NON-COERCIVE LYAPUNOV FUNCTIONS FOR INFINITE-DIMENSIONAL SYSTEMS

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ABSTRACT. We show that the existence of a non-coercive Lyapunov function is sufficient for uniform global asymptotic stability (UGAS) of infinite-dimensional systems with external disturbances provided the speed of decay is measured in terms of the norm of the state and an additional mild assumption is satisfied. For evolution equations in Banach spaces with Lipschitz continuous nonlinearities these additional assumptions become especially simple. The results encompass some recent results on linear switched systems on Banach spaces. Some examples show the necessity of the assumptions which are made.

1. INTRODUCTION

The theory of Lyapunov functions is one of the cornerstones in the analysis and synthesis of dynamical systems. Since its origins due to Lyapunov there have been numerous developments leading first to sufficient and later on to necessary conditions for various dynamical properties expressed in terms of Lyapunov functions, see [42, 18, 29, 27]. The uses of Lyapunov functions are manifold. Originally invented by Lyapunov to characterize stability properties of fixed points, or more complex attractors, they have become useful in other contexts. In existence theory they provide bounds on the transients of solutions and so in some situations conditions for forward completeness of trajectories, [1]. A criterion for the existence of a bounded absorbing ball is formulated in [8, Theorem 2.1.2]. In the finite-dimensional case they have also been used to investigate the geometric structure of the general solution of differential equations [40] and to analyze coordinate-free notions of growth rates [16]. The complete characterization of attractor-repeller pairs of continuous-time dynamical systems has been obtained in [41, 10]. Some of these uses extend from finite-dimensional applications to the infinite-dimensional case, while others use distinct finite-dimensional arguments.

On the other hand numerous converse results have been obtained which prove the existence of certain types of Lyapunov functions characterizing different stability notions, [27]. Before starting to look for a Lyapunov function it is highly desirable to know in advance that such a Lyapunov function for a given class of systems exists. The first results guaranteeing existence of Lyapunov functions for asymptotically stable systems appeared in the works of Kurzweil [28] and Massera [30]. These have been generalized in different directions [21, 13, 25, 9, 27, 34, 38]. In particular, in [25, 38] the well-known Sontag $\mathcal{KL}$-lemma [36] has been elegantly used in order to

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prove global Lyapunov theorems with the help of Yoshizawa's method [42, Theorem 19.3], [21, Theorem 4.2.1].

In this paper we address a particular aspect of infinite-dimensional systems by providing Lyapunov stability results for non-coercive Lyapunov functions and converse theorems. Throughout the paper we will be dealing with a fixed point at the origin, $x^* = 0$, which imposes no restrictions as long as we are interested in the stability properties of fixed points. The question of coercivity of Lyapunov functions is subtle and statements that are obvious in finite dimensions become intricate or even wrong in infinite dimensions. The standard definition of a Lyapunov function V, found in many textbooks on finite-dimensional dynamical systems, is that it should be a continuous (or more regular) positive definite and proper function, i.e. a function for which there exist $\mathcal{K}_\infty$[1] functions ψ_1, ψ_2, α such that

$$\psi_1(\|x\|) \leq V(x) \leq \psi_2(\|x\|) \quad \forall x \in X, \tag{1.1}$$

and such that

$$\dot{V}(x) < -\alpha(\|x\|) \quad \forall x \in X, \tag{1.2}$$

where $\dot{V}(x)$ is some sort of generalized derivative of V along the trajectories of the system, see below for a precise definition. Of course, if we have (1.1) then we may just as well require that there is a $\gamma \in \mathcal{K}_\infty$ such that

$$\dot{V}(x) < -\gamma(V(x)), \tag{1.3}$$

because in the presence of (1.1) we clearly have an equivalence of (1.2) and (1.3).

The inequality (1.3) shows that $V(x(t))$ converges to zero in a uniform way as $t \to \infty$ (by the "comparison principle"), and (1.1) implies that $\|x(t)\|$ has the same asymptotic behavior. This simple argument remains (up to some minor technicalities) the same also for infinite-dimensional systems and has been applied for stability analysis, e.g. in [32, 13, 5].

On the other hand, converse Lyapunov theorems proved for wide classes of infinite-dimensional systems show that asymptotic stability guarantees the existence of a proper and positive definite Lyapunov function.

A first inkling that this is not the complete story comes from the study of linear systems. In a seminal paper [12] Datko proved the following. If X is a Hilbert space and A the generator of a C_0 semigroup on X, then the system $\dot{x} = Ax$ is exponentially stable if and only if there exists a positive definite bilinear form on X (generated by a certain bounded positive definite linear operator P) such that for all $x \in D(A)$ we have the following estimate for the scalar product of Px, Ax:

$$\langle Px, Ax \rangle < -\|x\|^2.$$

This is a natural extension of the finite-dimensional Lyapunov inequality.

At the same time the operator P need not be coercive, so the natural Lyapunov function $V : x \mapsto \langle Px, x \rangle$ for the linear system $\dot{x} = Ax$ does not satisfy (1.1). In fact, there exist exponentially stable C_0-semigroups on Hilbert spaces such that there does not exist an equivalent scalar product under which the semigroup is a strict contraction semigroup, [7]. Hence, the non-coercivity of P cannot be avoided in general. In this situation, it appears that the left inequality in (1.1) is an artifact of the finite-dimensional origin of the theory. In infinite dimensions it may sometimes

[1] Increasing, unbounded, continuous, positive definite function.

be easier and more natural to derive Lyapunov functions which have the weaker property that

$$0 < V(x) \leq \psi_2(\|x\|)\,, \quad x \neq 0. \tag{1.4}$$

The next question then becomes under which decay conditions the existence of such a non-coercive Lyapunov function guarantees uniform asymptotic stability.

In this paper we provide a positive and a negative answer. We show that for a large class of dynamical systems on Banach spaces, the existence of a non-coercive Lyapunov function satisfying (1.4) and the decay estimate (1.2) guarantees global uniform asymptotic stability of the origin provided a forward completeness property is satisfied and the fixed point is robust. This is achieved using uniform Barbalat-like estimates. On the other hand, we show by examples that the results cannot be extended much further. In particular, non-coercive Lyapunov functions cannot be used in conjunction with a decay estimate in terms of the Lyapunov function itself as in (1.3). To show this, in Section 7 we give an example of an exponentially unstable linear system with a bounded generator admitting a Lyapunov function satisfying conditions (1.4) and (1.3). Also it is a disadvantage that we have to assume forward completeness whereas in many cases coercive Lyapunov functions give this property for free. Example 7.1 shows that non-coercive Lyapunov functions do not guarantee forward completeness even in the finite dimensional case. Also they cannot be used to obtain bounds on the transient behavior.

To compare this with a case of coercive Lyapunov functions, note that to obtain boundedness of solutions on bounded intervals it is sufficient that

$$\dot{V}(x) < \alpha(\|x\|), \tag{1.5}$$

where again we may appeal to (1.1) to see that this is equivalent to $\dot{V}(x) < \gamma(V(x))$ for a suitable function γ. In [1] it is shown for a class of finite-dimensional systems that forward completeness is equivalent to the existence of a coercive Lyapunov function satisfying $\dot{V}(x) < V(x)$. This argument is also applicable to several classes of nonlinear infinite dimensional systems, see [32, Chapter 6], [25, Chapter 1]. As we will see in the examples the assumption of coercivity cannot be relaxed if it is the aim to conclude that solutions exist for all positive times from the fact that solutions do not escape to infinity in finite-time.

If the origin is not a robust equilibrium, the existence of a non-coercive Lyapunov function for a robustly forward complete system does not ensure uniform global asymptotic stability of the origin. However, in this case the origin still exhibits a form of attractivity. Motivated by the classical notion of weak attractivity [3, 4, 35] we introduce the stronger concept of uniform weak attractivity. In the finite-dimensional case, this does not add anything new, but in infinite dimensions uniform weak attractivity is essentially stronger than weak attractivity. For robustly forward complete systems we characterize uniform global asymptotic stability as uniform global weak attractivity together with uniform stability. Finally, we show that the existence of a non-coercive Lyapunov function for a robustly forward complete system ensures that 0 is uniformly globally weakly attractive and Lagrange stable.

In addition to the non-coercive Lyapunov theorem we also derive a converse theorem for system which have flows with a Lipschitz continuity property. The construction is motivated by classical converse theorems and Yoshizawa's method [42, Theorem 19.3], [21, Theorem 4.2.1]. We derive in Section 6 a novel "integral" construction of a non-coercive Lipschitz continuous Lyapunov function for a UGAS system with a Lipschitz continuous flow map. The construction appears to be novel, maybe because so far it has been unclear what the interest in a non-coercive Lyapunov function is, in general.

The paper is organized as follows. In Section 2 we introduce an abstract class of dynamical systems subsject to time-varying disturbances and define the various stability concepts used in this paper. In particular the concepts of robust forward completeness and robustness of the equilibrium point will be of importance and we introduce characterizations of these. Section 3 is devoted to the derivation of sufficient conditions for uniform global and local asymptotic stability of the fixed point in terms of non-coercive Lyapunov functions. In this section the assumptions of robust forward completeness and robustness of the equilibrium are vital. In Section 4 we investigate what conclusions can be drawn from the existence of a non-coercive Lyapunov function if only forward completeness of the system is assumed. In this context the notions of Lagrange stability and uniform weak attractivity become important. In order to showcase the applicability of the results, Section 5 discusses system more concrete system classes which satisfy our general assumption. In Section 6 we derive a converse theorem for systems which have a Lipschitz continuous flow. Section 7 discusses examples which give counterexamples showing that some of our results cannot be extended much further. We conclude in Section 8.

1.1. **Notation.** The following notation will be used throughout these notes. By $\mathbb{R}_+$ we denote the set of nonnegative real numbers. For an arbitrary set S and $n \in \mathbb{N}$ the n-fold Cartesian product is $S^n := S \times \ldots \times S$. For normed linear spaces X, Y we denote by $\mathcal{L}(X,Y)$ the space of linear bounded operators acting from X to Y and we abbreviate $\mathcal{L}(X) := \mathcal{L}(X,X)$. The open ball in a normed linear space X with radius r and center in $y \in X$ is denoted by $B_r(y) := \{x \in X \mid \|x-y\|_X < r\}$ (the space X in which the ball is taken, will always be clear from the context). For short, we denote $B_r := B_r(0)$. The (norm)-closure of a set $S \subset X$ will be denoted by $\overline{S}$.

For the formulation of stability properties the following classes of comparison functions are useful, see [18, 26]:

$$
\begin{array}{ll}
\mathcal{K} & := \left\{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \ \gamma \text{ is continuous and strictly increasing, } \gamma(0) = 0\right\}, \\
\mathcal{K}_\infty & := \left\{\gamma \in \mathcal{K} \mid \gamma \text{ is unbounded}\right\}, \\
\mathcal{L} & := \left\{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous and decreasing with } \lim\limits_{t\to\infty} \gamma(t) = 0\right\}, \\
\mathcal{KL} & := \left\{\beta : \mathbb{R}_+^2 \to \mathbb{R}_+ \mid \ \beta(\cdot,t) \in \mathcal{K},\ \forall t \geq 0,\ \beta(r,\cdot) \in \mathcal{L},\ \forall r > 0\right\}.
\end{array}
$$

2. Problem statement

In this paper we consider abstract axiomatically defined time-invariant and forward complete systems on the state space X which are subject to a shift-invariant set of disturbances $\mathcal{D}$.

Definition 2.1. Consider the triple $\Sigma = (X, \mathcal{D}, \phi)$, consisting of

(i) A normed linear space $(X, \|\cdot\|_X)$, called the state space, endowed with the norm $\|\cdot\|_X$.
(ii) A set of disturbance values D, which is a nonempty subset of a normed linear space S_d.
(iii) A space of disturbances $\mathcal{D} \subset \{d : \mathbb{R}_+ \to D\}$ satisfying the following two axioms.

The axiom of shift invariance states that for all $d \in \mathcal{D}$ and all $\tau \geq 0$ the time shift $d(\cdot + \tau)$ is in $\mathcal{D}$.

The axiom of concatenation is defined by the requirement that for all $d_1, d_2 \in \mathcal{D}$ and for all $t > 0$ the concatenation of d_1 and d_2 at time t

$$d(\tau) := \begin{cases} d_1(\tau), & \text{if } \tau \in [0,t], \\ d_2(\tau - t), & \text{otherwise,} \end{cases} \tag{2.1}$$

belongs to $\mathcal{D}$.

(iv) A transition map $\phi : \mathbb{R}_+ \times X \times \mathcal{D} \to X$.

The triple Σ is called a (forward complete) dynamical system, if the following properties hold:

(Σ1) forward completeness: for every $(x,d) \in X \times \mathcal{D}$ and for all $t \geq 0$ the value $\phi(t,x,d) \in X$ is well-defined.

(Σ2) The identity property: for every $(x,d) \in X \times \mathcal{D}$ it holds that $\phi(0,x,d) = x$.

(Σ3) Causality: for every $(t,x,d) \in \mathbb{R}_+ \times X \times \mathcal{D}$, for every $\tilde{d} \in \mathcal{D}$, such that $d(s) = \tilde{d}(s)$, $s \in [0,t]$ it holds that $\phi(t,x,d) = \phi(t,x,\tilde{d})$.

(Σ4) Continuity: for each $(x,d) \in X \times \mathcal{D}$ the map $t \mapsto \phi(t,x,d)$ is continuous.

(Σ5) The cocycle property: for all $t,h \geq 0$, for all $x \in X$, $d \in \mathcal{D}$ we have $\phi(h, \phi(t,x,d), d(t+\cdot)) = \phi(t+h,x,d)$.

Here $\phi(t,x,d)$ denotes the state of a system at the moment $t \in \mathbb{R}_+$ corresponding to the initial condition $x \in X$ and the disturbance $d \in \mathcal{D}$.

