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A non-coercive Lyapunov framework for stability of distributed parameter systems

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Abstract— **We show that the existence of a non-coercive Lyapunov function is sufficient for uniform global asymptotic stability (UGAS) of infinite-dimensional systems with external disturbances provided the speed of decay is measured in terms of the norm of the state and an additional mild assumption is satisfied. These additional assumptions cannot be dropped in general. However, for the special classes of systems as evolution equations in Banach spaces with Lipschitz continuous nonlinearities, delay systems, or linear switched systems on Banach spaces they become much simpler to check. Moreover, for time-delay systems we prove an alternative non-coercive Lyapunov theorem, which is close in spirit to the Lyapunov-Krasovskii method.**

Keywords: nonlinear control systems, infinite-dimensional systems, Lyapunov methods, global asymptotic stability.

I. Introduction

The theory of Lyapunov functions is one of the cornerstones in the analysis and synthesis of dynamical systems. Since its origins due to Lyapunov there have been numerous developments leading first to sufficient and later on to necessary conditions for various dynamical properties expressed in terms of Lyapunov functions, see [26], [8], [16], [14]. Originally invented to characterize stability properties of fixed points, or more complex attractors, Lyapunov functions have become useful in other contexts: to derive conditions for forward completeness of trajectories [1], to analyze coordinate-free notions of growth rates [7], for the controller and observer design of nonlinear systems [2], [15] etc.

In this paper we concentrate ourselves on a question of a coercivity of Lyapunov functions. The standard definition of a Lyapunov function V, found in many textbooks on finite-dimensional dynamical systems, is that it should be a continuous (or more regular) positive definite and proper function, i.e. a function for which there exist $\mathcal{K}_\infty$[1] functions ψ_1, ψ_2, α such that

$$\psi_1(\|x\|) \leq V(x) \leq \psi_2(\|x\|) \quad \forall x \in X, \tag{1}$$

and such that

$$\dot{V}(x) < -\alpha(\|x\|) \quad \forall x \in X \backslash \{0\}, \tag{2}$$

where $\dot{V}(x)$ is some sort of generalized derivative of V along the trajectories of the system, see below for a precise definition. Of course, if we have (1) then we may just as well require that there is a $\gamma \in \mathcal{K}_\infty$ such that

$$\dot{V}(x) < -\gamma(V(x)), \tag{3}$$

because in the presence of (1) we clearly have an equivalence of (2) and (3).

The inequality (3) shows that $V(x(t))$ converges to zero in a uniform way as $t \to \infty$ (by the "comparison principle"), and (1) implies that $\|x(t)\|$ has the same asymptotic behavior. This simple argument remains (up to some minor technicalities) the same also for infinite-dimensional systems and has been applied for stability analysis, e.g. in [21], [6], [3].

On the other hand, converse Lyapunov theorems proved for wide classes of infinite-dimensional systems show that asymptotic stability guarantees the existence of a proper and positive definite Lyapunov function.

A first inkling that this is not the complete story comes from the study of linear systems. In a seminal paper [5] Datko proved the following. If A is the generator of a C_0-semigroup on a Hilbert space X, then the system $\dot{x} = Ax$ is exponentially stable if and only if there is a positive definite bilinear form on X (generated by a certain bounded positive definite linear operator P) such that for all $x \in D(A)$ we have the following estimate for the scalar product of Px, Ax:

$$\langle Px, Ax \rangle < -\|x\|^2.$$

This is a natural extension of the finite-dimensional Lyapunov inequality.

At the same time the operator P need not be coercive, so the natural Lyapunov function $V : x \mapsto \langle Px, x \rangle$ for the linear system $\dot{x} = Ax$ does not satisfy (1). In fact, there exist exponentially stable C_0-semigroups on Hilbert spaces such that there does not exist an equivalent scalar product under which the semigroup is a strict contraction semigroup, [4]. Hence, the non-coercivity of P cannot be avoided in general. In this situation, it appears that the left inequality in (1) is an artifact of the finite-dimensional origin of the theory. In infinite dimensions it may sometimes be easier and more natural to derive Lyapunov functions which have the weaker property that

$$0 < V(x) \leq \psi_2(\|x\|), \quad x \neq 0. \tag{4}$$

In this note we thoroughly study this question. We consider closed bounded invariant sets of systems defined on Banach

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[1] Increasing, unbounded, continuous, positive definite function.

spaces and formulate conditions for a non-coercive Lyapunov function with respect to such an invariant set satisfying the analogue of (4) and the decay estimate (2). We prove in Section III that for a wide class of dynamical systems on normed linear spaces, the existence of such a non-coercive Lyapunov function guarantees global uniform asymptotic stability of the invariant set provided a robust forward completeness property is satisfied and the invariant set is robust. This is achieved using uniform Barbalat-like estimates.

