity is ultimately helpful for verification of UGAS.

Definition 9. A control system $\Sigma = (X, \mathcal{D}, \phi)$ is called uniformly globally attractive, if for any $r, \varepsilon > 0$ there exists $\tau = \tau(r, \varepsilon)$ so that for all $d \in \mathcal{D}$ it holds that

$$\|x\|_X \le r,\ t \ge \tau(r,\varepsilon) \quad \Rightarrow \quad \|\phi(t,x,d)\|_X \le \varepsilon. \quad (5)$$

We need the following characterization of UGAS property, which can be found e.g. in (Karafyllis and Jiang, 2011b, Theorem 2.2):

Proposition 10. Let $\Sigma = (X, \mathcal{D}, \phi)$ be a control system and let 0 be a robust equilibrium point for Σ. Then Σ is UGAS if and only if Σ is robustly forward complete and uniformly globally attractive.

It may be surprising at first glance that the characterization of UGAS does not directly require a stability property, whereas in the usual context of ODEs it is well known that attractivity on its own does not imply asymptotic stability. The point to notice here is that uniform attractivity is a far stronger concept than attractivity as it requires convergence rates that are uniform for all initial conditions from a ball around the fixed point. This excludes examples that are attractive but not stable.

2.1 Direct Lyapunov theorems

Lyapunov functions provide a predominant tool to study UGAS. In our context they are defined as follows. Let $V : X \to \mathbb{R}$ be continuous. Given $x \in X, d \in \mathcal{D}$, we consider the upper Dini derivative of the continuous function $t \mapsto V(\phi(t,x,d))$ at $t = 0$:

$$\dot{V}_d(x) := \overline{\lim_{t \to +0}} \frac{1}{t}(V(\phi(t,x,d)) - V(x)). \quad (6)$$

We call this the Dini derivative of V along the trajectories of Σ.

Definition 11. A continuous function $V : X \to \mathbb{R}_+$ is called a *Lyapunov function* for a control system $\Sigma = (X, \mathcal{D}, \phi)$, if there exist $\psi_1, \psi_2 \in \mathcal{K}_\infty$ and $\alpha \in \mathcal{K}_\infty$ such that

$$\psi_1(\|x\|_X) \le V(x) \le \psi_2(\|x\|_X) \quad \forall x \in X \quad (7)$$

holds and the Dini derivative of V along the trajectories of Σ satisfies

$$\dot{V}_d(x) \le -\alpha(\|x\|_X) \quad (8)$$

for all $x \in X$ and all $d \in \mathcal{D}$. We call V a *noncoercive Lyapunov function*, if instead of (7) we have $V(0) = 0$ and

$$0 < V(x) \le \psi_2(\|x\|_X) \quad \forall x \in X \backslash \{0\}. \quad (9)$$

If we want to emphasize that (7) holds we will also speak of a coercive Lyapunov function.

The following result is well-known:

Proposition 12. Let $\Sigma = (X, \mathcal{D}, \phi)$ be a control system. If there exists a coercive Lyapunov function for Σ, then Σ is UGAS.

The proof of Proposition 12 is analogous to the proof of its finite-dimensional counterpart, see (Lin et al., 1996, p. 160). Note however, that we use continuous Lyapunov functions and the trajectories of the system (19) are merely continuous, therefore we cannot use the standard comparison principle, see (Lin et al., 1996, Lemma 4.4) in the proof of Proposition 12. Instead one can exploit the following generalized comparison principle from (Mironchenko and Ito, 2016, Lemma 1):

Lemma 13. Let $\alpha \in \mathcal{P}$ and consider the differential inequality

$$\dot{y}(t) \le -\alpha(y(t)), \quad t > 0. \quad (10)$$

There exists a $\beta \in \mathcal{KL}$ so that for all continuous functions $y : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying (10) in the sense of the upper Dini derivative (defined in (6)) we have

$$y(t) \le \beta(y(0), t) \quad \forall t \ge 0. \quad (11)$$

Next we show that already the existence of a non-coercive Lyapunov function is sufficient for UGAS of a control system provided another mild assumption is satisfied.