Definition 2.2. We say that the flow of $\Sigma = (X, \mathcal{D}, \phi)$ is Lipschitz continuous on compact intervals, if for any $\tau > 0$ and any $r > 0$ there exists $L > 0$ so that

$$x,y \in \overline{B_r},\ t \in [0,\tau], d \in \mathcal{D} \quad \Rightarrow \quad \|\phi(t,x,d) - \phi(t,y,d)\|_X \leq L\|x-y\|_X. \tag{2.2}$$

We exploit the following stronger version of forward completeness:

Definition 2.3. The system $\Sigma = (X, \mathcal{D}, \phi)$ is called robustly forward complete (RFC) if for any $C > 0$ and any $\tau > 0$ it holds that

$$\sup\{\|\phi(t,x,d)\|_X \mid \|x\|_X \leq C,\ t \in [0,\tau],\ d \in \mathcal{D}\} < \infty.$$

The condition of robust forward completeness is satisfied by large classes of infinite-dimensional systems.

Definition 2.4. We call $0 \in X$ an equilibrium point of the system $\Sigma = (X, \mathcal{D}, \phi)$, if $\phi(t,0,d) = 0$ for all $t \geq 0$ and all $d \in \mathcal{D}$.

Definition 2.5. We call $0 \in X$ a robust equilibrium point of the system $\Sigma = (X, \mathcal{D}, \phi)$, if it is an equilibrium point such that for every $\varepsilon > 0$ and for any $h > 0$ there exists $\delta = \delta(\varepsilon, h) > 0$, so that

$$t \in [0,h],\ \|x\|_X \leq \delta,\ d \in \mathcal{D} \quad \Rightarrow \quad \|\phi(t,x,d)\|_X \leq \varepsilon. \tag{2.3}$$

Example 2.6. Let $X = D = \mathbb{R}$ and let $\mathcal{D} = L^\infty(\mathbb{R}_+, D)$. The following examples show the relations between forward completeness, robust forward completeness and robustness of the equilibrium point.

(i) Σ is RFC, but 0 is not a REP of Σ: $\quad \dot{x} = |d|(x - x^3)$.

(ii) Σ is forward complete, but not RFC and 0 is not a REP: $\quad \dot{x} = d \cdot x$.

(iii) 0 is a REP of Σ, Σ is forward complete, but not RFC:

$$\dot{x} = \frac{1}{|d|+1}x + d\max\{|x|-1, 0\}.$$

(iv) 0 is a REP of Σ and Σ is RFC: $\dot{x} = \frac{1}{|d|+1}x$.

For systems with a Lipschitz continuous flow the equilibrium point is necessarily robust, which is clear as the robustness property of an equilibrium is the continuity of the function $\xi : (x,h) \mapsto \sup_{t\in[0,h],\ d\in\mathcal{D}} \|\phi(t,x,d)\|_X$ with respect to its first argument at $x=0$.

Lemma 2.7. *Let $\Sigma = (X, \mathcal{D}, \phi)$ be a system with a flow which is Lipschitz continuous on compact intervals. If $0 \in X$ is an equilibrium point of Σ, then 0 is a robust equilibrium point of Σ.*

In this paper we investigate the following stability properties of equilibria of abstract systems.

Definition 2.8. Consider a system $\Sigma = (X, \mathcal{D}, \phi)$ with fixed point 0. The equilibrium position 0 is called

(i) Lagrange stable, if there exist $c > 0$ and $\sigma \in \mathcal{K}_\infty$ so that

$$x \in X,\ t \geq 0,\ d \in \mathcal{D} \quad \Rightarrow \quad \|\phi(t,x,d)\|_X \leq \sigma(\|x\|_X) + c.$$

(ii) (locally) uniformly stable (US), if for every $\varepsilon > 0$ there is a $\delta > 0$ so that

$$\|x\|_X \leq \delta,\ d \in \mathcal{D},\ t \geq 0 \quad \Rightarrow \quad \|\phi(t,x,d)\|_X \leq \varepsilon. \tag{2.4}$$

(iii) uniformly globally stable, if it is Lagrange stable with $c = 0$.

(iv) uniformly globally asymptotically stable (UGAS) if there exists a $\beta \in \mathcal{KL}$ such that for all $x \in X, d \in \mathcal{D}, t \geq 0$

$$\|\phi(t,x,d)\|_X \leq \beta(\|x\|_X, t). \tag{2.5}$$

(v) uniformly (locally) asymptotically stable (UAS) if there exist a $\beta \in \mathcal{KL}$ and an $r > 0$ such that the inequality (2.5) holds for all $x \in B_r, d \in \mathcal{D}, t \geq 0$.

(vi) globally weakly attractive, if

$$x \in X,\ d \in \mathcal{D} \quad \Rightarrow \quad \inf_{t\geq 0} \|\phi(t,x,d)\|_X = 0. \tag{2.6}$$

(vii) uniformly globally weakly attractive, if for every $\varepsilon > 0$ and for every $r > 0$ there exists a $\tau = \tau(\varepsilon, r)$ such that

$$\|x\|_X \leq r,\ d \in \mathcal{D} \quad \Rightarrow \quad \exists t = t(x,d,\varepsilon) \leq \tau :\ \|\phi(t,x,d)\|_X \leq \varepsilon. \tag{2.7}$$

(viii) uniformly globally attractive (UGATT), if for any $r, \varepsilon > 0$ there exists $\tau = \tau(r,\varepsilon)$ so that

$$\|x\|_X \leq r,\ d \in \mathcal{D},\ t \geq \tau(r,\varepsilon) \quad \Rightarrow \quad \|\phi(t,x,d)\|_X \leq \varepsilon. \tag{2.8}$$

Remark 2.9. According to Definition 2.8 the origin is globally weakly attractive if and only if for all $x \in X,\ d \in \mathcal{D}$ the origin belongs to the ω-limit set $\omega(x,d)$, defined by

$$\omega(x,d) := \{y \in X \mid \exists\ t_k \to \infty : \phi(t_k, x, d) \to y\}. \tag{2.9}$$

In this form 0 is a weak attractor, a notion originally introduced in [3], see also [4] and our nomenclature follows this reference.

On the other hand, in the language of [35], weak attractivity is the limit property with zero gain. Clearly, 0 is a global weak attractor of Σ if and only if for every $x \in X$, every $d \in \mathcal{D}$ and any $\varepsilon > 0$ there exists $t = t(x, d, \varepsilon)$ so that $\|\phi(t, x, d)\|_X \leq \varepsilon$. This justifies the interpretation of (vii) as a uniform version of weak attractivity.

We stress that in some works "weak" stability concepts are discussed which are related to convergence properties in the weak topology on X, e.g. [14]. This is not the intended meaning here.

The following characterization of UGAS, [25, Theorem 2.2], shows that the concept of uniform global attractivity is very helpful in the verification of UGAS.

Proposition 2.10. *Let $\Sigma = (X, \mathcal{D}, \phi)$ be a control system and let 0 be a robust equilibrium point for Σ. Then 0 is UGAS if and only if Σ is robustly forward complete and uniformly globally attractive.*

It may be surprising at first glance that the characterization of UGAS does not directly require a stability property, whereas in the usual context of ODEs it is well known that attractivity on its own does not imply asymptotic stability. The point to notice here is that uniform attractivity is a far stronger concept than attractivity as it requires convergence rates that are uniform for all initial conditions from a ball around the fixed point. This ensures that uniformly attractive systems, having 0 as a robust equilibrium point are uniformly stable. Note however, that uniform attractivity without robustness of the equilibrium point does not imply stability anymore. This is illustrated by the following example.

Example 2.11. Let $\mathcal{D} := L^\infty(\mathbb{R}_+, \mathbb{R})$, $x, y \in \mathbb{R}$ and consider

$$\dot{x}(t) = d(t)x(t)y(t) - x^3(t) - x^{1/3}(t), \tag{2.10a}$$

$$\dot{y}(t) = \; - y^3(t) - y^{1/3}(t). \tag{2.10b}$$

It is easy to see that (2.10) is forward complete and 0 is its equilibrium. This system is also UGATT in $x^* = 0$. To see this, note that there exists a time $t^* > 0$, so that for any initial condition y_0 the solution of the y-subsystem attains the value 0 in time less or equal than t^*. Hence, $d(t)x(t)y(t) = 0$ for $t \geq t^*$, any $d \in \mathcal{D}$ and any initial conditions $(x_0, y_0) \in \mathbb{R}^2$. Consequently, the first component of the solution $\phi_1(t, (x_0, y_0), d) = 0$ for all $t \geq 2t^*$ regardless of the initial condition. In particular, (2.10) is UGATT in $x^* = 0$.

On the other hand, it is easy to see that for any initial condition $z_0 := (x_0, y_0)$ with $x_0 > 0$, $y_0 > 0$ and for any $K > 0$, and any small enough $\tau > 0$ one can find $d \in \mathcal{D}$ so that $|\phi(\tau, z_0, d)| > K$. In particular, this shows that zero is not a robust equilibrium of (2.10) and that (2.10) is not RFC. The strict analysis of this example is straightforward, and we skip it. □

2.1. **Characterizations of Robust Forward Completeness.** In this preliminary section we provide useful restatements in terms of the comparison functions of robust forward completeness and of robust equilibrium points.

We call a function $h : \mathbb{R}^2_+ \to \mathbb{R}_+$ increasing, if $(r_1, R_1) \leq (r_2, R_2)$ implies that $h(r_1, R_1) \leq h(r_2, R_2)$, where we use the component-wise partial order on $\mathbb{R}^2_+$. We call h strictly increasing if $(r_1, R_1) \leq (r_2, R_2)$ and $(r_1, R_1) \neq (r_2, R_2)$ imply $h(r_1, R_1) < h(r_2, R_2)$.

Lemma 2.12. *Consider a forward complete system $\Sigma = (X, \mathcal{D}, \phi)$. The following statements are equivalent:*

(i) Σ *is robustly forward complete.*
(ii) there exists a continuous, increasing function $\mu : \mathbb{R}_+^2 \to \mathbb{R}_+$*, such that*

$$x \in X,\ d \in \mathcal{D},\ t \geq 0 \quad \Rightarrow \quad \|\phi(t,x,d)\|_X \leq \mu(\|x\|_X, t). \tag{2.11}$$

(iii) there exists a continuous function $\mu : \mathbb{R}_+^2 \to \mathbb{R}_+$ *such that the implication* (2.11) *holds.*

Proof. (i) $\Rightarrow$ (ii). Let Σ be robustly forward complete. Define $\tilde{\mu} : \mathbb{R}_+^2 \to \mathbb{R}_+$ by

$$\tilde{\mu}(C,\tau) := \sup\{\|\phi(t,x,d)\|_X \mid \|x\|_X \leq C,\ t \in [0,\tau],\ d \in \mathcal{D}\} \tag{2.12}$$

which is well-defined due to the robust forward completeness of Σ. Clearly, $\tilde{\mu}$ is increasing by definition. In particular, it is locally integrable.

Now define $\hat{\mu} : (0,+\infty)^2 \to \mathbb{R}_+$ by setting for $C > 0$ and $\tau > 0$

$$\hat{\mu}(C,\tau) := \frac{1}{C\tau}\int_C^{2C}\int_\tau^{2\tau} \tilde{\mu}(r,s)drds + C\tau. \tag{2.13}$$

By construction, $\hat{\mu}$ is strictly increasing and continuous on $(0,+\infty) \times (0,+\infty)$. We can also enlarge the domain of definition of $\hat{\mu}$ to all of $\mathbb{R}_+^2$ using monotonicity. To this end we define for $\tau > 0$: $\hat{\mu}(0,\tau) := \lim_{C\to+0}\hat{\mu}(C,\tau)$ and for $C \geq 0$ we define $\hat{\mu}(C,0) := \lim_{\tau\to+0}\hat{\mu}(C,\tau)$. This is well-defined as $\hat{\mu}$ is increasing on $(0,+\infty)^2$ and we obtain that the resulting function is increasing on $\mathbb{R}_+^2$. Note that the construction does not guarantee that $\hat{\mu}$ is continuous. In order to obtain continuity choose a continuous, strictly increasing function $\nu : \mathbb{R}_+ \to \mathbb{R}_+$ with $\nu(r) > \max\{\hat{\mu}(0,r), \hat{\mu}(r,0)\}$, $r \geq 0$ and define for $(C,\tau) \geq (0,0)$

$$\mu(C,\tau) := \max\left\{\nu\big(\max\{C,\tau\}\big), \hat{\mu}(C,\tau)\right\} + C\tau. \tag{2.14}$$

It is easy to see that μ is continuous as $\mu(C,\tau) = \nu\big(\max\{C,\tau\}\big) + C\tau$ whenever C or τ is small enough. At the same time we have for $C > 0, \tau > 0$ that

$$\mu(C,\tau) \geq \hat{\mu}(C,\tau) \geq \frac{1}{C\tau}\int_C^{2C}\int_\tau^{2\tau} drds\ \tilde{\mu}(C,\tau) + C\tau \geq \tilde{\mu}(C,\tau).$$

This implies that (ii) holds with this μ.

(ii) $\Rightarrow$ (iii) is evident.