As we argue in Section IV, for some classes of systems as evolution equations in Banach spaces or linear switched systems in Banach spaces some of assumptions can be dropped, which makes the applicability of non-coercive Lyapunov functions easier. For time-delay systems we prove an alternative non-coercive Lyapunov theorem, which is close in spirit to the framework of Lyapunov-Krasovskii functionals.

On the other hand, from [19] we know that the results cannot be extended much further in such a general setting. E.g., non-coercive Lyapunov functions cannot be used in conjunction with a decay estimate in terms of the Lyapunov function itself as in (3), even for linear undisturbed systems with bounded generators. Also it is a disadvantage that we have to assume forward completeness whereas in many cases coercive Lyapunov functions give this property for free.

Due to the page limits, the proofs of most of the results are omitted. An extended version of this paper is currently submitted for publication [19].

Recently non-coercive Lyapunov functions have been used to good effect: in [17] non-coercive Lyapunov sufficient conditions for practical UGAS of infinite-dimensional systems are obtained and in [20] the counterpart of the non-coercive Lyapunov theorem (Theorem 3.4) has been derived in the context of infinite-dimensional input-to-state stability theory.

A. Notation

By $\mathbb{R}_+$ we denote the set of nonnegative real numbers. For an arbitrary set S and $n \in \mathbb{N}$ the n-fold Cartesian product is $S^n := S \times \ldots \times S$. The open ball in a normed linear space X with radius r and center in $y \in X$ is denoted by $B_r(y) := \{x \in X \mid \|x-y\|_X < r\}$ (the space X in which the ball is taken, will always be clear from the context). For short, we denote $B_r := B_r(0)$. The (norm)-closure of a set $S \subset X$ will be denoted by $\overline{S}$. Given a closed set $C \subset X$ we denote the distance of a point x to C by $|x|_C := \min\{\|x-y\|_X \mid y \in C\}$.

For the formulation of stability properties the following classes of comparison functions are useful, see [8], [13]:

$$\begin{aligned}
\mathscr{P} &:= \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous, } \gamma(0) = 0 \text{ and } \gamma(r) > 0 \text{ for } r > 0\} \\
\mathscr{K} &:= \{\gamma \in \mathscr{P} \mid \gamma \text{ is strictly increasing}\} \\
\mathscr{K}_\infty &:= \{\gamma \in \mathscr{K} \mid \gamma \text{ is unbounded}\} \\
\mathscr{L} &:= \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous and strictly decreasing with } \lim_{t\to\infty} \gamma(t) = 0\} \\
\mathscr{KL} &:= \{\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \mid \beta \text{ is continuous, } \beta(\cdot,t) \in \mathscr{K},\ \forall t \geq 0,\ \beta(r,\cdot) \in \mathscr{L},\ \forall r > 0\}
\end{aligned}$$

II. Problem statement

We consider abstract axiomatically defined time-invariant and forward complete systems on the state space X which are subject to shift-invariant disturbances in $\mathscr{D}$.

Definition 2.1: Consider the triple $\Sigma = (X, \mathscr{D}, \phi)$ with

(i) A normed linear space $(X, \|\cdot\|_X)$, called the state space, endowed with the norm $\|\cdot\|_X$.

(ii) A set of disturbance values D, which is a nonempty subset of a certain normed linear space.

(iii) A space of disturbances $\mathscr{D} \subset \{d : \mathbb{R}_+ \to D\}$ satisfying the following two axioms.
The axiom of shift invariance states that for all $d \in \mathscr{D}$ and all $\tau \geq 0$ the time shift $d(\cdot + \tau)$ is in $\mathscr{D}$.
The axiom of concatenation is defined by the requirement that for all $d_1, d_2 \in \mathscr{D}$ and for all $t > 0$ the concatenation of d_1 and d_2 at time t

$$d(\tau) := \begin{cases} d_1(\tau), & \text{if } \tau \in [0,t], \\ d_2(\tau - t), & \text{otherwise,} \end{cases} \tag{5}$$

belongs to $\mathscr{D}$.

(iv) A transition map $\phi : \mathbb{R}_+ \times X \times \mathscr{D} \to X$.

The triple Σ is called a (forward complete) dynamical system, if the following properties hold:

(Σ1) forward completeness: for every $(x,d) \in X \times \mathscr{D}$ and for all $t \geq 0$ the value $\phi(t,x,d) \in X$ is well-defined.

(Σ2) The identity property: for every $(x,d) \in X \times \mathscr{D}$ it holds that $\phi(0,x,d) = x$.

(Σ3) Causality: for every $(t,x,d) \in \mathbb{R}_+ \times X \times \mathscr{D}$, for every $\tilde{d} \in \mathscr{D}$, such that $d(s) = \tilde{d}(s)$, $s \in [0,t]$ it holds that $\phi(t,x,d) = \phi(t,x,\tilde{d})$.

