2017 IEEE 56th Annual Conference on Decision and Control (CDC)
December 12-15, 2017, Melbourne, Australia

Input-to-state stability of time-delay systems: criteria and open problems

Andrii Mironchenko and Fabian Wirth

Abstract— **We state a characterization of input-to-state stability (ISS) for a broad class of control systems, including time-delay systems, partial differential equations, ordinary differential equations, switched systems etc. Next we specify this result for a general class of nonlinear time-delay systems. We show that in this case some additional refinements are possible. Finally, we throw some light on several important open problems in ISS theory of time-delay systems: whether time-invariant forward complete systems necessarily have bounded reachability sets and whether limit property and uniform limit properties are equivalent for nonlinear time-delay systems.**

Keywords: nonlinear control systems, input-to-state stability, time-delay systems, infinite-dimensional systems.

I. Introduction

The concept of input-to-state stability (ISS), introduced in [21], has become indispensable for various branches of nonlinear control theory, such as robust stabilization of nonlinear systems [6], design of nonlinear observers [1], analysis of large-scale networks [10], [5] etc.

For input-to-state stability of time-delay systems two different Lyapunov-type sufficient conditions have been proposed: via ISS Lyapunov-Razumikhin functions [24] and by ISS Lyapunov-Krasovskii functionals [19]. These results together with small-gain theorems [12], [14], [4] make possible the study of ISS of large-scale interconnections of control systems as e.g. chemostat model with time-delays [14], [25].

Characterization of ISS in terms of other stability properties are among the fundamental results in ISS theory. In the case of ordinary differential equations (ODEs) in [22] Sontag and Wang have shown that ISS is equivalent to the existence of a smooth ISS Lyapunov function and in [23] the same authors proved an ISS superposition theorem, saying that ISS is equivalent to the limit property combined with local stability. Related ISS superposition theorems have been proved for hybrid systems in [2]. Characterizations of ISS greatly simplify the proofs of other fundamental results, such as small-gain theorems for ODEs [5] and hybrid systems [2], [3], Lyapunov-Razumikhin [24], [4] and Lyapunov-Krasovskii [9], [8] theory for time-delay systems, to name a few examples.

Whereas there are several criteria for ISS of time-delay systems in terms of Lyapunov functions [11], [20], there are virtually no results on non-Lyapunov characterizations for time-delay systems. On the way of obtaining these results there appear several difficulties: noncompactness of closed bounded balls in infinite-dimensional normed linear spaces, possible unboundedness of reachability sets even for nonuniformly globally asymptotically stable nonlinear systems, etc [18]. Because of these facts the generalization of the characterizations of ISS for ODEs from [23] is far from being straightforward.

Recent results in [18], [17], [16] overcome these problems and develop a characterization of ISS for a broad class of infinite-dimensional control systems, encompassing many evolution PDEs, time-delay systems, differential equations in Banach spaces, switched systems etc. This class is closely related to abstract definitions of control systems as presented e.g. in [13]. In Section III of this note we concentrate ourselves on time-delay systems and discuss the question whether in this particular case even stronger criteria for ISS can be achieved. We prove some properties that are of interest in this respect. The complete picture is as yet unclear.

For general infinite-dimensional systems [18] develops a hierarchy of concepts. The strongest notion is that of ISS, which is equivalent to the uniform asymptotic gain property, and equivalent to the uniform limit property in combination with suitable local stability properties. Weaker notions are that of strong ISS (so named because of its relation to strong stability of C_0-semigroups) and its characterizations. Because of the lack of local compactness of the state space other familiar characterizations of ISS no longer hold in infinite dimensions. For instance, the familiar equivalence of ISS with the asymptotic gain property together with global asymptotic stability of the origin for the system with zero input is a strictly finite-dimensional result.

For a characterization of ISS an important role is played by the boundedness of reachability sets (for uniformly bounded inputs). According to the knowledge of the authors, it is an open problem, whether (time invariant) forward complete time-delay systems possess bounded reachability sets (for the ODE case such a claim is true [15] and for general infinite-dimensional systems it is false [18]). Some insights to this problem are presented in Section III-B.

In the characterization of ISS the uniform limit property is of importance. To the best of the knowledge of the authors it is first introduced in [18]. It is of interest to note that in the ODE case it is equivalent to the limit property which has already been used in [23]. We show that this is the case in Section IV.

A. Mironchenko and F. Wirth are with Faculty of Computer Science and Mathematics, University of Passau, Innstraße 33, 94032 Passau, Germany. Emails: andrii.mironchenko@uni-passau.de, fabian.(lastname)@uni-passau.de

This research has been supported by the German Research Foundation (DFG) within the project "Input-to-state stability and stabilization of distributed parameter systems" (grant Wi 1458/13-1).

A. Notation

The following notation will be used throughout these notes. By $\mathbb{R}_+$ we denote the set of nonnegative real numbers. For an arbitrary set S and $n \in \mathbb{N}$ the n-fold Cartesian product is $S^n := S \times \ldots \times S$.