Definition 14. Consider a system $\Sigma = (X, \mathcal{D}, \phi)$. We say that Σ satisfies a uniform growth bound, if there exists a continuous function $\chi : \mathbb{R}^2_+ \to \mathbb{R}_+$ such that $\chi(r, \cdot) \in \mathcal{K}_\infty$ for all $r \ge 0$ and such that for all $x \in X, d \in \mathcal{D}$ and all $t \ge 0$ we have

$$\|\phi(t,x,d)\|_X - \|x\|_X \le \chi(\|x\|_X, t). \quad (12)$$

We note that the uniform bounded growth condition immediately implies that 0 is a robust equilibrium point, if it is an equilibrium point in the sense of Definition 5. In addition, a system Σ is RFC if it satisfies the uniform growth bound.

The condition of a uniform growth bound thus requires that solutions cannot grow arbitrarily fast on a finite interval. On the other, arbitrarily fast decay is admitted (and can in fact be observed in many infinite-dimensional systems). This assumption is quite mild and is satisfied by large classes of infinite-dimensional systems. In the linear case, even for C_0-semigroups with unbounded input operators and uniformly bounded disturbances this is implied by admissibility assumptions, and also for nonlinear infinite dimensional systems this is common. The system class treated in Section 3 satisfies this property as long as there is a uniform bound on the inputs.

In the following result we need variants of the fundamental theorem of calculus for Dini derivatives. For this see Saks (1947); Hagood and Thomson (2006).

Theorem 15. **(Noncoercive Lyapunov UGAS theorem)** Consider a system $\Sigma = (X, \mathcal{D}, \phi)$ and assume that V is a noncoercive Lyapunov function. If Σ satisfies a uniform growth bound then Σ is UGAS.

Proof. Let V be a non-coercive Lyapunov function and let $\alpha \in \mathcal{K}$ be such that we have the decay estimate (8). Along any trajectory ϕ of Σ we have the inequality

$$\frac{d}{dt} V(\phi(t,x,d)) \le -\alpha(\|\phi(t,x,d)\|_X).$$

It follows from (Saks, 1947, pp. 204-205) that

$$V(\phi(t,x,d)) - V(x) \le -\int_0^t \alpha(\|\phi(s,x,d)\|_X) ds,$$

which implies that for all $t \geq 0$ we have

$$\int_0^t \alpha(\|\phi(s,x,d)\|_X)ds \leq V(x). \tag{13}$$

We by showing stability. Seeking a contradiction, assume that Σ is not uniformly stable in $x^* = 0$. Then there exists an $\varepsilon > 0$ and sequences $\{x_k\}_{k\in\mathbb{N}}$ in X, $\{d_k\}_{k\in\mathbb{N}}$ in $\mathcal{D}$, and $t_k \geq 0$ such that $x_k \to 0$ as $k \to \infty$ and

$$\|\phi(t_k, x_k, d_k)\|_X = \varepsilon \quad \forall k \geq 1.$$

By the bound on V given by (9) if follows that $V(x_k) \to 0$. Let χ be the function characterizing the growth bound. Appealing to continuity we may choose $\tau > 0$ such that $\chi(r,\tau) \leq \varepsilon/2$ for all $0 \leq r \leq \varepsilon$. Using the growth bound condition we obtain that for all $k \in \mathbb{N}$ and for all $t \in [t_k - \tau, t_k]$ we have $\|\phi(t,x_k,d_k)\|_X > \varepsilon$ or

$$\|\phi(t,x_k,d_k)\|_X \geq$$

$$\|\phi(t_k,x_k,d_k)\|_X - \chi(\|\phi(t,x_k,d_k)\|_X, t_k - t) \geq \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2}.$$

Now (13) implies for every k

$$V(x_k) \geq \int_{t_k-\tau}^{t_k} \alpha(\|\phi(s,x,d)\|_X)ds \geq \alpha(\varepsilon/2)\tau > 0.$$

This contradiction proves uniform stability of Σ.