(Σ4) Continuity: for each $(x,d) \in X \times \mathscr{D}$ the map $t \mapsto \phi(t,x,d)$ is continuous.

(Σ5) The cocycle property: for all $t,h \geq 0$, for all $x \in X$, $d \in \mathscr{D}$ we have $\phi(h, \phi(t,x,d), d(t+\cdot)) = \phi(t+h,x,d)$.

Here $\phi(t,x,d)$ denotes the state of a system at the moment $t \in \mathbb{R}_+$ corresponding to the initial condition $x \in X$ and the disturbance $d \in \mathscr{D}$.

Definition 2.2: The flow of $\Sigma = (X, \mathscr{D}, \phi)$ is called Lipschitz continuous on compact intervals, if for any $\tau > 0$ and any $r > 0$ there is a $L > 0$ s.t. for any $x, y \in \overline{B_r}$ it holds that

$$t \in [0,\tau],\ d \in \mathscr{D} \quad \Rightarrow \quad \|\phi(t,x,d) - \phi(t,y,d)\|_X \leq L\|x-y\|_X.$$

We exploit the following stronger version of forward completeness:

Definition 2.3: The system $\Sigma = (X, \mathscr{D}, \phi)$ is called robustly forward complete (RFC) if

$$R > 0 \ \wedge\ \tau > 0 \quad \Rightarrow \quad \sup_{\|x\|_X \leq R,\, t \in [0,\tau],\, d \in \mathscr{D}} \|\phi(t,x,d)\|_X < \infty.$$

The condition of robust forward completeness is satisfied by large classes of infinite-dimensional systems.

Definition 2.4: Let $C \subset X$ be nonempty, closed and bounded. We call C an invariant set of the system $\Sigma = (X, \mathscr{D}, \phi)$, if for all $x \in C$, $t \geq 0$ and all $d \in \mathscr{D}$ we have $\phi(t,x,d) \in C$.

Definition 2.5: Let $C \subset X$ be a nonempty closed bounded invariant set of Σ. We call C a robust invariant set if for every $\varepsilon > 0$ and for any $h > 0$ there is a $\delta = \delta(\varepsilon, h) > 0$ such that

$$t \in [0,h],\ |x|_C \leq \delta,\ d \in \mathscr{D} \quad \Rightarrow \quad |\phi(t,x,d)|_C \leq \varepsilon. \qquad (6)$$

Lemma 2.6: Let $\Sigma = (X, \mathscr{D}, \phi)$ be a system with a flow which is Lipschitz continuous on compact intervals. If $C \subset X$ is an invariant set of Σ, then C is a robust invariant set of Σ.

Proof: Robustness of an invariant set is the continuity of the function $\xi : (x,h) \mapsto \sup_{t\in[0,h],\ d\in\mathscr{D}} |\phi(t,x,d)|_C$ in its first argument at $|x|_C = 0$. As C is bounded by assumption we may choose a Lipschitz constant for ϕ on a bounded neighborhood of C and then the claim follows directly. ∎

The following result will be useful in the sequel.

Proposition 2.7: Consider a forward complete system $\Sigma = (X, \mathscr{D}, \phi)$. The following statements are equivalent:

(i) Σ is RFC and $C \subset X$ is a robust invariant set for Σ.

(ii) there is a $\sigma \in \mathscr{K}_\infty$ and a continuous function $\chi : \mathbb{R}_+^2 \to \mathbb{R}_+$ such that $\chi(r,\cdot) \in \mathscr{K}$ for all $r > 0$, $\chi(0,t) = 0$ for all $t \in \mathbb{R}_+$ and such that for all $x \in X, d \in \mathscr{D}$ and all $t \geq 0$ we have

$$|\phi(t,x,d)|_C \leq \sigma(|x|_C) + \chi(|x|_C, t). \qquad (7)$$

In this paper we investigate the following stability properties of invariant sets of abstract systems.

Definition 2.8: Consider a system $\Sigma = (X, \mathscr{D}, \phi)$ with a closed, bounded invariant set C. The set C is called

(i) (locally) uniformly stable (US), if $\forall \varepsilon > 0\ \exists \delta > 0$ s.t.

$$|x|_C \leq \delta,\ d \in \mathscr{D},\ t \geq 0 \quad \Rightarrow \quad |\phi(t,x,d)|_C \leq \varepsilon. \qquad (8)$$

(ii) uniformly globally asymptotically stable (UGAS) if there is a $\beta \in \mathscr{KL}$ s.t. for all $x \in X$, $d \in \mathscr{D}$, $t \geq 0$

$$|\phi(t,x,d)|_C \leq \beta(|x|_C, t). \qquad (9)$$

(iii) uniformly globally attractive (UGATT), if for any $r, \varepsilon > 0$ there exists $\tau = \tau(r,\varepsilon)$ so that

$$|x|_C \leq r,\ d \in \mathscr{D},\ t \geq \tau(r,\varepsilon) \quad \Rightarrow \quad |\phi(t,x,d)|_C \leq \varepsilon. \qquad (10)$$

The relation between UGAS and UGATT is given by:

Proposition 2.9: Let $\Sigma = (X, \mathscr{D}, \phi)$ be a control system and let C be a robust invariant set for Σ. Then C is UGAS if and only if C is UGATT and Σ is RFC.