(iii) $\Rightarrow$ (i) follows due to continuity of μ. $\square$

Lemma 2.13. *Consider a forward complete system* $\Sigma = (X, \mathcal{D}, \phi)$*. The following statements are equivalent:*

(i) 0 is a robust equilibrium point of Σ*.*
(ii) For any $\tau \geq 0$ *there exists a* $\delta = \delta(\tau) > 0$ *such that the function* $\tilde{\mu}(\cdot,\tau) : [0,\delta) \to \mathbb{R}_+$ *defined by*

$$\tilde{\mu}(r,\tau) := \sup\{\|\phi(t,x,d)\|_X \mid \|x\|_X \leq r,\ t \in [0,\tau],\ d \in \mathcal{D}\}, \quad r \in [0,\delta) \tag{2.15}$$

is continuous at $r = 0$ *with* $\tilde{\mu}(0,\tau) = 0$*.*

Proof. (i) $\Rightarrow$ (ii). Let 0 be a robust equilibrium point of Σ. For $\tau = 0$ we have $\tilde{\mu}(r,0) = r$ by (Σ2) and there is nothing to show. By the assumption, for any $\tau > 0$ the function $\tilde{\mu}(\cdot,\tau)$ defined by (2.15) is well-defined on an interval $[0,\delta(\tau))$ with $\delta(\tau) > 0$. In addition, $\tilde{\mu}(0,\tau) = 0$ for any $\tau > 0$. Assume (ii) does not hold. Then there exists a $\tau^* > 0$ so that the corresponding function $\tilde{\mu}(\cdot,\tau^*)$, is not continuous at zero.

According to the definition, $\tilde{\mu}(\cdot,\tau^*)$ is nondecreasing on its domain of definition. Therefore $a := \lim_{r\to+0}\tilde{\mu}(r,\tau^*)$ exists. Then there exist sequences $\{t_k\}_{k\geq 1}\subset[0,\tau]$, $\{x_k\}_{k\geq 1}\subset X$ with $\|x_k\|_X\to 0$ as $k\to\infty$, and $\{d_k\}_{k\geq 1}\subset\mathcal{D}$ so that

$$\|\phi(t_k,x_k,d_k)\|_X \geq \frac{a}{2} \quad \forall k\geq 1.$$

This contradicts the fact that 0 is a robust equilibrium point of Σ.

(ii) $\Rightarrow$ (i). Pick any $\tau>0$ and any $\varepsilon>0$. Due to the continuity of $\tilde{\mu}(\cdot,\tau)$ at $r=0$ there exists a $\delta>0$ so that

$$\sup\{\|\phi(t,x,d)\|_X \mid \|x\|_X\leq\delta,\ t\in[0,\tau],\ d\in\mathcal{D}\}<\varepsilon,$$

which shows that 0 is a robust equilibrium point of Σ. □

Proposition 2.14. *Consider a forward complete system $\Sigma=(X,\mathcal{D},\phi)$. The following statements are equivalent:*

(i) Σ is robustly forward complete and 0 is a robust equilibrium point of Σ.
(ii) there exists a continuous, radially unbounded function $\mu:\mathbb{R}^2_+\to\mathbb{R}_+$ such that $\mu(\cdot,h)\in\mathcal{K}_\infty$ for all $h\geq 0$ and the implication (2.11) holds.
(iii) there exist $\sigma\in\mathcal{K}_\infty$ and a continuous function $\chi:\mathbb{R}^2_+\to\mathbb{R}_+$ such that $\chi(r,\cdot)\in\mathcal{K}$ for all $r>0$, $\chi(0,t)=0$ for all $t\in\mathbb{R}_+$ and such that for all $x\in X, d\in\mathcal{D}$ and all $t\geq 0$ we have

$$\|\phi(t,x,d)\|_X\leq\sigma(\|x\|_X)+\chi(\|x\|_X,t). \tag{2.16}$$

Proof. (i) $\Rightarrow$ (ii). Since Σ is RFC, Lemma 2.12 implies that there exists a continuous, increasing function $\mu:\mathbb{R}^2_+\to\mathbb{R}_+$ such that the implication (2.11) holds. It remains to show that we may choose μ such that $\mu(0,h)=0$ for all $h\geq 0$.

Consider the function $\tilde{\mu}:\mathbb{R}^2_+\to\mathbb{R}_+$ defined by (2.12). Since 0 is a robust equilibrium point of Σ, we have by Lemma 2.13 for all $\tau\geq 0$ that $\tilde{\mu}(0,\tau)=0$ and $r=0$ is a point of continuity of $\tilde{\mu}(\cdot,\tau)$. For $\hat{\mu}:\mathbb{R}^2_+\to\mathbb{R}_+$ defined by (2.13) it thus holds for all $C>0,h>0$ that

$$\hat{\mu}(C,h)\leq\tilde{\mu}(2C,2h)+Ch\to 0, \text{ as } C\to+0,$$

which shows that $\hat{\mu}(0,h)=0$ for every $h\geq 0$ and $r=0$ is a point of continuity of $\hat{\mu}(\cdot,h)$ for any $h\geq 0$.

Consider $\nu:\mathbb{R}_+\to\mathbb{R}_+$ from the proof of Lemma 2.12, which was chosen such that $\nu(r)>\max\{\hat{\mu}(0,r),\hat{\mu}(r,0)\}=\hat{\mu}(r,0)$ for all $r>0$. Since, by Lemma 2.13, $\hat{\mu}(r,0)\to 0$ as $r\to+0$, we may choose ν with $\nu(0)=0$, and thus $\nu\in\mathcal{K}$.

Now consider the following function $\mu:\mathbb{R}^2_+\to\mathbb{R}_+$ (which is different from that of (2.14)):

$$\mu(C,\tau):=\max\big\{\nu(C),\hat{\mu}(C,\tau)\big\}+C\tau. \tag{2.17}$$

Since ν is continuous on $\mathbb{R}_+$ and $\hat{\mu}$ is continuous over $[0,+\infty)\times(0,+\infty)$, μ is also continuous over $[0,+\infty)\times(0,+\infty)$. At the same time for τ small enough $\mu(C,\tau)=\nu(C)+C\tau$, and again it is not hard to show that μ is continuous over $\mathbb{R}^2_+$. Moreover, it is clear that μ satisfies item (ii) of Lemma 2.12.

It follows from the definition that $\mu(0,h)=0$. Since $\hat{\mu}(C,\tau)\geq C$ for any $C\geq 0$ and $\tau\geq 0$ (due to the identity axiom (Σ2)), and since both ν and $\hat{\mu}$ are increasing, $\mu(\cdot,h)\in\mathcal{K}_\infty$ for any $h>0$.

(ii) $\Rightarrow$ (iii). Let μ as in (ii) exist. Then for all $r \in \mathbb{R}_+$ and all $t \in \mathbb{R}_+$ we have that

$$\mu(r,t) = \underbrace{\mu(r,0)}_{\sigma(r)} + \underbrace{\mu(r,t) - \mu(r,0)}_{\chi(r,t)}.$$

Now (iii) is satisfied with these σ and χ.

(iii) $\Rightarrow$ (i). Let σ and μ be as in (iii). Then $\phi(t,0,d) = 0$ for all $t \geq 0$ and $d \in \mathcal{D}$. Now pick any $\varepsilon > 0$ and any $\tau > 0$. Set $\delta_1 := \sigma^{-1}(\frac{\varepsilon}{2})$ and choose δ_2 so that

$$\sup_{0 \leq r \leq \delta_2,\ 0 \leq t \leq \tau} \chi(r,t) = \sup_{0 \leq r \leq \delta_2} \chi(r,\tau) \leq \frac{\varepsilon}{2}.$$

Now define $\delta := \min\{\delta_1, \delta_2\}$. Then

$$t \in [0,\tau],\ \|x\|_X \leq \delta,\ d \in \mathcal{D} \quad \Rightarrow \quad \|\phi(t,x,d)\|_X \leq \varepsilon,$$

which shows that 0 is a robust equilibrium point of Σ.

Obviously, (iii) implies robust forward completeness of Σ. $\square$

Remark 2.15. In order to ensure the existence of $\sigma \in \mathcal{K}_\infty$ and χ as in item (iii) of Proposition 2.14, it is not sufficient to assume that Σ is RFC and 0 is an equilibrium point. Indeed, for the system from item (i) in Example 2.6 the function $\tilde{\mu}$ from (2.12) can be computed for all $C > 0$ and $\tau > 0$ as $\tilde{\mu}(C,\tau) = \max\{1, C\}$ and for all $C \geq 0$, $\tau \geq 0$ as $\tilde{\mu}(C,0) = C$, $\tilde{\mu}(0,\tau) = 0$. Clearly, one cannot majorize this function by any functions $\sigma \in \mathcal{K}_\infty$ and χ as in item (iii) of Proposition 2.14.

Remark 2.16. Setting $t := 0$ in (2.16) and using the identity axiom (Σ2) we see that for $\sigma \in \mathcal{K}_\infty$ in (2.16) it holds that $\sigma(r) \geq r$ for all $r \geq 0$. For some systems it is possible to choose $\sigma(r) := r$ for all $r \in \mathbb{R}_+$, but in general such a choice is not possible. Consider a linear system $\dot{x} = Ax$, where A is the generator of a C_0-semigroup $T(\cdot)$ over X, satisfying $\lim_{t \to +0} \|T(t)\|_{\mathcal{L}(X)} > 1$ (there are many examples of such semigroups). Then there exist a sequence $\{x_k\} \subset X$: $\|x_k\|_X = 1$ for all $k \in \mathbb{N}$ and a corresponding sequence of times $t_k \to +0$ as $k \to \infty$ so that $\|T(t_k)x_k\|_X > 1 + \varepsilon$ for some $\varepsilon > 0$. Hence we have

$$1 + \varepsilon < \lim_{k \to \infty} \|T(t_k)x_k\|_X \leq \lim_{k \to \infty} \big(\sigma(\|x_k\|_X) + \chi(\|x_k\|_X, t_k)\big) = \sigma(1).$$

3. Non-coercive Lyapunov theorems

Lyapunov functions provide a predominant tool to study UGAS. In our context they are defined as follows. Recall that for a continuous function $h : \mathbb{R} \to \mathbb{R}$ the (right-hand lower) Dini derivative at a point t is defined by, see [37],

$$D_+ h(t) := \underline{\lim}_{\tau \to +0} \frac{1}{\tau}\big(h(t+\tau) - h(t)\big). \tag{3.1}$$

Consider a system $\Sigma = (X, \mathcal{D}, \phi)$ and let $V : X \to \mathbb{R}$ be continuous. Given $x \in X, d \in \mathcal{D}$, we consider the (right-hand lower) Dini derivative of the continuous function $t \mapsto V(\phi(t,x,d))$ at $t = 0$:

$$\dot{V}_d(x) := \underline{\lim}_{t \to +0} \frac{1}{t}\big(V(\phi(t,x,d)) - V(x)\big). \tag{3.2}$$

We call this the Dini derivative of V along the trajectories of Σ.

Definition 3.1. A continuous function $V : X \to \mathbb{R}_+$ is called a *Lyapunov function* for system $\Sigma = (X, \mathcal{D}, \phi)$, if there exist $\psi_1, \psi_2 \in \mathcal{K}_\infty$ and $\alpha \in \mathcal{K}$ such that

$$\psi_1(\|x\|_X) \leq V(x) \leq \psi_2(\|x\|_X) \quad \forall x \in X \tag{3.3}$$

holds and the Dini derivative of V along the trajectories of Σ satisfies

$$\dot{V}_d(x) \leq -\alpha(\|x\|_X) \tag{3.4}$$

for all $x \in X$ and all $d \in \mathcal{D}$. We call V a *non-coercive Lyapunov function*, if instead of (3.3) we have $V(0) = 0$ and

$$0 < V(x) \leq \psi_2(\|x\|_X) \quad \forall x \in X \backslash \{0\}. \tag{3.5}$$

If we want to emphasize that (3.3) holds we will also speak of a coercive Lyapunov function. The following result is well-known:

Proposition 3.2. *Let $\Sigma = (X, \mathcal{D}, \phi)$ be a dynamical system. If there exists a coercive Lyapunov function for Σ, then 0 is UGAS.*

The proof of Proposition 3.2 is analogous to the proof of its finite-dimensional counterpart, see [29, p. 160]. Note however, that we use continuous Lyapunov functions and the trajectories of the system Σ are merely continuous, therefore we cannot use the standard comparison principle, see [29, Lemma 4.4] in the proof of Proposition 3.2. Instead one can exploit the following generalized comparison principle from [37, Lemma 6.1], [31, Lemma 1]:

Lemma 3.3. *Let $\alpha \in \mathcal{P}$ and consider the differential inequality*

$$\dot{y}(t) \leq -\alpha(y(t)), \quad t > 0. \tag{3.6}$$

There exists a $\beta \in \mathcal{KL}$ so that for all continuous functions $y : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying (3.6) *in the sense of the Dini derivative (defined in* (3.2)*) we have*

$$y(t) \leq \beta(y(0), t) \quad \forall t \geq 0. \tag{3.7}$$

Next we show that already the existence of a non-coercive Lyapunov function is sufficient for UGAS of a system provided another mild assumption is satisfied. To this end we need the following property of Dini derivatives.

Lemma 3.4. *Let $f, g : [0, \infty) \to \mathbb{R}$ be continuous. If for all $t \geq 0$ we have $D_+ f(t) \leq -g(t)$, then for all $t \geq 0$ it follows that*

$$f(t) - f(0) \leq -G(t) := -\int_0^t g(s)ds. \tag{3.8}$$

Proof. As g is continuous, it follows that $D_+ (f(t) + G(t)) = (D_+ f(t)) + g(t) \leq 0$ for all $t \geq 0$. It follows from [37, Theorem 2.1] that $f + G$ is decreasing. As $G(0) = 0$ the claim follows. □

Alternative arguments for this simple property may be found in [33, pp. 204-205], [17].