The proof of uniform global attractivity uses the fact that stability has been established in combination with arguments similar to the ones discussed so far. It is omitted for reasons of space. ■

2.2 Converse Lyapunov theorems

In this section we consider two constructions of Lyapunov functions for UGAS systems $\Sigma = (X, \mathcal{D}, \phi)$. First we recall a construction of Lipschitz continuous coercive Lyapunov functions, due to Karafyllis and Jiang (2011b), and then we use these ideas in order to derive another construction of a global non-coercive Lyapunov function for a UGAS control system.

The following theorem has been shown in (Karafyllis and Jiang, 2011b, Section 3.4). To formulate it we need the following constructions. First, by Sontag's $\mathcal{KL}$ lemma, given the UGAS bound $\beta \in \mathcal{KL}$, we can find $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

$$\beta(r,t) \leq \alpha_2(\alpha_1(r)e^{-t}), \quad \forall r \geq 0,\ \forall t \geq 0. \tag{14}$$

Furthermore, by (Karafyllis and Jiang, 2011b, p.130) we may choose a globally Lipschitz $\rho \in \mathcal{K}_\infty$ with Lipschitz constant $L = 1$ such that $\rho \leq \alpha_2^{-1}$. We also need, for $k \in \mathbb{N}$, the functions $G_k : \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$G_k(r) := \max\left\{r - \frac{1}{k}, 0\right\}, \quad r \geq 0.$$

Theorem 16. Consider a control system $\Sigma = (X, \mathcal{D}, \phi)$ with a flow, which is Lipschitz continuous on compact intervals. If Σ is UGAS, then for any $\eta \in (0,1)$ there exists a sequence of positive real numbers a_k, $k = 1, 2, \ldots$ so that $W^\eta : X \to \mathbb{R}_+$, defined by

$$W^\eta(x) := \sum_{k=1}^{\infty} a_k V_k^\eta(x) \qquad \forall x \in X, \tag{15}$$

where, using G_k and an arbitrary ρ as discussed above,

$$V_k^\eta(x) := \sup_{d\in\mathcal{D}} \max_{s\in[0,+\infty)} e^{\eta s} G_k\big(\rho(\|\phi(s,x,d)\|_X)\big). \tag{16}$$

is a UGAS Lyapunov function for (19) which is Lipschitz continuous on bounded balls.

The proof of this result is based on earlier local converse Lyapunov theorems, see e.g. (Yoshizawa, 1966, Theorem 19.3), (Henry, 1981, Theorem 4.2.1) and on using Sontags' $\mathcal{KL}$-Lemma (Sontag, 1998, Proposition 7).

Next, exploiting ideas from (Karafyllis and Jiang, 2011b, Section 3.4) we construct a Lipschitz continuous non-coercive Lyapunov function for a UGAS control system using integration instead of maximization.

Theorem 17. Consider a system $\Sigma = (X, \mathcal{D}, \phi)$ with a flow which is Lipschitz continuous on compact intervals. If Σ is UGAS, then there exists a sequence of positive real numbers a_k, $k = 1, 2, \ldots$, so that $W : X \to \mathbb{R}_+$, defined by

$$W(x) := \sum_{k=1}^{\infty} a_k V_k(x) \qquad \forall x \in X. \tag{17}$$

where, using G_k and an arbitrary ρ as above,

$$V_k(x) = \sup_{d\in\mathcal{D}} \int_0^\infty G_k\big(\rho(\|\phi(t,x,d)\|_X)\big)dt. \tag{18}$$

is a non-coercive UGAS Lyapunov function for (19) which is Lipschitz continuous on bounded balls.

3. INPUT-TO-STATE STABILITY AND WEAK ROBUST STABILITY

In this section we consider infinite-dimensional systems of the form

$$\dot{x}(t) = Ax(t) + f(x(t), u(t)),\ x(t) \in X, u(t) \in U, \tag{19}$$

where A generates a strongly continuous semigroup (of bounded operators), X is a Banach space and U is a normed linear space of external inputs. As the space of admissible inputs we consider the space $\mathcal{U}$ of globally bounded, piecewise continuous functions from $\mathbb{R}_+$ to U.

In this paper we consider mild solutions of (19), i.e. solutions of the integral equation

$$x(t) = T(t)x(0) + \int_0^t T(t-s)f(x(s),u(s))ds \tag{20}$$

belonging to the class $C([0,\tau], X)$ for certain $\tau > 0$. Here $\{T(t),\ t \geq 0\}$ is a C_0-semigroup over X, generated by A.