Proof: Follows from [12, Theorem 2.2] when choosing the output $H(x) = |x|_C$. ∎

III. Non-coercive Lyapunov theorems

Lyapunov functions provide a predominant tool to study UGAS. In our context they are defined as follows. Recall that for a continuous function $h : \mathbb{R} \to \mathbb{R}$ the (right-hand lower) Dini derivative at a point t is defined by, see [25],

$$D_+h(t) := \underline{\lim}_{\tau\to+0} \frac{1}{\tau}\big(h(t+\tau) - h(t)\big). \qquad (11)$$

Consider a system $\Sigma = (X, \mathscr{D}, \phi)$ and let $V : X \to \mathbb{R}$ be continuous. Given $x \in X, d \in \mathscr{D}$, we consider the (right-hand lower) Dini derivative of the continuous function $t \mapsto V(\phi(t,x,d))$ at $t = 0$:

$$\dot{V}_d(x) := \underline{\lim}_{t\to+0} \frac{1}{t}\big(V(\phi(t,x,d)) - V(x)\big). \qquad (12)$$

We call this the Dini derivative of V along the trajectories of Σ.

Definition 3.1: A continuous function $V : X \to \mathbb{R}_+$ is called a *Lyapunov function* for system $\Sigma = (X, \mathscr{D}, \phi)$ with respect to the nonempty, closed invariant set C, if there exist $\psi_1, \psi_2 \in \mathscr{K}_\infty$ and $\alpha \in \mathscr{K}$ such that

$$\psi_1(|x|_C) \leq V(x) \leq \psi_2(|x|_C) \quad \forall x \in X \qquad (13)$$

holds and the Dini derivative of V along the trajectories of Σ satisfies

$$\dot{V}_d(x) \leq -\alpha(|x|_C) \qquad (14)$$

for all $x \in X$ and all $d \in \mathscr{D}$. We call V a *non-coercive Lyapunov function*, if instead of (13) we have $V(0) = 0$ and

$$0 < V(x) \leq \psi_2(|x|_C) \quad \forall x \in (X \setminus C). \qquad (15)$$

To emphasize that (13) holds we will sometimes speak of coercive Lyapunov functions. The next result is well-known:

Proposition 3.2: Let $\Sigma = (X, \mathscr{D}, \phi)$ be a dynamical system with a nonempty, closed invariant set C. If there is a coercive Lyapunov function for Σ with respect to C, then C is UGAS.

Proposition 3.2 can be proved analogously to its finite-dimensional counterpart, see [16, p. 160]. Note however, that we use continuous Lyapunov functions and the trajectories of the system Σ are merely continuous, therefore we cannot use the standard comparison principle [16, Lemma 4.4] but need to apply a generalized comparison principle from [25, Lemma 6.1], [18, Lemma 1].

Next we show that already the existence of a non-coercive Lyapunov function is sufficient for UGAS of a system provided another mild assumption is satisfied. We exploit the following well-known property, see e.g. [24, pp. 204-205]:

Lemma 3.3: Let $f, g : [0,\infty) \to \mathbb{R}$ be continuous. If for all $t \geq 0$ we have $D_+f(t) \leq -g(t)$, then for all $t \geq 0$ it follows that $f(t) - f(0) \leq -G(t) := -\int_0^t g(s)ds$.

Theorem 3.4: **(Non-coercive UGAS Lyapunov theorem)** Consider an RFC system $\Sigma = (X, \mathscr{D}, \phi)$. Let $C \subset X$ be a nonempty, closed, robustly invariant set for Σ. If V is a non-coercive Lyapunov function for Σ w.r.t. C, then C is UGAS.

Proof: Let V be a non-coercive Lyapunov function and let $\alpha \in \mathscr{K}$ be such that we have the decay estimate (14). Along any trajectory ϕ of Σ we have the inequality

$$\dot{V}_{d(t+\cdot)}(\phi(t,x,d)) \leq -\alpha(|\phi(t,x,d)|_C), \quad \forall t \geq 0. \qquad (16)$$

It follows from Lemma 3.3 that

$$V(\phi(t,x,d)) - V(x) \leq -\int_0^t \alpha(|\phi(s,x,d)|_C)ds, \qquad (17)$$

which implies that for all $t \geq 0$ we have

$$\int_0^t \alpha\big(|\phi(s,x,d)|_C\big)ds \quad \leq \quad V(x). \qquad (18)$$