Theorem 3.5. ***(Non-coercive UGAS Lyapunov theorem)*** *Consider a system $\Sigma = (X, \mathcal{D}, \phi)$ and assume that Σ is robustly forward complete and 0 is a robust equilibrium of Σ. If V is a non-coercive Lyapunov function for Σ, then 0 is UGAS.*

Proof. Let V be a non-coercive Lyapunov function and let $\alpha \in \mathcal{K}$ be such that we have the decay estimate (3.4). Along any trajectory ϕ of Σ we have the inequality

$$\dot{V}_{d(t+\cdot)}(\phi(t,x,d)) \leq -\alpha(\|\phi(t,x,d)\|_X), \quad \forall t \geq 0. \tag{3.9}$$

It follows from Lemma 3.4 that

$$V(\phi(t,x,d)) - V(x) \leq -\int_0^t \alpha(\|\phi(s,x,d)\|_X) ds, \tag{3.10}$$

which implies that for all $t \geq 0$ we have

$$\int_0^t \alpha(\|\phi(s,x,d)\|_X) ds \quad \leq \quad V(x). \tag{3.11}$$

Step 1: (Stability) Seeking a contradiction, assume that Σ is not uniformly stable in $x^* = 0$. Then there exist an $\varepsilon > 0$ and sequences $\{x_k\}_{k\in\mathbb{N}}$ in X, $\{d_k\}_{k\in\mathbb{N}}$ in $\mathcal{D}$, and $t_k \geq 0$ such that $x_k \to 0$ as $k \to \infty$ and

$$\|\phi(t_k, x_k, d_k)\|_X = \varepsilon \quad \forall k \geq 1.$$

By the bound on V given by (3.5) it follows that $V(x_k) \to 0$.

Since Σ is RFC and 0 is a robust equilibrium point of Σ, Proposition 2.14 implies that there exist $\sigma \in \mathcal{K}_\infty$ and χ as in item (iii) of Proposition 2.14 so that (2.16) holds.

Appealing to continuity of χ we may choose $\tau > 0$ such that $\chi(r,\tau) \leq \varepsilon/2$ for all $0 \leq r \leq \varepsilon$.

Using (2.16) we obtain that for all $k \in \mathbb{N}$ and for all $t \in [t_k - \tau, t_k]$ we have either $\|\phi(t, x_k, d_k)\|_X > \varepsilon$ or

$$\sigma\big(\|\phi(t,x_k,d_k)\|_X\big) \geq \|\phi(t_k,x_k,d_k)\|_X - \chi\big(\|\phi(t,x_k,d_k)\|_X, t_k - t\big) \geq \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2}.$$

Setting $t := 0$ in (2.16) and using the identity axiom (Σ2) we see that $\sigma(r) \geq r$ for all $r \geq 0$, and thus $\sigma^{-1}(r) \leq r$ for all $r \in \mathbb{R}_+$.

Hence $\min\big\{\|\phi(s,x_k,d)\|_X \mid s \in [t_k - \tau, t_k]\big\} \geq \min\{\varepsilon, \sigma^{-1}(\frac{\varepsilon}{2})\} = \sigma^{-1}(\frac{\varepsilon}{2})$ and (3.11) implies for every k

$$V(x_k) \geq \int_{t_k-\tau}^{t_k} \alpha(\|\phi(s,x_k,d)\|_X) ds \geq \alpha \circ \sigma^{-1}\Big(\frac{\varepsilon}{2}\Big)\tau > 0.$$

This contradiction proves uniform stability of 0.

Step 2: (Uniform global attractivity) Again we assume that 0 is not uniformly globally attractive. This implies that there are $r, \varepsilon > 0$ and sequences $\{x_k\}_{k\in\mathbb{N}}$ in X, $\{d_k\}_{k\in\mathbb{N}}$ in $\mathcal{D}$ and times $t_k \to \infty$, as $k \to \infty$ such that

$$\|x_k\|_X \leq r \text{ and } \|\phi(t_k, x_k, d_k)\|_X \geq \varepsilon. \tag{3.12}$$

As we have already shown that 0 is uniformly stable we may choose for the above ε a $\delta = \delta(\varepsilon) > 0$ such that $\|z\|_X < \delta$ implies

$$\|\phi(t,z,d)\|_X \leq \frac{\varepsilon}{2} \quad \forall\, t \geq 0, \forall\, d \in \mathcal{D}. \tag{3.13}$$

Now assume that there exist a certain $k \in \mathbb{N}$ and $s_k \in [0, t_k]$ so that $\|\phi(s_k, x_k, d_k)\|_X \leq \delta$. Since Σ satisfies the cocycle property (Σ5), (3.13) and (3.12) lead us to

$$\varepsilon \leq \|\phi(t_k,x_k,d_k)\|_X = \big\|\phi\big(t_k - s_k, \phi(s_k,x_k,d_k), d_k(s+\cdot)\big)\big\|_X \leq \frac{\varepsilon}{2},$$

which is a contradiction. We conclude that for all k and all $t \in [0, t_k]$ we have

$$\|\phi(t, x_k, d_k)\|_X \geq \delta. \tag{3.14}$$

It then follows with (3.12), (3.5) and (3.11) that for all $k \geq 1$

$$\psi_2(r) \geq \psi_2(\|x_k\|_X) \geq V(x_k) \geq \int_0^{t_k} \alpha\left(\|\phi(t, x_k, d_k)\|_X\right) dt \geq \alpha(\delta) t_k. \tag{3.15}$$

As $t_k \to \infty$, this is a contradiction and hence 0 is uniformly globally attractive.

Since Σ is robustly forward complete and 0 is a robust equilibrium point of Σ, Proposition 2.10 ensures that 0 is UGAS. □

Theorem 3.6. ***(Non-coercive local UAS Lyapunov theorem)***
Consider a system $\Sigma = (X, \mathcal{D}, \phi)$ and assume that Σ is robustly forward complete and 0 is a robust equilibrium of Σ. Let V be a non-coercive Lyapunov function in the sense of Definition 3.1 and assume that α in (3.4) is positive definite. Then Σ is locally uniformly asymptotically stable in $x^* = 0$.

Proof. We just point out the necessary modifications in the proof of Theorem 3.5. The derivation of the integral bound (3.11) does not depend on a particular property of α, so that it also holds under the assumptions of the present theorem. To obtain stability we argue as in Step 1 of the previous proof. We only need to change the final estimate to obtain

$$V(x_k) \;\geq\; \int_{t_k-\tau}^{t_k} \alpha(\|\phi(s, x, d)\|_X) ds \;\geq\; \min\left\{\alpha(\xi) \;\middle|\; \xi \;\in\; [\sigma^{-1}\left(\frac{\varepsilon}{2}\right), \varepsilon]\right\}\tau \;>\; 0.$$

This again yields the contradiction which proves uniform stability.

Given that we have shown uniform stability, we may choose an $R > 0$ and a corresponding $r > 0$ such that all solutions with initial condition in B_r remain in B_R for all times. The proof of locally uniform attractivity uses precisely the same arguments as before but restricted to solutions with initial conditions in B_r. The only necessary change in the argument is then to replace the term $\alpha(\delta)t_k$ on the right hand side of (3.15) by

$$\min\{\alpha(\xi) \mid \xi \in [\delta, R]\} t_k,$$

which uses the invariance argument we have just noted. Again this expression is unbounded as $k \to \infty$ and we obtain the desired contradiction. □

4. Uniform weak attractivity

Theorem 3.5 shows that the existence of a non-coercive Lyapunov function implies UGAS in 0 provided Σ is robustly forward complete and 0 is a robust equilibrium of Σ. At the same time the examples in Section 7 show that the existence of a non-coercive Lyapunov function alone does not even imply forward completeness. And together with forward completeness it still does not imply robust forward completeness and robustness of the zero equilibrium, see [20, Remark 4] and equation (7.1) in this paper. Nevertheless, we show in this section that forward complete systems which fail to satisfy RFC or REP, may still enjoy nice stability properties if a non-coercive Lyapunov function for such systems exists. To obtain these results we first study the relation of Lagrange stability and attractivity.

The relation between uniform global attractivity and uniform global weak attractivity is given by the following criterion.

Proposition 4.1. *Consider a forward complete system $\Sigma = (X, \mathcal{D}, \phi)$. The fixed point 0 is a robust equilibrium and UGATT if and only if 0 is uniformly stable and uniformly globally weakly attractive.*

Proof. $\Rightarrow$. It follows directly from Definition 2.8 (vii) and (viii) that UGATT implies uniform global weak attractivity. To show uniform stability of 0 fix $\varepsilon > 0$ and $r > 0$. By Definition 2.8 (viii) there exists a $\tau = \tau(\varepsilon, r) > 0$ so that

$$\|x\|_X \leq r,\ d \in \mathcal{D},\ t \geq \tau(r, \varepsilon) \quad \Rightarrow \quad \|\phi(t, x, d)\|_X \leq \varepsilon.$$

Since 0 is a robust equilibrium of Σ, there exists a $\hat{\delta} > 0$ corresponding to τ so that

$$t \in [0, \tau],\ \|x\|_X \leq \hat{\delta},\ d \in \mathcal{D} \quad \Rightarrow \quad \|\phi(t, x, d)\|_X \leq \varepsilon.$$

The combination of the two implications shows that 0 is uniformly stable with $\delta := \min\{r, \hat{\delta}\}$.

$\Leftarrow$. Fix $\varepsilon > 0$. Since 0 is uniformly stable, there is a $\delta > 0$ so that (2.4) holds. This shows that 0 is a robust equilibrium point. Now pick any $r > 0$. Since 0 is a uniform global weak attractor, for the above δ there exists a $\tau = \tau(\delta, r)$ so that

$$\|x\|_X \leq r,\ d \in \mathcal{D} \quad \Rightarrow \quad \exists \bar{t} = \bar{t}(x, d, \delta) \leq \tau :\ \|\phi(t, x, d)\|_X \leq \delta.$$

Finally, the cocycle property and uniform stability imply that

$$\|x\|_X \leq r,\ d \in \mathcal{D},\ t \geq \bar{t}(x, d, \delta) \quad \Rightarrow \quad \|\phi(t, x, d)\|_X \leq \varepsilon.$$

Specifying this to $t \geq \tau(\varepsilon, r)$, we see that 0 is UGATT. $\square$

The following corollary summarizes several equivalent conditions of stability.

Corollary 4.2. *Consider a forward complete system $\Sigma = (X, \mathcal{D}, \phi)$. The following statements are equivalent:*

(i) 0 is UGAS.
(ii) Σ is RFC and 0 is UGATT and REP.
(iii) Σ is RFC and 0 is uniformly stable and uniformly globally weakly attractive.
(iv) 0 is uniformly globally stable and UGATT.
(v) 0 is uniformly globally stable and uniformly globally weakly attractive.

Proof. The equivalence of (i) and (ii) is a consequence of Proposition 2.10. Proposition 4.1 shows (ii) $\Leftrightarrow$ (iii). It is clear that global stability implies that 0 is a robust equilibrium point, so (iv) $\Leftrightarrow$ (v) again follows from Proposition 4.1. Finally, (iv) $\Rightarrow$ (ii) is obvious, as is (i) $\Rightarrow$ (v). $\square$

The following gives a sufficient condition for Lagrange stability in terms of uniform weak attractivity of 0 and robust forward completeness.

Proposition 4.3. *Assume that $\Sigma = (X, \mathcal{D}, \phi)$ is robustly forward complete and 0 is uniformly globally weakly attractive. Then 0 is Lagrange stable.*

Proof. Pick any $r > 0$ and set $\varepsilon := \frac{r}{2}$. Since 0 is uniformly globally weakly attractive, there exists a $\tau = \tau(r)$ so that

$$\|x\|_X \leq r,\ d \in \mathcal{D} \quad \Rightarrow \quad \exists t = t(x, d, r) \leq \tau(r) :\ \|\phi(t, x, d)\|_X \leq \varepsilon. \tag{4.1}$$

Without loss of generality we can assume that τ is increasing in r, in particular, it is locally integrable. Defining $\bar{\tau}(r) := \frac{1}{r}\int_r^{2r} \tau(s)ds$ we see that $\bar{\tau}(r) \geq \tau(r)$ and $\bar{\tau}$ is continuous.

Since Σ is RFC, by Lemma 2.12 there exists a continuous, increasing function $\mu : \mathbb{R}_+^2 \to \mathbb{R}_+$, such that for all $x \in X, d \in \mathcal{D}$ and all $t \geq 0$ we have (2.11). In particular,

$$\|x\|_X \leq r,\ d \in \mathcal{D},\ t \in [0, \tau(r)] \quad \Rightarrow \quad \|\phi(t,x,d)\|_X \leq \mu(r, \bar{\tau}(r)). \tag{4.2}$$

Since μ and $\bar{\tau}$ are both continuous and increasing, $\tilde{\sigma} : r \mapsto \mu(r, \bar{\tau}(r))$ is again continuous and increasing. Also from (4.1) and (4.2) it is clear that $\tilde{\sigma}(r) \geq \frac{r}{2}$ for any $r > 0$.

Seeking a contradiction, assume that there exist $x \in X$ with $\|x\|_X \leq r$, $d \in \mathcal{D}$ and $t \geq 0$ so that $\|\phi(t,x,d)\|_X > \tilde{\sigma}(r)$. Define $t_m := \sup\{s \in [0,t] \mid \|\phi(s,x,d)\|_X \leq r\}$ Note that t_m is well-defined, since (Σ2) implies that $\|\phi(0,x,d)\|_X = \|x\|_X \leq r$.

In view of the cocycle property, it holds that

$$\phi(t,x,d) = \phi\big(t - t_m, \phi(t_m,x,d), d(\cdot + t_m)\big),$$

and $d(\cdot + t_m) \in \mathcal{D}$ due to the axiom of shift invariance.

Assume that $t - t_m \leq \tau(r)$. Since $\|\phi(t_m,x,d)\|_X \leq r$, (4.2) implies that $\|\phi(t,x,d)\|_X \leq \tilde{\sigma}(r)$ for all $t \in [t_m, t]$. Otherwise, if $t - t_m > \tau(r)$, we may apply (4.1) and there exists $t^* < \tau(r)$, so that

$$\|\phi\big(t^*, \phi(t_m,x,d), d(\cdot + t_m)\big)\|_X = \|\phi(t^* + t_m, x, d)\|_X \leq \varepsilon = \frac{r}{2}.$$

This contradicts the definition of t_m, since $t_m + t^* < t$. Hence

$$\|x\|_X \leq r,\ d \in \mathcal{D},\ t \geq 0 \quad \Rightarrow \quad \|\phi(t,x,d)\|_X \leq \tilde{\sigma}(r). \tag{4.3}$$

Denote $\sigma(r) := \tilde{\sigma}(r) - \tilde{\sigma}(0)$, for any $r \geq 0$. Clearly, $\sigma \in \mathcal{K}_\infty$. Then from (4.3) we have for all $x \in X,\ d \in \mathcal{D},\ t \geq 0$ that

$$\|\phi(t,x,d)\|_X \leq \sigma(\|x\|_X) + \tilde{\sigma}(0), \tag{4.4}$$

which shows Lagrange stability of 0. □

It turns out that for a forward complete system the existence of a non-coercive Lyapunov function implies that 0 is uniformly globally weakly attractive.