We will need the following lemma:

Lemma 18. Let:

(i) (19) be robustly forward-complete.
(ii) f be Lipschitz continuous on bounded subsets of X, uniformly w.r.t. the second argument.

Then (19) has a flow which is Lipschitz continuous on compact intervals.

In what follows we suppose that the nonlinearity f satisfies the following assumption:

Assumption 1. Let $f : X \times U \to X$ be bi-Lipschitz continuous on bounded subsets, which means that two following properties hold:

(1) $\forall C > 0\ \exists L_f^1(C) > 0$, such that $\forall x, y \in X : \|x\|_X \leq C,\ \|y\|_X \leq C$ and $\forall v \in U$, it holds that

$$\|f(x,v) - f(y,v)\|_X \leq L_f^1(C)\|x-y\|_X. \tag{21}$$

(2) $\forall C > 0 \; \exists L_f^2(C) > 0$, such that $\forall u, v \in U : \|u\|_U \leq C, \; \|v\|_U \leq C$ and $\forall x \in X$, it holds that

$$\|f(x,u) - f(x,v)\|_X \leq L_f^2(C)\|u - v\|_U. \quad (22)$$

Due to standard arguments, Assumption 1 implies that mild solutions corresponding to any $x(0) \in X$ and any $u \in \mathcal{U}$ exist and are unique (actually, the second condition is too strong for existence and uniqueness, but we need it for the further development).

Now we are going to treat u not as a 'disturbance', but as an external input, which may have a significant influence on the dynamics of the system. For the stability analysis of such systems a fundamental role is played by the concept of input-to-state stability, which is defined as follows:

Definition 19. System (19) is called *input-to-state stable (ISS)*, if it is forward-complete and there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that $\forall x \in X$, $\forall u \in \mathcal{U}$ and $\forall t \geq 0$ the following holds

$$\|\phi(t,x,u)\|_X \leq \beta(\|x\|_X, t) + \gamma(\|u\|_{\mathcal{U}}). \quad (23)$$

Similarly to the case of uniform global asymptotic stability, we can define a notion of ISS Lyapunov functions:

Definition 20. A continuous function $V : X \to \mathbb{R}_+$ is called a (coercive) *ISS Lyapunov function*, if there exist $\psi_1, \psi_2 \in \mathcal{K}_\infty$, $\alpha \in \mathcal{P}$ and $\chi \in \mathcal{K}$ so that

$$\psi_1(\|x\|_X) \leq V(x) \leq \psi_2(\|x\|_X) \quad \forall x \in X \quad (24)$$

and so that the Dini derivative of V along the trajectories of the system (19) satisfies the implication

$$\|x\|_X \geq \chi(\|u(0)\|_U) \quad \Rightarrow \quad \dot{V}_u(x) \leq -\alpha(\|x\|_X) \quad (25)$$

for all $x \in X$ and $u \in \mathcal{U}$.

As in the case of UGAS, we have the following result, see (Dashkovskiy and Mironchenko, 2013, Theorem 1).

Proposition 21. If there exists an ISS Lyapunov function for (19), then (19) is ISS.

We are going to establish a converse ISS Lyapunov theorem for the systems (19) and relate ISS to the robust stability of (19). On this way we follow the method developed in Sontag and Wang (1995) for systems described by ordinary differential equations.

In order to formalize the robust stability property of (19) we consider in (19) feedback laws of the form

$$u(t) := d(t)\varphi(x(t)),$$

where $d \in \mathcal{D} = \{d : \mathbb{R}_+ \to D, \text{ piecewise continuous}\}$, $D = \{d \in U : \|d\|_U \leq 1\}$, and $\varphi : X \to \mathbb{R}_+$ is Lipschitz continuous on bounded balls.