Step 1: (Stability) Seeking a contradiction, assume that Σ is not uniformly stable in C. Then there exist an $\varepsilon > 0$ and sequences $\{x_k\}_{k\in\mathbb{N}}$ in X, $\{d_k\}_{k\in\mathbb{N}}$ in $\mathscr{D}$, and $t_k \geq 0$ such that $|x_k|_C \to 0$ as $k \to \infty$ and

$$|\phi(t_k, x_k, d_k)|_C = \varepsilon \quad \forall k \geq 1.$$

By the bound on V given by (15) it follows that $V(x_k) \to 0$. Since Σ is RFC and C is a robust invariant set of Σ, Proposition 2.7 implies that there exist $\sigma \in \mathcal{K}_\infty$ and χ as in item (ii) of Proposition 2.7 so that (7) holds.

Appealing to continuity of χ we may choose $\tau > 0$ such that $\chi(r,\tau) \le \varepsilon/2$ for all $0 \le r \le \varepsilon$.

Using (7) we obtain that for all $k \in \mathbb{N}$ and for all $t \in [t_k - \tau, t_k]$ we have either $|\phi(t,x_k,d_k)|_C > \varepsilon$ or

$$\begin{aligned}\sigma\big(|\phi(t,x_k,d_k)|_C\big) \ge& |\phi(t_k,x_k,d_k)|_C - \chi\big(|\phi(t,x_k,d_k)|_C, t_k - t\big)\\ \ge& \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2}.\end{aligned}$$

Setting $t := 0$ in (7) and using the identity axiom ($\Sigma 2$) we see that $\sigma(r) \ge r$ for all $r \ge 0$, and thus $\sigma^{-1}(r) \le r$ for all $r \in \mathbb{R}_+$. Hence

$$\min_{s\in[t_k-\tau,t_k]} |\phi(s,x_k,d)|_C \ge \min\left\{\varepsilon, \sigma^{-1}\left(\frac{\varepsilon}{2}\right)\right\} = \sigma^{-1}\left(\frac{\varepsilon}{2}\right)$$

and (18) implies for every k

$$V(x_k) \ge \int_{t_k-\tau}^{t_k} \alpha\big(|\phi(s,x_k,d_k)|_C\big) ds \ge \alpha \circ \sigma^{-1}\left(\frac{\varepsilon}{2}\right)\tau > 0.$$

This contradiction proves uniform stability of C.

Step 2: (Uniform global attractivity) Again we assume that C is not uniformly globally attractive. This implies that there are r, $\varepsilon > 0$ and sequences $\{x_k\}_{k\in\mathbb{N}}$ in X, $\{d_k\}_{k\in\mathbb{N}}$ in $\mathcal{D}$ and times $t_k \to \infty$, as $k \to \infty$ such that

$$|x_k|_C \le r \text{ and } |\phi(t_k,x_k,d_k)|_C \ge \varepsilon. \tag{19}$$

As we have already shown that C is uniformly stable we may choose for the above ε a $\delta = \delta(\varepsilon) > 0$ such that

$$|z|_C < \delta,\ t \ge 0,\ d \in \mathcal{D} \ \Rightarrow\ |\phi(t,z,d)|_C \le \frac{\varepsilon}{2}. \tag{20}$$

Now assume that there exist a certain $k \in \mathbb{N}$ and $s_k \in [0,t_k]$ so that $|\phi(s_k,x_k,d_k)|_C \le \delta$. Since Σ satisfies the cocycle property ($\Sigma 5$), (20) and (19) lead us to

$$\varepsilon \le |\phi(t_k,x_k,d_k)|_C = \left|\phi\left(t_k - s_k, \phi(s_k,x_k,d_k), d_k(s+\cdot)\right)\right|_C \le \frac{\varepsilon}{2},$$

which is a contradiction. We conclude that for all k and all $t \in [0,t_k]$ we have $|\phi(t,x_k,d_k)|_C \ge \delta$. It then follows with (19), (15) and (18) that for all $k \ge 1$

$$\psi_2(r) \ge \psi_2(|x_k|_C) \ge V(x_k) \ge \int_0^{t_k} \alpha\left(|\phi(t,x_k,d_k)|_C\right) dt \ge \alpha(\delta) t_k.$$

As $t_k \to \infty$, this is a contradiction and hence C is uniformly globally attractive. Since Σ is RFC and C is a robust invariant set for Σ, Proposition 2.9 ensures that C is UGAS. ■

Remark 3.5: Contrary to the coercive case where the existence of a Lyapunov function implies REP and RFC, the existence of a non-coercive Lyapunov function implies neither REP nor RFC. This follows from [10, Remark 4][2].