Proposition 4.4. *Consider a forward complete system $\Sigma = (X, \mathcal{D}, \phi)$. If there exists a non-coercive Lyapunov function V for Σ, then 0 is uniformly globally weakly attractive.*

Proof. Let $\alpha \in \mathcal{K}$ and $\psi_2 \in \mathcal{K}_\infty$ be the functions characterizing the decay condition in (3.4) and the upper bound of V in (3.3), respectively. As in the proof of Theorem 3.5, forward completeness and the existence of a non-coercive Lyapunov function imply the existence of a $\psi_2 \in \mathcal{K}_\infty$ such that for all $x \in X, d \in \mathcal{D}, t \geq 0$

$$\int_0^t \alpha(\|\phi(s,x,d)\|_X) ds \leq V(x) \leq \psi_2(\|x\|_X). \tag{4.5}$$

Fix $\varepsilon > 0$ and $r > 0$ and define $\tau(\varepsilon, r) := \frac{\psi_2(r)+1}{\alpha(\varepsilon)}$. We claim that this choice of $\tau = \tau(\varepsilon, r)$ yields an appropriate time to conclude uniform weak attracticivity of 0. Assume to the contrary that there exist $x \in X$ with $\|x\|_X \leq r$ and some $d \in \mathcal{D}$ with the property $\|\phi(t,x,d)\|_X \geq \varepsilon$ for all $t \in [0, \tau(\varepsilon, r)]$. In view of (4.5) and since $\alpha \in \mathcal{K}$, we obtain

$$\psi_2(r) + 1 = \tau(\varepsilon, r)\alpha(\varepsilon) \leq \int_0^{\tau(\varepsilon,r)} \alpha(\|\phi(s,x,d)\|_X) ds \leq \psi_2(r),$$

a contradiction. □

As a corollary of Propositions 4.4 and 4.3 we obtain:

Corollary 4.5. *Consider a robustly forward complete system $\Sigma = (X, \mathcal{D}, \phi)$. If V is a non-coercive Lyapunov function for Σ, then 0 is Lagrange stable and uniformly globally weakly attractive.*

Remark 4.6. With Propositions 4.1 and 4.4 in mind, a shorter proof of Theorem 3.5 is possible. Consider an RFC system $\Sigma = (X, \mathcal{D}, \phi)$ which has 0 as a robust equilibrium and which admits a non-coercive Lyapunov function. In view of the first part of the proof of Theorem 3.5, these assumptions imply uniform stability of 0. At the same time Proposition 4.4 shows that 0 is uniformly globally weakly attractive. Now Corollary 4.2 shows that 0 is UGAS.

For linear infinite-dimensional systems without disturbances uniform weak attractivity of 0 can be characterized as follows:

Proposition 4.7. *Let T be a strongly continuous semigroup of linear bounded operators generated by the operator A. T is exponentially stable if and only if 0 is uniformly weakly attractive for $\dot{x} = Ax$.*

In particular, if there exists a non-coercive Lyapunov function V for $\dot{x} = Ax$ then by Corollary 4.5 and the previous considerations T is exponentially stable.

Proof. It is immediate from the definitions that exponential stability of T implies that 0 is uniformly weakly attractive.

Conversely, assume that 0 is uniformly weakly attractive for the C_0-semigroup T. It is well-known that every strongly continuous semigroup satisfies a bound of the type $\|T(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}$, $t \geq 0$ for suitable $M, \omega \in \mathbb{R}$, [32]. Hence it is robustly forward-complete. By Proposition 4.3, T is Lagrange stable. In particular, for any $x \in X$ it holds that $\sup_{t\geq 0} \|T(t)x\|_X < \infty$ and the Banach-Steinhaus theorem implies $\sup_{t\geq 0} \|T(t)\|_{\mathcal{L}(X)} < \infty$, which means that T is globally stable. Now Corollary 4.2 shows that $\dot{x} = Ax$ is UGAS. Finally, exponential stability of a strongly continuous semigroup is equivalent to UGAS of the underlying dynamical system, see e.g. [11, Lemma 1].

If there exists a non-coercive Lyapunov function we can now apply Corollary 4.5 to obtain exponential stability. □

Remark 4.8. Proposition 4.7 immediately shows that uniform weak attractivity is a stronger requirement than weak attractivity. In fact, every strongly stable C_0-semigroup T (a semigroup, satisfying $T(t)x \to 0$ as $t \to \infty$ for any $x \in X$), which is not exponentially stable is globally weakly attractive.

In contrast to Remark 4.8, for systems of nonlinear ordinary differential equations with uniformly bounded disturbances the notions of global weak attractivity and uniform global weak attractivity coincide. This is a direct consequence of an important result by Sontag and Wang ([35, Corollary III.3]):

Proposition 4.9. *Let D be a compact subset of $\mathbb{R}^m$, $f : \mathbb{R}^n \times D \to \mathbb{R}^n$ be locally Lipschitz continuous and $\mathcal{D} := L_\infty(\mathbb{R}_+, D)$. Assume the system*

$$\dot{x} = f(x, d) \tag{4.6}$$

has the fixed point $x^* = 0$. Then 0 is uniformly globally weakly attractive if and only if it is globally weakly attractive.

Proof. Pick any $r, \varepsilon > 0$. An application of [35, Corollary III.3] with $C := \overline{B_r}$, $\Omega := B_\varepsilon$, $K := \overline{B_{\frac{\varepsilon}{2}}} = \{x \in \mathbb{R}^n \mid |x| \le \frac{\varepsilon}{2}\}$ shows the existence of a $\tau(r, \varepsilon)$ such that for all $x \in C$ and any $d \in \mathcal{D}$ there exist $t \le \tau$ with $|\phi(t, x, d)| < \varepsilon$. □

Remark 4.10. It is easy to see that Proposition 4.9 is not valid for systems with disturbances, which are not uniformly bounded. Indeed, the system $\dot{x} = -\frac{1}{|d|+1}x$ with $\mathcal{D} = L_\infty(\mathbb{R}_+, \mathbb{R})$ has 0 as a global weak attractor but not in a uniform way.

5. Applications

In this section we give a few examples of system classes that are covered by our assumptions. Subsumed are, of course, the systems of ordinary differential equations with uniformly bounded disturbances studied in [35] and briefly introduced in Proposition 4.9. This class can be extended to systems of ordinary differential equations on Banach spaces using the tools described in [2]. We do not dwell on this and prefer to present examples in which the unboundedness of generators of C_0 semigroups may play a role.

5.1. **Homogeneous systems.** Some of the arguments can be simplified if the system exhibits a homogeneity property. We call a system $\Sigma = (X, \mathcal{D}, \phi)$ homogeneous in x (of order one) if for all $\lambda \ge 0$, all initial conditions $x \in X$, $d \in \mathcal{D}$ and $t \ge 0$ we have

$$\phi(t, \lambda x, d) = \lambda \phi(t, x, d).$$

In particular, systems which are linear in x are homogeneous in x. In the sequel we will simply speak of homogeneous systems. Some results on Lyapunov functions for homogeneous systems are already available in [13, Theorem 14.3].

For homogeneous systems robustness of the equilibrium point and robust forward completeness are equivalent.

Lemma 5.1. *Consider a homogeneous system $\Sigma = (X, \mathcal{D}, \phi)$. The equilibrium point $x^* = 0$ is robust if and only if Σ is robustly forward complete.*

Proof. Let 0 be a robust equilibrium of Σ. Then for the choice $\varepsilon = 1$ and any $\tau > 0$ there is a $\delta > 0$ so that

$$\|x\|_X \le \delta,\ d \in \mathcal{D},\ t \in [0, \tau] \quad \Rightarrow \quad \|\phi(t, x, d)\|_X \le 1. \tag{5.1}$$

Let $C > 0$ and $\tau > 0$ be arbitrary and choose a $\delta = \delta(\tau)$ such that (5.1) is satisfied. Consider $x \in B_C$ and an arbitrary $d \in \mathcal{D}$. If $\|x\|_X \le \delta$, then the solution is bounded by 1 on $[0, \tau]$ and there is nothing to show. Otherwise, let $\lambda := \delta / \|x\|_X$. Then we obtain from homogeneity and (5.1)

$$\|\phi(t, x, d)\|_X = \frac{1}{\lambda}\|\phi(t, \lambda x, d)\|_X \le \frac{\|x\|_X}{\delta} \le \frac{C}{\delta}.$$

This shows that Σ is RFC.

Conversely, assume that Σ is RFC and fix $\varepsilon > 0$ and $\tau > 0$. By assumption we have

$$\sup\{\|\phi(t, x, d)\|_X \mid \|x\|_X \le 1,\ t \in [0, \tau],\ d \in \mathcal{D}\} := \Delta < \infty. \tag{5.2}$$

If $\Delta \le \varepsilon$ we are done. Otherwise, let $\delta := \varepsilon / \Delta$. If $x \in X$ with $\|x\|_X \le \delta$ and $t \in [0, \tau]$ it follows for all $d \in \mathcal{D}$ that

$$\|\phi(t, x, d)\|_X = \delta \left\| \phi\left(t, \frac{1}{\delta}x, d\right) \right\|_X \le \frac{\varepsilon}{\Delta}\Delta = \varepsilon.$$

This shows robustness of the equilibrium point. □

Corollary 5.2. *Consider a homogeneous system $\Sigma = (X, \mathcal{D}, \phi)$ and assume that V is a non-coercive Lyapunov function in the sense of Definition 3.1 and assume that α in (3.4) is positive definite. If 0 is a robust equilibrium of Σ, then Σ is uniformly globally asymptotically stable in $x^* = 0$.*

Proof. By Lemma 5.1 Σ is RFC. We then have local uniform asymptotic stability by Theorem 3.5. Global uniform attractivity then follows by homogeneity using a simple scaling argument similar to those in the proof of Lemma 5.1. □

5.2. **Switched linear systems in Banach spaces.** This class of infinite-dimensional switched linear systems has been studied in [20]. Further results for the special case of switched linear delay systems are obtained in [19]. Here we briefly outline how to recover some of the results of [20] with the arguments proposed here. Let X be a Banach space. Consider a set of closed linear operators $\{A_q \mid q \in Q\}$ where Q is some index set. We assume that each A_q generates a C_0-semigroup T_q on X. Let

$$\mathcal{D} := \{d : \mathbb{R}_+ \to Q \mid d \text{ is piece-wise constant}\}, \tag{5.3}$$

where piece-wise constant means here that any half-open bounded interval $[a, b) \subset \mathbb{R}_+$ can be partitioned into finitely many half open intervals $[a_i, b_i)$ such that d is constant on each $[a_i, b_i)$. (The choice of using right-closed intervals is mere convention and nothing would change, if we were to use left-closed intervals instead.) We consider the family of differential equations

$$\dot{x} = A_{d(t)}x(t) \tag{5.4}$$

which generates evolution operators in the following manner. For $d \in \mathcal{D}$ and an interval $[s, t]$ with a partition $s = b_0 < b_1 < \ldots < b_k = t$ such that $d \equiv d_j \in Q$ on $[b_{j-1}, b_j)$, $j = 1, \ldots, k$ we set

$$\Phi_d(t, s) = T_{d_k}(t - b_{k-1})T_{d_{k-1}}(b_{k-1} - b_{k-2}) \ldots T_{d_1}(b_1 - s). \tag{5.5}$$

With this notation we have $\phi(t, x, d) = \Phi_d(t, 0)x$ for all $x \in X, d \in \mathcal{D}, t \geq 0$. It is easy to check that the conditions of Definition 2.1 are all satisfied. We also have the following lemma.

Lemma 5.3. *Consider the system $\Sigma = (X, \mathcal{D}, \phi)$ given by switched linear system (5.4) with $\mathcal{D}$ as defined in (5.3). The following statements are equivalent.*

(i) Σ is robustly forward complete.
(ii) $x^* = 0$ is a robust equilibrium point of Σ.
(iii) There exist constants $M \geq 1$, $\omega \in \mathbb{R}$ such that

$$\|\Phi_d(t, s)\| \leq Me^{\omega(t-s)} \quad \forall\, d \in \mathcal{D},\ t \geq s \geq 0. \tag{5.6}$$

Proof. The equivalence (i) ⇔ (ii) is an immediate consequence of Lemma 5.1.

(iii) ⇒ (i). If (iii) holds then we can define $\mu(C, \tau) := CMe^{\omega\tau}$ to see that for all solutions we have $\|\phi(t, x, d)\|_X \leq \mu(\|x\|_X, t)$. Thus Σ is RFC by Lemma 2.12.