Applying this feedback law to (19) we obtain the system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + f(x(t), d(t)\varphi(x(t))) \\ &=: Ax(t) + g(x(t), d(t)). \end{aligned} \quad (26)$$

Let us denote the solution of (26) at time t, starting at $x \in X$ and with disturbance $d \in \mathcal{D}$ by $\phi_\varphi(t,x,d)$. On the interval of its existence $\phi_\varphi(t,x,d)$ coincides with the solution of (19) for the input $u(t) = d(t)\varphi(x(t))$. [1]

[1] Forward-completeness of (19) does not imply forward-completeness of (26). For example, consider $\dot{x} = -x + u$, $u(t) = d \cdot x^2(t)$ for $d > 0$.

Now, in order to use the converse UGAS Lyapunov theorems from the previous section, we need to ensure that the flow of (26) is Lipschitz continuous on compact intervals.

The next lemma shows that g in (26) is well-behaved.

Lemma 22. Let f be bi-Lipschitz continuous on bounded balls. Then g is Lipschitz continuous on bounded subsets of X, uniformly with respect to the second argument, i.e. $\forall C > 0 \; \exists L_g(C) > 0$, such that $\forall x, y : \|x\|_X \leq C, \; \|y\|_X \leq C$, $\forall d \in D$, it holds that

$$\|g(x,d) - g(y,d)\|_X \leq L_g(C)\|x - y\|_X. \quad (27)$$

Remark 23. Lipschitz continuous feedbacks do not necessarily lead to Lipschitz continuous g if f is not Lipschitz w.r.t. inputs. Consider e.g. $\dot{x}(t) = (u(t))^{1/3}$ and $u(t) := x(t)$.

Definition 24. System (19) is called weakly uniformly robustly asymptotically stable (WURS), if there exist a function $\varphi : X \to \mathbb{R}_+$, Lipschitz continuous on bounded balls, and a $\psi \in \mathcal{K}_\infty$ such that $\varphi(x) \geq \psi(\|x\|_X)$ and (26) is uniformly globally asymptotically stable over $\mathcal{D}$.

The objective of this section is to prove that for systems (19) the notions depicted in Figure 1 are equivalent.

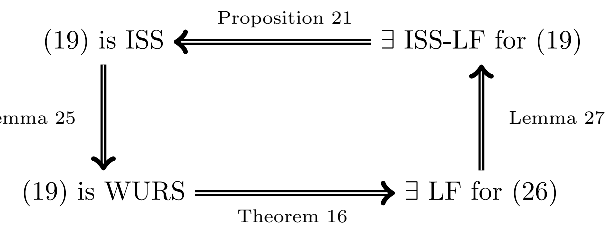

Fig. 1. ISS Converse Lyapunov Theorem

First we show in Lemma 25 that ISS implies WURS. Next we apply Theorem 16 in order to prove that WURS of (19) implies the existence of a Lipschitz continuous coercive ISS Lyapunov function for (19). Finally, the direct Lyapunov theorem (Proposition 21) finishes the proof.

As depicted in Figure 1, we start with

Lemma 25. If (19) is ISS, then it is WURS.

The following proposition establishes that (26) is a well-defined system with nice properties of the flow:

Proposition 26. Assume that

(i) (19) is forward-complete.
(ii) f is bi-Lipschitz on bounded subsets of X
(iii) (19) is WURS.

Then (26) is a control system with a flow which is Lipschitz continuous on compact intervals.

The next ingredient for the proof of our main result is:

Lemma 27. If system (19) is WURS and Assumption 1 is satisfied then there exists an ISS Lyapunov function for (19), which is Lipschitz continuous on bounded balls.

The key step in the proof of this Lemma is an application of Theorem 16. Finally, we can characterize ISS in the following way:

Theorem 28. Let Assumption 1 hold. Then the following statements are equivalent:

(1) (19) is ISS
(2) (19) is WURS
(3) There exists a Lipschitz continuous on bounded balls coercive ISS Lyapunov function for (19).

Proof. Follows from Prop. 21 and Lemmas 25 and 27. ■

4. CONCLUSIONS

We have given new constructions of global Lipschitz continuous Lyapunov functions for infinite dimensional systems with disturbances and have applied the obtained results in order to prove that input-to-state stability of the infinite-dimensional systems is (under reasonable assumptions) equivalent to the existence of a Lipschitz continuous coercive ISS-Lyapunov function.

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