[2]We note that a few modifications are necessary in [10, Remark 4] in order to obtain the desired example. Most importantly, the semigroups $T_j(t)$ on $X = L^p(0,1)$, $j \in \mathbb{N}$ should be defined by setting for $f \in X$

$$(T_j(t)f)(s) := \begin{cases} 2^{\frac{1}{p}} f(s+t), & \text{if } s \in [0,1-t] \cap [4^{-j} - t, 4^{-j}),\\ f(s+t), & \text{if } s \in [0,1-t] \backslash [4^{-j} - t, 4^{-j}),\\ 0, & \text{if } s \in (1-t,1] \cap [0,1]. \end{cases} \tag{21}$$

Moreover, it is possible to show by means of an example (see [19]) that even for finite-dimensional undisturbed systems the following properties are possible: (i) the system has a unique fixed point, (ii) there exist non-coercive Lyapunov functions which satisfy the decay condition (14) and (iii) the system is not forward complete.

IV. Applications

In this section we give a few examples of system classes that are covered by our assumptions and in which the unboundedness of generators of C_0-semigroups and nonlinearity w.r.t. time-delays may play a role.

A. Switched linear systems in Banach spaces

This class of infinite-dimensional switched linear systems has been studied in [10]. Here we briefly outline how to recover some of the results of [10] with the arguments proposed here. Let X be a Banach space. Consider a set of closed linear operators $\{A_q \mid q \in Q\}$, Q some index set. Assume that each A_q generates a C_0-semigroup T_q on X. Let

$$\mathcal{D} := \{d : \mathbb{R}_+ \to Q \mid d \text{ is piece-wise constant}\}, \tag{22}$$

where piece-wise constant means here that any half-open bounded interval $[a,b) \subset \mathbb{R}_+$ can be partitioned into finitely many half open intervals $[a_i,b_i)$ such that d is constant on each $[a_i,b_i)$. We consider the family of differential equations

$$\dot{x} = A_{d(t)} x(t) \tag{23}$$

which generates evolution operators in the following manner. For $d \in \mathcal{D}$ and an interval $[s,t]$ with a partition $s = b_0 < b_1 < \ldots < b_k = t$ s.t. $d \equiv d_j \in Q$ on $[b_{j-1}, b_j)$, $j = 1,\ldots,k$ we set

$$\Phi_d(t,s) = T_{d_k}(t - b_{k-1}) T_{d_{k-1}}(b_{k-1} - b_{k-2}) \ldots T_{d_1}(b_1 - s). \tag{24}$$

With this notation we have $\phi(t,x,d) = \Phi_d(t,0)x$ for all $x \in X$, $d \in \mathcal{D}$, $t \ge 0$. It is easy to check that the conditions of Definition 2.1 are all satisfied. We also have the following lemma.

Lemma 4.1: Consider the system $\Sigma = (X, \mathcal{D}, \phi)$ given by switched linear system (23) with $\mathcal{D}$ as defined in (22). The following statements are equivalent.

(i) Σ is robustly forward complete.
(ii) $x^* = 0$ is a robust equilibrium point of Σ.
(iii) There exist constants $M \ge 1$, $\omega \in \mathbb{R}$ such that

$$\|\Phi_d(t,s)\| \le M e^{\omega(t-s)} \quad \forall\, d \in \mathcal{D},\ t \ge s \ge 0. \tag{25}$$

Remark 4.2: An immediate consequence of the characterization (iii) of the previous lemma is that for linear switched systems the flow of Σ is Lipschitz continuous if and only if the system is robustly forward complete.

For switched linear systems on Banach space we thus obtain the following result, which recovers some of the results of [10, Theorem 3].

Corollary 4.3: Consider an RFC switched linear system $\Sigma = (X, \mathcal{D}, \phi)$ as described by (22)-(24). Then the following two statements are equivalent:

(a) there exists a non-coercive Lyapunov function V for Σ.

(b) 0 is uniformly globally asymptotically stable and hence exponentially stable.

Proof: It is clear that switched linear systems as described by (22)-(24) are forward complete. The sufficiency part "(a) $\Rightarrow$ (b)" follows from Theorem 3.4 and Lemma 4.1. Necessity is a consequence of standard converse Lyapunov theorems, e.g. of [12, Section 3.4]. ∎

We note that Remark 4 in [10] also shows that even for this system class the assumption of robust forward completeness cannot be removed in order to conclude uniform global asymptotic stability.

B. Time-delay systems

Consider the following class of time-delay systems

$$\dot{x}(t) = f(x_t, d), \tag{26}$$

where

(i) $X := C([-r,0],\mathbb{R}^n)$ and $r>0$ is a given maximal delay,
(ii) $x_t : s \mapsto x(t+s)$, $s\in[-r,0]$ is the state of (26) at time t,
(iii) d belongs to the space $L_\infty(\mathbb{R}_+,\mathbb{R}^m)$ of globally essentially bounded functions endowed with the essential supremum norm $\|\cdot\|_\infty$,
(iv) $f : X\times\mathbb{R}^m$ is completely continuous[3] and in addition Lipschitz continuous in $x\in X$ on bounded balls uniformly with respect to d.