(ii) ⇒ (iii). Fix $h > 0$. Using Lemma 2.13 there exists a $\delta > 0$ and a function $\tilde{\mu}(\cdot, h) : [0, \delta) \to \mathbb{R}_+$ such that for all $x \in X$ with $\|x\|_X = \delta/2$ we have

$$\|\Phi_d(t, 0)x\|_{\mathcal{L}(X)} \leq \tilde{\mu}\left(\frac{\delta}{2}, h\right) \quad \forall d \in \mathcal{D}, t \in [0, h]. \tag{5.7}$$

By linearity we thus obtain for all $x \in X$ a bound of $\|\Phi_d(t,0)x\|$ which is uniform in $d \in \mathcal{D}$. An application of the Banach-Steinhaus theorem yields that there is a constant $\tilde{M} > 0$ such that

$$\sup\left\{\|\Phi_d(t,0)\|_{\mathcal{L}(X)} \mid t \in [0,h], d \in \mathcal{D}\right\} \le \tilde{M}. \tag{5.8}$$

Take an arbitrary $d \in \mathcal{D}$ and $t \ge 0$. Write $t = kh + \tau$, where $\tau \in [0,h]$. For each $j = 1, \dots, k$ the shift $d(\cdot + jh) \in \mathcal{D}$ so that we obtain

$$\|\Phi_d(t,0)\|_{\mathcal{L}(X)} \le \|\Phi_{d(\cdot+kh)}(\tau,0)\|_{\mathcal{L}(X)} \prod_{j=0}^{k-1} \|\Phi_{d(\cdot+jh)}(h,0)\|_{\mathcal{L}(X)} \le \tilde{M}^{k+1}. \tag{5.9}$$

From the last inequality it is easy to arrive at the desired (5.6). □

Remark 5.4. An immediate consequence of the characterization (iii) of the previous lemma is that for linear switched systems the flow of Σ is Lipschitz continuous if and only if the system is robustly forward complete.

For switched linear systems on Banach space we thus obtain the following results

Corollary 5.5. *Consider a switched linear system $\Sigma = (X, \mathcal{D}, \phi)$ as described by* (5.3)-(5.5).

(i) If there exists a non-coercive Lyapunov function V for Σ then 0 *is uniformly globally weakly attractive.*

(ii) If Σ is RFC, then the following two statements are equivalent
   - *(a) there exists a non-coercive Lyapunov function V for Σ.*
   - *(b)* 0 *is uniformly globally asymptotically stable and hence exponentially stable.*

Proof. It is clear that switched linear systems as described by (5.3)-(5.5) are forward complete. Then Proposition 4.4 shows (i). The sufficiency part "(a) $\Rightarrow$ (b)" in (ii) follows from Theorem 3.5 and Lemma 5.3. Necessity is a consequence of Theorem 6.6. □

While item (i) of the previous result is new (to the best knowledge of the authors), item (ii) recovers some of the results of Theorem 3 in [20]. We note that Remark 4 in [20] also shows that even for this system class the assumption of robust forward completeness cannot be removed in order to conclude uniform global asymptotic stability. This will be discussed in the examples of Section 7. A further property is that the Lyapunov function may in fact be chosen to be a norm on X (or the square of a norm, which really makes no difference). In the non-coercive case, this norm is of course not equivalent to the original norm.

5.3. **Strongly continuous semigroups with admissible input operators.** Let X be a Banach space and A be the generator of a C_0-semigroup $T(\cdot)$ on X. Consider the extrapolation space X_{-1} defined as the completion of X with respect to the norm

$$\|x\|_{-1} := \|(sI - A)^{-1}x\|_X \,, \tag{5.10}$$

where s is an arbitrary element of the resolvent set of A. In this case $T(\cdot)$ extends to a C_0-semigroup on X_{-1} which we denote by the same symbol. Consider a linear operator $B \in \mathcal{L}(U, X_{-1})$, which may therefore be unbounded as an operator from

U to X. Given the space of input functions $L^p(\mathbb{R}_+, U)$, the operator B is called p-admissible if for all $t > 0$ the map

$$G_t : u(\cdot) \mapsto \int_0^t T(t-s)u(s)ds \tag{5.11}$$

defines a bounded operator from $L^p(\mathbb{R}_+, U)$ to X. Note that the integral is a priori defined in X_{-1} and it is a requirement that the integral yields a value in X. In the particular case that $p = \infty$ we require the further condition that $\|G_t\|_{\mathcal{L}(L^\infty(\mathbb{R}_+,U),X)} \to 0$ as $t \to +0$. If this is the case then B is called zero-class admissible [24].

Consider a linear system

$$\dot{x} = Ax + Bu, \tag{5.12}$$

where A is the generator of a C_0-semigroup over the Banach space X, and B is admissible for A. It is known that if B is p-admissible for $1 \le p < \infty$ or ∞-admissible and zero-class, then the solutions $\phi(\cdot, x, d)$ are continuous, due to an argument similar to [39, Proposition 2.3].

This system class gives rise to a system with disturbances if we consider uncertain time-varying feedbacks. In this way we can study unbounded perturbations of the generator A, see e.g. [15]. To this end consider a system of the form

$$\dot{x} = Ax + B\Delta(t)Cx, \tag{5.13}$$

where (A, B) is (for instance) ∞-admissible and zero-class, $C \in \mathcal{L}(X, Y)$, $D \subset \mathcal{L}(Y, U)$ and the set of disturbances satisfies

$$\mathcal{D} = \{\Delta : [0, \infty) \to D \mid \Delta \text{ is piecewise continuous } \}.$$

We consider mild solutions to (5.13), i.e. solutions to the integral equation

$$\phi(t, x_0, \Delta) = T(t)x_0 + \int_0^t T(t-s)B\Delta(s)Cx(s)ds.$$

Conditions for the existence of such solutions are discussed in [22, 23, 6]. Here we will assume that (5.13) defines a system $\Sigma = (X, \mathcal{D}, \phi)$.

System (5.13) is homogeneous, so that we obtain that provided the equilbrium $x^* = 0$ is robust, the existence of a non-coercive Lyapunov function guarantees uniform global asymptotic stability of 0.

5.4. **Nonlinear evolution equations in Banach spaces.** We consider nonlinear infinite-dimensional systems of the form

$$\dot{x}(t) = Ax(t) + g(x(t), d(t)), \;\; x(t) \in X, \;\; d(t) \in D, \tag{5.14}$$

where A generates the C_0-semigroup $T(\cdot)$ of bounded operators, X is a Banach space and D is a nonempty subset of a normed linear space. As the space of admissible inputs we consider the space $\mathcal{D}$ of globally bounded, piecewise continuous functions from $\mathbb{R}_+$ to D.

Assumption 1. *The function $g : X \times D \to X$ is Lipschitz continuous on bounded subsets of X, uniformly with respect to the second argument, i.e. for all $C > 0$ there exists $L_f(C) > 0$, such that for all x, y with $\|x\|_X \le C$, $\|y\|_X \le C$ and $d \in D$ it holds that*

$$\|g(y, d) - g(x, d)\|_X \le L_f(C)\|y - x\|_X. \tag{5.15}$$

Assume also that $g(x, \cdot)$ is continuous for all $x \in X$.

We consider mild solutions of (5.14), i.e. solutions of the integral equation

$$x(t) = T(t)x(0) + \int_0^t T(t-s)g(x(s), d(s))ds \tag{5.16}$$

belonging to the class $C([0,\tau], X)$ for certain $\tau > 0$.

It is well known that the system (5.14) is well-posed if Assumption 1 is satisfied. Moreover it satisfies all the axioms of the Definition 2.1, possibly with exception of forward completeness. Thus, (5.14) gives rise to a control system $\Sigma = (X, \mathcal{D}, \phi)$.

We are going to show that for system (5.14) some of the assumptions of Theorem 3.5 are automatically satisfied.

Lemma 5.6. *Assume that*

(i) (5.14) *is robustly forward complete,*
(ii) Assumption 1 holds.

Then (5.14) *has a flow which is Lipschitz continuous on compact intervals.*

Proof. Let $C > 0$, $d \in \mathcal{D}$, and x_1^0, x_2^0 with $\|x_i^0\|_X \leq C$, $i = 1,2$. Denote $x_i(t) := \phi(t, x_i^0, d)$, $i = 1,2$ be the solutions of (5.14) which are defined for $t \geq 0$ since we assume that (5.14) is forward complete.

There exist $M > 0, \lambda \in \mathbb{R}$ such that $\|T(t)\| \leq Me^{\lambda t}$ for all $t \geq 0$. Pick any $\tau > 0$ and set

$$K(C,\tau) := \sup_{\|x\|_X \leq C, d \in \mathcal{D}, t \in [0,\tau]} \|\phi(t,x,d)\|_X,$$

which is finite due to assumption (i). For any $t \in [0,\tau]$ we have that

$$\begin{aligned}\|x_1(t) - x_2(t)\|_X &\leq \|T(t)\|\|x_1^0 - x_2^0\|_X \\ &\quad + \int_0^t \|T(t-r)\|\|g(x_1(r), d(r)) - g(x_2(r), d(r))\|_X dr \\ &\leq Me^{\lambda t}\|x_1^0 - x_2^0\|_X + \int_0^t Me^{\lambda(t-r)} L(K(C,\tau))\|x_1(r) - x_2(r)\|_X dr.\end{aligned}$$

Now define $z_i(t) := e^{-\lambda t}x_i(t)$, $i = 1, 2$, $t \geq 0$. Then

$$\|z_1(t) - z_2(t)\|_X \leq M\|x_1^0 - x_2^0\|_X + ML(K(C,\tau)) \int_0^t \|z_1(r) - z_2(r)\|_X dr.$$

According to Gronwall's inequality we obtain

$$\|z_1(t) - z_2(t)\|_X \leq M\|x_1^0 - x_2^0\|_X e^{ML(K(C,\tau))t}$$

or in the original variables, for $t \in [0,\tau]$

$$\|\phi(t, x_1^0, d) - \phi(t, x_2^0, d)\|_X \leq \|x_1^0 - x_2^0\|_X e^{(ML(K(C,\tau))+\lambda)\tau}.$$

which proves the lemma. □

Summarizing the results of Theorem 3.5, Lemma 5.6 and Lemma 2.7, we obtain:

Corollary 5.7. *Let Assumption 1 be satisfied. Let* (5.14) *be robustly forward complete and let* 0 *be an equilibrium of* (5.14)*. If there exists a non-coercive Lyapunov function for* (5.14)*, then* (5.14) *is UGAS.*

Proof. Lemma 5.6 and RFC property of (5.14) imply that the flow of (5.14) is Lipschitz continuous on compact intervals. Next Lemma 2.7 implies that 0 is a robust equilibrium point of (5.14). Finally, Theorem 3.5 shows that (5.14) is UGAS. □

6. Converse theorems

In this section we consider two constructions of Lyapunov functions for UGAS systems $\Sigma = (X, \mathcal{D}, \phi)$. First we recall a construction of Lipschitz continuous coercive Lyapunov functions, due to [25], and then we use these ideas in order to derive another construction of a non-coercive Lyapunov function for a system with a UGAS equilibrium. The construction requires Lipschitz continuity of the flow - a condition we have hardly used so far. For a detailed discussion of the properties required of the set of solutions of a system to obtain a converse Lyapunov theorem yielding continuous Lyapunov functions we refer to [34].

Both of the constructions exploit Sontag's $\mathcal{KL}$-lemma [36, Proposition 7], which can be stated as follows:

Lemma 6.1. *For each $\beta \in \mathcal{KL}$ there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that*

$$\beta(r,t) \leq \alpha_2(\alpha_1(r)e^{-t}) \quad \forall r \geq 0,\ \forall t \geq 0. \tag{6.1}$$

We will need several technical lemmas, which are well-known.

Lemma 6.2. *Let $f, g : \mathcal{D} \to \mathbb{R}_+$ be maps such that $\sup_{d\in\mathcal{D}} f(d)$ and $\sup_{d\in\mathcal{D}} g(d)$ are finite. Then*

$$\sup_{d\in\mathcal{D}} f(d) - \sup_{d\in\mathcal{D}} g(d) \leq \sup_{d\in\mathcal{D}} \big(f(d) - g(d)\big). \tag{6.2}$$

We note that Lemma 6.2 does not claim that the expression on the right in (6.2) is finite. The following lemma is taken from [25, p.130].

Lemma 6.3. *For any $\alpha \in \mathcal{K}_\infty$ there exist $\rho \in \mathcal{K}_\infty$ so that $\rho(s) \leq \alpha(s)$ for all $s \in \mathbb{R}_+$ and ρ is globally Lipschitz with a unit Lipschitz constant, i.e. for any $s_1, s_2 \geq 0$ it holds that*

$$|\rho(s_1) - \rho(s_2)| \leq |s_1 - s_2|. \tag{6.3}$$

We will also need the functions $G_k : \mathbb{R}_+ \to \mathbb{R}_+$ defined, for $k \in \mathbb{N}$, by

$$G_k(r) := \max\{r - \frac{1}{k}, 0\}, \quad r \geq 0.$$

Lemma 6.4. *For each $k \in \mathbb{N}$ the function G_k is Lipschitz continuous with a unit Lipschitz constant, i.e., for all $r_1, r_2 \geq 0$ it holds that*

$$\big|G_k(r_1) - G_k(r_2)\big| \leq |r_1 - r_2|. \tag{6.4}$$

Proof. This follows as each G_k is the maximum of two Lipschitz continuous functions with Lipschitz constant (at most) 1. □

The following theorem has been shown in [25, Section 3.4].

Theorem 6.5. *Consider a system $\Sigma = (X, \mathcal{D}, \phi)$ with a flow, which is Lipschitz continuous on compact intervals. If 0 is UGAS, then for any $\eta \in (0,1)$ there exists a sequence of positive real numbers $\{a_k\}_{k\in\mathbb{N}}$ so that $W^\eta : X \to \mathbb{R}_+$, defined by*

$$W^\eta(x) := \sum_{k=1}^{\infty} a_k V_k^\eta(x) \qquad \forall x \in X, \tag{6.5}$$

with

$$V_k^\eta(x) := \sup_{d\in\mathcal{D}} \max_{s\in[0,+\infty)} e^{\eta s} G_k\big(\rho(\|\phi(s,x,d)\|_X)\big). \tag{6.6}$$

is a coercive Lyapunov function for Σ *which is Lipschitz continuous on bounded balls.*

The proof of this result is based on the earlier local converse Lyapunov theorems, see e.g. [42, Theorem 19.3], [21, Theorem 4.2.1] and on using Sontag's $\mathcal{KL}$-Lemma (Lemma 6.1).