In [23] Lipschitz continuous Lyapunov-Krasovskii functions were used to characterize the stability of an equilibrium position for (26). In [22] it was shown that for large classes of systems and Lyapunov functions the derivative $\dot{V}$ can be obtained without the knowledge of solutions, which is one of the drawbacks of the definition we use in (14).

In [11, p. 310] it is shown (in a somewhat more general setting) that forward complete time-delay systems (26) belong to the class of control systems as defined in Definition 2.1. Moreover, using the methods of this reference one can show that if C is a closed, bounded invariant set of (26), then C is a robust invariant set of (26). For more precise formulation of these results we refer to [11, Example 2.6].

For this class of systems we thus obtain the following:

Corollary 4.4: Consider a time-delay system as in (26) with bounded delay $r>0$. Assume the system is robustly forward complete on X. Let $C\subset X$ be a closed bounded invariant set for (26). If $V : X\to\mathbb{R}_+$ is a non-coercive Lyapunov function with respect to C, then C is UGAS.

We now specialize our results to equilibrium points, which we take without loss of generality to be $x^*=0$. This is a special case of an invariant set. In the classical presentation of Lyapunov-Krasovskii functions, see e.g. [9], the conditions which guarantee stability of an equilibrium position are

$$\psi_1(\|x(0)\|) \le V(x) \le \psi_2(\|x\|_X) \tag{27}$$
$$\dot{V}(x) < -\alpha(\|x(0)\|), \tag{28}$$

where again $\psi_1,\psi_2\in\mathcal{K}_\infty$, $\alpha\in\mathcal{K}$ and the inequalities hold for all $x\in C([-r,0],\mathbb{R}^n)=X$. Note that $\|\cdot\|_X$ is the norm in X, while $\|\cdot\|$ is the norm in $\mathbb{R}^n$. We note that already this version presents a non-coercive Lyapunov function as the left hand side of (27) can be arbitrarily small for $x\in X$ with $\|x\|_X=1$. On the other hand the decay condition in (28) is weaker than the condition we have imposed as it is not in terms of the norm of x. The following result shows that in this particular case this is sufficient.

Theorem 4.5: **(Non-coercive UGAS Lyapunov theorem for time-delay systems)** Consider the time-delay system (26) with bounded delay $r>0$ and equilibrium point 0. Assume the system is robustly forward complete on X. If $V : X\to\mathbb{R}_+$ is continuous with $V(0)=0$ and satisfies:

1) There exists a $\psi_2\in\mathcal{K}_\infty$ such that
$$0 < V(x) \le \psi_2(\|x\|_X) \quad \forall\, x\in X,\ x\neq 0; \tag{29}$$
2) There exists an $\alpha\in\mathcal{K}$ such that
$$\dot{V}_d(x) < -\alpha(\|x(0)\|) \quad \forall x\in X,\ d\in\mathcal{D}; \tag{30}$$

then the fixed point 0 is uniformly locally asymptotically stable for system (26). If, in addition, solutions of (26) are uniformly bounded (Lagrange stable), i.e. if there exists a $\eta\in\mathcal{K}_\infty$, $q\ge 0$ such that for all $x\in X$

$$\sup_{t\ge 0,\ d\in\mathcal{D}} \|\phi(t,x,d)\|_X \le \eta(\|x\|_X)+q,$$

then the fixed point 0 is UGAS for the system (26).

Proof: For reasons of space we just outline the necessary modifications in the proof of Theorem 3.4. We abbreviate $\hat{\phi}(t,x,d) := \phi(t,x,d)(0)$, which is the evaluation of the function $\phi(t,x,d)\in C([-r,0],\mathbb{R}^n)$ at $s=0$. Following the steps (16)-(18) we obtain for all $t\ge 0$ and all solutions

$$\int_0^t \alpha(\|\hat{\phi}(s,x,d)\|)ds \quad\le\quad V(x). \tag{31}$$

Step 1: (Stability) Seeking a contradiction as before there exist an $\varepsilon>0$ and sequences $\{x_k\}_{k\in\mathbb{N}}$ in X, $\{d_k\}_{k\in\mathbb{N}}$ in $\mathcal{D}$, and $t_k\ge r$ such that $\|x_k\|_X\to 0$ as $k\to\infty$ and

$$\|\phi(t_k,x_k,d_k)\|_X=\varepsilon \quad \forall k\ge 1.$$

By the bound on V given by (15) it follows that $V(x_k)\to 0$. By modifying t_k slightly in the interval $[t_k-r,t_k]$ we may assume that $\|\hat{\phi}(t_k,x_k,d_k)\| = \|\phi(t_k,x_k,d_k)\|_X=\varepsilon$, $\forall k\ge 1$. As f is bounded on the ball $B_{2\varepsilon}$ in X, it follows that there is a $\tau>0$ such that

$$\|\hat{\phi}(t_k+t,x_k,d_k)\| \ge \frac{\varepsilon}{2} \quad \forall t\in[0,\tau].$$

We obtain for every k that

$$V(x_k) \ge \int_{t_k}^{t_k+\tau} \alpha(\|\hat{\phi}(s,x_k,d_k)\|)ds \ge \alpha\left(\frac{\varepsilon}{2}\right)\tau > 0.$$

This contradiction proves uniform stability of 0.