Next, exploiting ideas from [25, Section 3.4] we construct a Lipschitz continuous non-coercive Lyapunov function using integration instead of maximization.

Theorem 6.6. *Consider a forward complete system* $\Sigma = (X, \mathcal{D}, \phi)$ *with a flow, which is Lipschitz continuous on compact intervals. If* 0 *is UGAS, then there exists a non-coercive Lyapunov function for* Σ *which is Lipschitz continuous on bounded balls.*

Proof. Let the system $\Sigma = (X, \mathcal{D}, \phi)$ be UGAS in $x^* = 0$. Then there exists a $\beta \in \mathcal{KL}$, so that (2.5) holds for all $x \in X$, all $d \in \mathcal{D}$ and all $t \geq 0$. Due to Lemma 6.1 there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ so that (6.1) holds.

Pick any Lipschitz continuous function $\rho \in \mathcal{K}_\infty$ with a unit Lipschitz constant so that $\rho(r) \leq \alpha_2^{-1}(r)$, $r \in \mathbb{R}_+$. This is possible due to Lemma 6.3. For any $k \in \mathbb{N}$ define $G_k(r) := \max\{r - \frac{1}{k}, 0\}$ and consider the function

$$V_k(x) = \sup_{d\in\mathcal{D}} \int_0^\infty G_k\big(\rho\big(\|\phi(t,x,d)\|_X\big)\big)dt. \tag{6.7}$$

For all $x \in X$ it holds that

$$\begin{aligned} V_k(x) &\leq \sup_{d\in\mathcal{D}} \int_0^\infty \rho\big(\|\phi(t,x,d)\|_X\big)dt \\ &\leq \int_0^\infty \alpha_2^{-1}\big(\beta(\|x\|_X, t)\big)dt \\ &\leq \int_0^\infty \alpha_1(\|x\|_X)e^{-t}dt \\ &= \alpha_1(\|x\|_X). \end{aligned}$$

Clearly, $V_k(x) \geq 0$ for all $x \in X$. Moreover, if $\|x\|_X > \rho^{-1}(\frac{1}{k})$ holds, then, due to the continuity axiom (Σ4) for any given $d \in \mathcal{D}$ there exists $t_0 = t_0(d) > 0$ so that $\|\phi(t,x,d)\|_X > \rho^{-1}(\frac{1}{k})$ for all $t \in [0, t_0)$. Thus $G_k\big(\rho\big(\|\phi(t,x,d)\|_X\big)\big) > 0$ for all $t \in [0, t_0)$, which implies that $V_k(x) > 0$ provided that $\|x\|_X > \rho^{-1}(\frac{1}{k})$.

The Dini derivative of V_k at any $x \in X$ for $v \in \mathcal{D}$ is given by

$$\begin{aligned} \dot{V}_{kv}(x) &= \overline{\lim_{h\to+0}} \frac{1}{h}\Big(V_k\big(\phi(h,x,v)\big) - V_k(x)\Big) \\ &= \overline{\lim_{h\to+0}} \frac{1}{h}\Big(\sup_{d\in\mathcal{D}} \int_0^\infty G_k\big(\rho(\|\phi(t,\phi(h,x,v),d)\|_X)\big)dt \\ &\qquad - \sup_{d\in\mathcal{D}} \int_0^\infty G_k\big(\rho(\|\phi(t,x,d)\|_X)\big)dt\Big) \\ &= \overline{\lim_{h\to+0}} \frac{1}{h}\Big(\sup_{d\in\mathcal{D}} \int_0^\infty G_k\big(\rho(\|\phi(t+h,x,\tilde{d})\|_X)\big)dt \\ &\qquad - \sup_{d\in\mathcal{D}} \int_0^\infty G_k\big(\rho(\|\phi(t,x,d)\|_X)\big)dt\Big), \end{aligned}$$

where the disturbance function $\tilde{d}$ is defined as

$$\tilde{d}(t) := \begin{cases} v(t), & \text{if } t \in [0,h] \\ d(t-h) & \text{otherwise.} \end{cases}$$

The supremum in the first integral is taken over disturbances of a particular form. The supremum cannot decrease, if we allow general disturbances and hence

$$\begin{aligned} \dot{V}_{kv}(x) &\leq \overline{\lim_{h\to+0}} \frac{1}{h}\Big(\sup_{d\in\mathcal{D}} \int_0^\infty G_k\big(\rho(\|\phi(t+h,x,d)\|_X)\big)dt \\ &\qquad - \sup_{d\in\mathcal{D}} \int_0^\infty G_k\big(\rho(\|\phi(t,x,d)\|_X)\big)dt\Big) \\ &\leq \overline{\lim_{h\to+0}} \frac{1}{h}\Big(\sup_{d\in\mathcal{D}} \int_h^\infty G_k\big(\rho(\|\phi(t,x,d)\|_X)\big)dt \\ &\qquad - \sup_{d\in\mathcal{D}} \int_0^\infty G_k\big(\rho(\|\phi(t,x,d)\|_X)\big)dt\Big). \end{aligned}$$

Using the inequality (6.2) we obtain for all $v \in \mathcal{D}$ that

$$\begin{aligned} \dot{V}_{kv}(x) &\leq \overline{\lim_{h\to+0}} \frac{1}{h} \sup_{d\in\mathcal{D}} \Big(\int_h^\infty G_k\big(\rho(\|\phi(t,x,d)\|_X)\big)dt \\ &\qquad - \int_0^\infty G_k\big(\rho(\|\phi(t,x,d)\|_X)\big)dt\Big) \\ &= -\overline{\lim_{h\to+0}} \frac{1}{h} \sup_{d\in\mathcal{D}} \int_0^h G_k\big(\rho(\|\phi(t,x,d)\|_X)\big)dt \\ &\leq -\overline{\lim_{h\to+0}} \frac{1}{h} \int_0^h G_k\big(\rho(\|\phi(t,x,0)\|_X)\big)dt \\ &= -G_k\big(\rho(\|x\|_X)\big), \end{aligned}$$

where the final equality holds due to axiom (Σ4) and by continuity of G_k and ρ.

Now we show that the V_k are Lipschitz continuous on bounded balls. Since 0 is UGAS, it holds for any $d \in \mathcal{D}$, $x \in X$ and $t \geq 0$ that

$$\rho\big(\|\phi(t,x,d)\|_X\big) \leq \alpha_2^{-1}\big(\beta(\|x\|_X,t)\big) \leq e^{-t}\alpha_1(\|x\|_X). \tag{6.8}$$

Pick any $R > 0$ and $x \in X$ with $\|x\|_X \leq R$. Then for any $d \in \mathcal{D}$ and for any $t \geq T(R,k) := \ln(1 + k\alpha_1(R))$ it holds that

$$\rho(\|\phi(t,x,d)\|_X) \leq \frac{1}{k}.$$

Thus, the domain of integration in the definition of V_k has a finite length, i.e. for $R > 0$ and all $x \in X$ with $\|x\|_X \leq R$ the function V_k can be equivalently defined by

$$V_k(x) = \sup_{d\in\mathcal{D}} \int_0^{T(R,k)} G_k\big(\rho(\|\phi(t,x,d)\|_X)\big)dt.$$

Now pick any $x, y \in X$ such that $\|x\|_X \leq R$ and $\|y\|_X \leq R$ and consider

$$\begin{aligned}
|V_k(x) - V_k(y)| &= \Big| \sup_{d\in\mathcal{D}} \int_0^{T(R,k)} G_k\big(\rho\big(\|\phi(t,x,d)\|_X\big)\big)dt \\
&\qquad - \sup_{d\in\mathcal{D}} \int_0^{T(R,k)} G_k\big(\rho\big(\|\phi(t,y,d)\|_X\big)\big)dt\Big| \\
&\leq \sup_{d\in\mathcal{D}} \Big| \int_0^{T(R,k)} G_k\big(\rho\big(\|\phi(t,x,d)\|_X\big)\big)dt \\
&\qquad - \int_0^{T(R,k)} G_k\big(\rho\big(\|\phi(t,y,d)\|_X\big)\big)dt\Big| \\
&\leq \sup_{d\in\mathcal{D}} \int_0^{T(R,k)} \Big|G_k\big(\rho\big(\|\phi(t,x,d)\|_X\big)\big) \\
&\qquad - G_k\big(\rho\big(\|\phi(t,y,d)\|_X\big)\big)\Big|dt \\
&\leq \sup_{d\in\mathcal{D}} \int_0^{T(R,k)} \|\phi(t,x,d) - \phi(t,y,d)\|_X dt.
\end{aligned}$$

Exploiting the Lipschitz continuity of the flow on compact intervals we obtain

$$\begin{aligned}
|V_k(x) - V_k(y)| &\leq \sup_{d\in\mathcal{D}} \int_0^{T(R,k)} L(R,k)\|x-y\|_X dt \\
&= T(R,k)L(R,k)\|x-y\|_X,
\end{aligned}$$

where $L(R,k)$ is strictly increasing in both arguments. Define $M(R,k) := T(R,k)L(R,k)$, which is also strictly increasing in both arguments.

Define $W : X \to \mathbb{R}_+$ by

$$W(x) := \sum_{k=1}^{\infty} \frac{2^{-k}}{1+M(k,k)} V_k(x) \qquad \forall x \in X. \tag{6.9}$$

We have $W(x) \leq \alpha_1(\|x\|_X)$ for all $x \in X$. Since $V_k(x) > 0$ if $\|x\|_X > \rho^{-1}(\frac{1}{k})$ we obtain that $W(x) = 0 \Leftrightarrow x = 0$.

Differentiating W along the trajectory, we obtain:

$$\dot{W}(x) = \sum_{k=1}^{\infty} \frac{2^{-k}}{1+M(k,k)} \dot{V}_k(x) \leq -\sum_{k=1}^{\infty} \frac{2^{-k}}{1+M(k,k)} G_k(\rho(\|x\|_X)) \leq -\psi_1(\|x\|_X),$$

where $\psi_1 \in \mathcal{K}_\infty$ is defined for any $r \geq 0$ as

$$\psi_1(r) := \sum_{k=1}^{\infty} \frac{2^{-k}}{1+M(k,k)} G_k\big(\rho(r)\big). \tag{6.10}$$

Now pick any $R > 0$ and $x, y \in X$ such that $\|x\|_X \leq R$ and $\|y\|_X \leq R$. Then it holds that

$$\begin{aligned}\big|W^\eta(x) - W^\eta(y)\big| &= \Big|\sum_{k=1}^{\infty} \frac{2^{-k}}{1+M(k,k)}\big(V_k^\eta(x) - V_k^\eta(y)\big)\Big| \\ &\leq \sum_{k=1}^{\infty} \frac{2^{-k}M(R,k)}{1+M(k,k)}\|x-y\|_X \\ &\leq \Big(1 + \sum_{k=1}^{[R]+1} \frac{2^{-k}M(R,k)}{1+M(k,k)}\Big)\|x-y\|_X.\end{aligned}$$

This shows that W^η is a Lyapunov function for Σ which is Lipschitz continuous on bounded balls. □

To conclude the developments of this paper, we state the following characterization of uniform global asymptotic stability.

Corollary 6.7. *Consider a system $\Sigma = (X, \mathcal{D}, \phi)$ with a Lipschitz continuous flow. Assume Σ is RFC and 0 is a robust equilibrium of Σ. Then the following statements are equivalent:*

(i) 0 *is UGAS.*
(ii) 0 *is UGATT.*
(iii) *there exists a coercive Lyapunov function for Σ.*
(iv) *there exists a non-coercive Lyapunov function for Σ.*

7. Examples

In this section we discuss the limits of the results which can be obtained using non-coercive Lyapunov functions. Theorem 3.5 names three conditions for the conclusion of uniform global asymptotic stability: (i) the fixed point should be robust, (ii) the system should be robustly forward complete, (iii) there exists a non-coercive Lyapunov function. Contrary to the coercive case where the existence of a Lyapunov function implies REP and RFC, the existence of a non-coercive Lyapunov function does not imply either REP or RFC. This is a consequence of [20, Remark 4][2].

Example 7.1. In this example we show that even for finite-dimensional undisturbed systems the following properties are possible: (i) the system has a unique fixed point, (ii) there exist non-coercive Lyapunov functions which satisfy the decay condition (3.4), (iii) the fixed point is not globally asymptotically stable.

Since non-coercive Lyapunov functions for a finite-dimensional system are coercive in the neighborhood of zero, existence of such a Lyapunov function implies

[2] We note that a few modifications are necessary in [20, Remark 4] in order to obtain the desired example. Most importantly, the semigroups $T_j(t)$ on $X = L^p(0,1)$, $j \in \mathbb{N}$ should be defined by setting for $f \in X$

$$(T_j(t)f)(s) := \begin{cases} 2^{\frac{1}{p}} f(s+t), & \text{if } s \in [0, 1-t] \cap [4^{-j} - t, 4^{-j}), \\ f(s+t), & \text{if } s \in [0,1-t] \backslash [4^{-j} - t, 4^{-j}), \\ 0, & \text{if } s \in (1-t, 1] \cap [0,1]. \end{cases} \tag{7.1}$$

that the fixed point is locally asymptotically stable. Also it is impossible that solutions which are not attracted to the fixed point exist for all positive times, as then Theorem 3.5 would imply global asymptotic stability (for undisturbed ODE systems forward completeness and robust forward completeness are equivalent, see e.g. [29, Proposition 5.1]).