Step 2: (Uniform local attractivity) As we know that 0 is uniformly stable, we may choose a $\delta>0$ such that $\|x\|_X\le\delta$ implies $\|\phi(t,x,d)\|_X\le 1$ for all $t\ge 0$, $d\in\mathcal{D}$. Fix $x\in B_\delta$ and $d\in\mathcal{D}$. By the assumption on f this implies that $\frac{d}{dt}\hat{\phi}(t,x,d)$

[3] I.e., f is continuous and for any bounded subset S of $X\times\mathbb{R}^m$ the closure of $f(S)$ is compact.

is bounded and thus $\hat{\phi}(t,x,d)$ is uniformly continuous on $[0,\infty)$. From the estimate

$$V(x) \geq \int_0^\infty \alpha(\hat{\phi}(s,x,d))ds \geq 0$$

it thus follows that $\lim_{t\to\infty} \hat{\phi}(t,x,d) = 0$ by an application of Barbalat's lemma. Consequently, we have $\lim_{t\to\infty} \phi(t,x,d) = 0$. The uniformity of the convergence now follows just as in Step 2 of the proof of Theorem 3.4.

The only problem in the above argument was to obtain boundedness of the derivative of $\hat{\phi}(t,x,d)$. If it is known a priori that the solutions of Σ are uniformly bounded, then uniform global attractivity is obtained in the same way. ■

C. Nonlinear evolution equations in Banach spaces

Consider infinite-dimensional systems of the form

$$\dot{x}(t) = Ax(t) + g(x(t),d(t)),\ x(t) \in X,\ d(t) \in D, \tag{32}$$

where A generates the C_0-semigroup $T(\cdot)$ of bounded operators, X is a Banach space and D is a nonempty subset of a normed linear space. As the space of admissible inputs we consider the space $\mathscr{D}$ of globally bounded, piecewise continuous functions from $\mathbb{R}_+$ to D.

Assumption 1: We suppose that the nonlinearity $g : X \times D \to X$ is Lipschitz continuous on bounded subsets of X, uniformly with respect to the second argument, i.e. for all $C > 0$ there exists $L_f(C) > 0$, such that for all x,y with $\|x\|_X \leq C,\ \|y\|_X \leq C$ and $z \in D$ it holds that

$$\|g(y,z) - g(x,z)\|_X \leq L_f(C)\|y - x\|_X. \tag{33}$$

Assume also that $g(x,\cdot)$ is continuous for all $x \in X$.

We consider mild solutions of (32), i.e. solutions of the integral equation

$$x(t) = T(t)x(0) + \int_0^t T(t-s)g(x(s),d(s))ds \tag{34}$$

belonging to the class $C([0,\tau],X)$ for certain $\tau > 0$.

It is well known that the system (32) is well-posed if Assumption 1 is satisfied. Moreover it satisfies all the axioms of the Definition 2.1, possibly with exception of forward completeness. Thus, (32) gives rise to a control system $\Sigma = (X,\mathscr{D},\phi)$. We show next that for system (32) some of the assumptions of Theorem 3.4 are automatically satisfied.

Lemma 4.6: Assume that (32) is robustly forward complete and Assumption 1 holds. Then (32) has a flow which is Lipschitz continuous on compact intervals.

We obtain from Theorem 3.4, and Lemmas 4.6 and 2.6:

Corollary 4.7: Let Assumption 1 be satisfied. Let (32) be robustly forward complete and let 0 be an equilibrium of (32). If there exists a non-coercive Lyapunov function for (32), then (32) is UGAS.

Proof: Lemma 4.6 and RFC property of (32) imply that the flow of (32) is Lipschitz continuous on compact intervals. Next Lemma 2.6 implies that 0 is a robust equilibrium point of (32). Finally, Theorem 3.4 shows that (32) is UGAS. ■

V. Conclusions

We have shown that the *existence of a non-coercive Lyapunov function is equivalent to uniform global asymptotic stability of an invariant set for nonlinear infinite-dimensional systems with disturbances*, provided the system is robustly forward complete and the invariant set is robust. In the linear case these two properties are always satisfied, but they are essential in the general case. Also it is essential, that the decay rate along trajectories is given in terms of the state: $\dot{V}(x) \leq -\gamma(\|x\|_X)$ as shown by an example in [19].

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