Consider a strictly increasing, bounded, Lipschitz continuous function $\delta : \mathbb{R} \to \mathbb{R}$. Assume furthermore that $\delta(x)x > 0$ for $x \neq 0$, and for the sake of convenience that $\lim_{x\to-\infty} |\delta(x)| = 1 = \sup_{x\in\mathbb{R}} |\delta(x)|$. Define $\varepsilon_1 = 2\delta$, $\varepsilon_2 = (1/2)\delta$, which implies in particular that

$$\lim_{x\to-\infty} |\varepsilon_1(x)| > 1 > \sup_{x\in\mathbb{R}} |\varepsilon_2(x)| . \tag{7.2}$$

With these data consider the system

$$\begin{aligned} \dot{x}_1 &= -\varepsilon_1(x_1) - \delta(x_2), \\ \dot{x}_2 &= \quad \delta(x_1) - \varepsilon_2(x_2). \end{aligned} \tag{7.3}$$

It is easy to check that the system (7.3) is globally asymptotically stable by considering the Lyapunov function V_1 given by $V_1(x_1, x_2) = \int_0^{x_1} \delta(r)dr + \int_0^{x_2} \delta(r)dr$. In this case we see that

$$\dot{V}_1(x_1, x_2) = -\varepsilon_1(x_1)\delta(x_1) - \varepsilon_2(x_2)\delta(x_2),$$

which is negative for $x = (x_1, x_2) \neq 0$. Now choose a smooth $\rho \in \mathcal{K}$ with positive derivative and so that $\lim_{r\to\infty} \rho(r) = 1$. The function $V_2 := \rho \circ V_1$ is then also a Lyapunov function with compact sublevel sets on its image and such that $\dot{V}_2(x) = \rho'(V_1(x))\dot{V}_1(x) < 0$ for $x \neq 0$.

We will now extend the Lyapunov function so that the limit for $x_1 \to -\infty$ has a certain increase property. To this end consider a decreasing smooth function $\eta : \mathbb{R} \to [0, 1]$ with support in $(-\infty, -c]$ for $c > 0$. We will choose c sufficiently large later on. We also assume that $\lim_{x\to-\infty} \eta(x) = 1$. Now consider the function

$$W(x_1, x_2) = \eta(x_1)\left(1 + \arctan(x_2)\right).$$

Along the solution of (7.3) we have

$$\dot{W}(x_1, x_2) = -\eta'(x_1)\left(1 + \arctan(x_2)\right)\left(\varepsilon_1(x_1) + \delta(x_2)\right) + \frac{\eta(x_1)}{1 + x_2^2}\left(\delta(x_1) - \varepsilon_2(x_2)\right)$$

We note that the factors $-\eta'(x_1)\left(1 + \arctan(x_2)\right)$ and $\frac{\eta(x_1)}{1+x_2^2}$ are both nonnegative. Also $\dot{W}(x_1, x_2) = 0$ for $x_1 > -c$. Using (7.2) we now choose $c > 0$ such that

$$\varepsilon_1(-c) < -1 \quad \text{and} \quad \delta(-c) < -\sup_{x\in\mathbb{R}} |\varepsilon_2(x)| . \tag{7.4}$$

With this choice and our assumptions on $\varepsilon_1, \varepsilon_2, \delta$ we have $\dot{W}(x_1, x_2) \leq 0$ for all $x \in \mathbb{R}^2$. For the Lyapunov function $V_3(x) = V_2(x) + W(x)$ we have thus that $\dot{V}_3(x) < 0$ for all $x \neq 0$, but the sublevel sets of $V_3 : \mathbb{R}^2 \to [0, 2 + \frac{\pi}{2})$ are no longer compact on its image.

Still Theorem 3.5 implies global asymptotic stability appealing to V_3, as trajectories exist for all positive times. The latter fact is obvious from the boundedness of the right hand side of (7.3).

We will now use the previous system as the basis for our counterexample. To this end let $\psi : \mathbb{R} \to (-1, \infty)$ be an increasing diffeomorphism with $\psi(x) = x$ for

$x \geq 0$ and such that $\operatorname{ess.}\limsup_{x\to-\infty} \delta'(x)/\psi'(x) < M$ for some constant $M > 0$. Then the transformation

$$T : \mathbb{R}^2 \to (-1,\infty) \times \mathbb{R}, \quad (x,y) \mapsto (\psi(x), y) \tag{7.5}$$

is a diffeomorphism. On $(-1,\infty)\times\mathbb{R}$ we may thus consider the differential equation

$$\dot{z} = F(z) := DT(T^{-1}(z))f(T^{-1}(z)) = \begin{cases} \psi'(\psi^{-1}(z_1))(-\varepsilon_1(\psi^{-1}(z_1)) - \delta(z_2)) \\ \delta(\psi^{-1}(z_1)) - \varepsilon_2(z_2) \end{cases}. \tag{7.6}$$

As $\psi'(x_1) \to 0$ for $x_1 \to -\infty$, it follows that F can be extended continuously to $\mathbb{R}^2$ by setting $F(z_1,z_2) := \begin{bmatrix}0, & -1-\varepsilon_2(z_2)\end{bmatrix}^\top$ for $z \in \mathbb{R}^2$ with $z_1 \leq -1$. The condition $\delta'(x)/\psi'(x) < M$ for all x sufficiently negative guarantees that this extension is Lipschitz continuous.

Transforming the Lyapunov function V_3 as well, we see that the transformed version may also be continuously extended by

$$V(z) := \begin{cases} V_3(T^{-1}(z)) & \text{if } z_1 > -1, \\ 2 + \arctan(z_2) & \text{if } z_1 \leq -1. \end{cases} \tag{7.7}$$

By construction, we have for the system $\dot{z} = F(z)$, $z \in \mathbb{R}^2$ that $\dot{V}(z) < 0$ for all $z \neq 0$. To arrive at an estimate of the form (3.4) we define

$$h(z) := \max\left\{1, \frac{\|z\|}{\left|\dot{V}(z)\right|}\right\}, \quad \|z\| \geq 2,$$

and extend h to a positive, Lipschitz continuous function on $\mathbb{R}^2$ which is identically equal to 1 on $B_1(0)$. The vector field $F_2 := hF$ is simply a time transformation of F. As $h > 0$ everywhere, it follows that the stability properties of $z^* = 0$ for the system

$$\dot{z} = F_2(z) \tag{7.8}$$

are the same as those of $\dot{z} = F(z)$. If we consider the decay of V along the trajectories of (7.8) we obtain (the index denotes the system with respect to which the time derivative is taken)

$$\dot{V}_{|F_2}(z) = h(z)\dot{V}_{|F}(z) \leq -\|z\|, \quad \|z\| \geq 2. \tag{7.9}$$

As $\dot{V}_{|F_2}(z) < 0$ for all $z \neq 0$ it is easy to find an $\alpha \in \mathcal{K}$ such that $\dot{V}_{|F_2}(z) \leq -\alpha(\|z\|)$ for $\|z\| < 2$. Combining this with (7.9), α may indeed be chosen so that the decay condition (3.4) holds.

It can be checked directly that solutions of (7.8) with an initial condition z with $z_1 \leq -1$ explode. This may also be concluded using the decay condition (7.9) together with the observation that for such solutions $V(\varphi(t,z)) \in (1,3)$ for all t in the interval existence. $\square$

The next example shows that even in the benign case of infinite-dimensional linear systems with a bounded generator A an estimate of the form $\dot{V}(x) \leq -\gamma V(x)$ does not exclude exponential instability if the Lyapunov function V is not coercive. This shows that in the non-coercive case it is essential to have a decay estimate in terms of the norm of x as in (1.2).

Example 7.2. We consider the Hilbert space $X = \ell^2(\mathbb{N}, \mathbb{C})$ and denote its elements by $x = (x_i)_{i\in\mathbb{N}}$, where $x_i \in \mathbb{R}^i$, $i = 1, 2, \dots$. By $\|\cdot\|_2$ we denote the Euclidean norm on $\mathbb{R}^i$ and the induced matrix norm.

We construct a linear operator A on X as follows. Let $N_i \in \mathbb{C}^{i\times i}$ be the nilpotent matrix of order $i-1$ given by

$$N_i = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ \vdots & & 0 & 1 \\ 0 & \dots & \dots & 0 \end{bmatrix}.$$

Define

$$A_i = -I_i + N_i$$

and the operator A by

$$Ax = (A_i x_i)_{i\in\mathbb{N}} .$$

As $\|A_i\|_2 \le 2$ for all i, it is clear that A is bounded on X. To consider the spectrum of A, note that for $\lambda \neq -1$ we have

$$(\lambda I_i - A_i)^{-1} = \frac{1}{\lambda+1} I_i + \frac{1}{(\lambda+1)^2} N_i + \frac{1}{(\lambda+1)^3} N_i^2 + \dots + \frac{1}{(\lambda+1)^i} N_i^{i-1} \tag{7.10}$$

Thus for $|\lambda+1| > 1$ it follows for all $i \in \mathbb{N}$ that $\|\left(\lambda I_i - A_i\right)^{-1}\|_2 \le \sum_{k=1}^\infty |\lambda+1|^{-k} = \frac{|\lambda+1|-1}{|\lambda+1|}$ and consequently, $\lambda I - A$ is invertible with bounded inverse.

On the other hand, if $\lambda \in B_1(-1)$, then we see from (7.10) that for the ith standard unit vector in $\mathbb{R}^i$ we have

$$\|\left(\lambda I_i - A_i\right)^{-1} e_i\|_2^2 = \sum_{k=1}^{i} \frac{1}{(\lambda+1)^k} .$$

This expression tends to ∞ as $i \to \infty$. As a consequence $\lambda I - A$ cannot have a bounded inverse on X. As the spectrum of A is closed, it follows that

$$\sigma(A) = \overline{B_1(-1)} .$$

In particular $0 \in \sigma(A)$, whence the linear system

$$\dot{x} = Ax \tag{7.11}$$

is not exponentially stable in $x^* = 0$. We will show that it is possible to construct a non-coercive function which decays exponentially along all nontrivial solutions.

As A_i is Hurwitz with eigenvalue -1, we may for each $i \in \mathbb{N}$ choose a symmetric positive definite solution P_i of the Lyapunov inequality

$$A_i^* P_i + P_i A_i \prec -P_i .$$

By rescaling, if necessary, we may assume that $\|P_i\|_2 = 1$ for all i. Define the function $V : X \to \mathbb{R}_+$ by setting for $x = (x_i)_{i\in\mathbb{N}}$

$$V(x) = \sum_{i=1}^\infty x_i^* P_i x_i .$$

Using the positive definiteness and the norm bound of the P_i, it is easy to see that for $x \neq 0$ we have

$$0 < V(x) \le \|x\|^2.$$

If we consider the decay of V along solutions we have that

$$\dot{V}(x) = \sum_{i=1}^{\infty} x_i^* \left(A_i^* P_i + P_i A_i\right) x_i \leq -\sum_{i=1}^{\infty} x_i^* P_i x_i = -V(x). \tag{7.12}$$

It follows that $V(e^{At}x) \leq e^{-t}V(x)$ for all $x \in X$, so that V decays exponentially along the solutions of (7.11).

We therefore have now two arguments that show that V is not coercive. If V were coercive, it would be a proper Lyapunov function and then we could conclude exponential stability of (7.11). But we know from the analysis of the spectrum that this is not the case.

On the other hand, it is well known that the smallest eigenvalue of the matrices P_i tends to 0 as $i \to \infty$. Choosing eigenvectors v_i of norm 1 corresponding to the smallest eigenvalues of P_i, and letting $y_i \in X$ be the vectors with only 0 entries except for the entries of v_i in position i, it follows that $V(y_i) \to 0$ but $\|y_i\| = 1$ for all i.

As a variant of this example consider operators $A_\varepsilon = A + \varepsilon I$ for $\varepsilon \in (0, 1/2)$. These operators generate exponentially unstable semigroups by our considerations of the spectrum of A. On the other hand, the calculations analogous to (7.12) show that $V(e^{A_\varepsilon t}x) \leq e^{(2\varepsilon-1)t}V(x)$, so that V is exponentially decaying along solutions of these systems. $\square$

8. Conclusions

A classical result in infinite-dimensional stability theory, which can be proved using a generalized version of Datko's lemma, states that the origin of an infinite-dimensional linear undisturbed system on a Banach space is uniformly globally asymptotically stable if and only if there exists a non-coercive Lyapunov function for this system.

In this paper we prove that the existence of a non-coercive Lyapunov function is equivalent to uniformly globally asymptotically stable of the equilibrium position for *nonlinear infinite-dimensional systems with disturbances*, provided the system is robustly forward complete and the equilibrium is robust. In the linear case these two properties are always satisfied, but they are essential for the validity of the theorem for nonlinear systems. As we show by means of a counterexample, a nonlinear system of two ordinary differential equations, possessing a non-coercive Lyapunov function, may fail to be forward complete. Also it is essential, that the decay rate of a non-coercive Lyapunov function along the trajectory is given in terms of the state: $\dot{V}(x) \leq -\gamma(\|x\|_X)$. In a further counterexample we show that linear undisturbed systems admitting a non-coercive function V, satisfying a decay estimate of the form $\dot{V}(x) \leq -\gamma V(x)$, for some $\gamma > 0$, may have exponentially diverging trajectories, even though V decays to zero exponentially along trajectories.

Motivated by the notion of weak attractivity [3, 4, 35] we introduce the concept of uniform weak attractivity, which is equivalent to weak attractivity for ODE systems but which is essentially stronger than weak attractivity for infinite-dimensional systems. We show that 0 is uniformly globally asymptotically stable if and only if 0 is uniformly globally weakly attractive, uniformly stable and the system is robustly forward complete. In addition, the existence of a non-coercive Lyapunov function for a robustly forward complete system ensures that 0 is uniformly globally weakly attractive and Lagrange stable. Finally, a new construction of a non-coercive global

Lyapunov function is presented, which is based on Sontag's $\mathcal{KL}$-Lemma and on Yoshizawa's method.

An interesting direction for future research is the development of non-coercive Lyapunov tools for input-to-state stability of infinite-dimensional systems. In the finite-dimensional case Lyapunov functions which characterize uniform asymptotic stability properties with respect to disturbances were a key step in this development.

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