Let $(X, \|\cdot\|)$ be a normed linear space. For each nonempty $\mathscr{A} \subset X$ and $x \in X$ we define the distance from x to $\mathscr{A}$ by $\|x\|_{\mathscr{A}} := \inf_{y \in \mathscr{A}} \|x - y\|$. Define also $\|\mathscr{A}\| := \sup_{x \in \mathscr{A}} \|x\|$. The open ball in a normed linear space X with radius r around $\mathscr{A} \subset X$ is denoted by $B_r(\mathscr{A}) := \{x \in X : \|x\|_{\mathscr{A}} < r\}$. For short, we denote $B_r := B_r(\{0\})$. Similarly, $B_{r,\mathscr{U}} := \{u \in \mathscr{U} : \|u\|_{\mathscr{U}} < r\}$. The closure of a set $S \subset X$ w.r.t. the norm $\|\cdot\|$ is denoted by $\overline{S}$.

For the formulation of stability properties the following classes of comparison functions are useful:

$$\begin{aligned}
\mathscr{K} &:= \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous, strictly increasing and } \gamma(0) = 0\},\\
\mathscr{K}_\infty &:= \{\gamma \in \mathscr{K} \mid \gamma \text{ is unbounded}\},\\
\mathscr{L} &:= \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous and strictly decreasing with } \lim_{t\to\infty} \gamma(t) = 0\},\\
\mathscr{KL} &:= \{\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \mid \beta \text{ is continuous, } \beta(\cdot,t) \in \mathscr{K},\ \beta(r,\cdot) \in \mathscr{L},\ \forall t \geq 0,\ \forall r > 0\}.
\end{aligned}$$

II. Abstract setting

In this paper we consider abstract, axiomatically defined, time-invariant, and forward complete systems.

Definition 2.1: Consider the triple $\Sigma = (X, \mathscr{U}, \phi)$ consisting of

(i) A normed linear space $(X, \|\cdot\|)$, called the state space, endowed with the norm $\|\cdot\|$.

(ii) A set of input values U, which is a nonempty subset of a normed linear space S_u.

(iii) A space of inputs $\mathscr{U} \subset \{f : \mathbb{R}_+ \to U\}$ endowed with a norm $\|\cdot\|_{\mathscr{U}}$ which satisfies the following two axioms:
The axiom of shift invariance states that for all $u \in \mathscr{U}$ and all $\tau \geq 0$ the time shift $u(\cdot + \tau) \in \mathscr{U}$ with $\|u\|_{\mathscr{U}} \geq \|u(\cdot + \tau)\|_{\mathscr{U}}$.
The axiom of concatenation is defined by the requirement that for all $u_1, u_2 \in \mathscr{U}$ and for all $t > 0$ the concatenation of u_1 and u_2 at time t

$$u(\tau) := \begin{cases} u_1(\tau), & \text{if } \tau \in [0,t], \\ u_2(\tau - t), & \text{otherwise}, \end{cases} \tag{1}$$

belongs to $\mathscr{U}$.

(iv) A transition map $\phi : \mathbb{R}_+ \times X \times \mathscr{U} \to X$.

The triple Σ is called a (forward complete) dynamical system, if the following properties hold:

(Σ1) Forward completeness: for every $(x,u) \in X \times \mathscr{U}$ and for all $t \geq 0$ the value $\phi(t,x,u) \in X$ is well-defined.

(Σ2) The identity property: for every $(x,u) \in X \times \mathscr{U}$ it holds that $\phi(0,x,u) = x$.

(Σ3) Causality: for every $(t,x,u) \in \mathbb{R}_+ \times X \times \mathscr{U}$, for every $\tilde{u} \in \mathscr{U}$, such that $u(s) = \tilde{u}(s)$, $s \in [0,t]$ it holds that $\phi(t,x,u) = \phi(t,x,\tilde{u})$.

(Σ4) Continuity: for each $(x,u) \in X \times \mathscr{U}$ the map $t \mapsto \phi(t,x,u)$ is continuous.

(Σ5) The cocycle property: for all $t,h \geq 0$, for all $x \in X$, $u \in \mathscr{U}$ we have $\phi(h, \phi(t,x,u), u(t+\cdot)) = \phi(t+h,x,u)$.

This class of systems encompasses control systems generated by ordinary differential equations (ODEs), switched systems, time-delay systems, evolution partial differential equations (PDEs), abstract differential equations in Banach spaces and many others.

For a set $\mathscr{S} \subset U$ define the set of inputs with values in $\mathscr{S}$ as $\mathscr{U}_{\mathscr{S}} := \{u \in \mathscr{U} : u(t) \in \mathscr{S},\ t \in \mathbb{R}_+\}$.

For each subset $\mathscr{S} \subset U$, each $T \geq 0$, and each subset $C \subseteq X$ we define the sets of the states which can be reached from C by inputs from $\mathscr{U}_{\mathscr{S}}$ at time not exceeding T:

$$\mathscr{R}^T_{\mathscr{S}}(C) := \{\phi(t,x,u) :\ 0 \leq t \leq T,\ u \in \mathscr{U}_{\mathscr{S}},\ x \in C\}$$

and

$$\mathscr{R}_{\mathscr{S}}(C) := \bigcup_{T \geq 0} \mathscr{R}^T_{\mathscr{S}}(C) = \{\phi(t,x,u) :\ t \geq 0,\ u \in \mathscr{U}_{\mathscr{S}},\ x \in C\}.$$

For short we denote $\mathscr{R}^T(C) := \mathscr{R}^T_U(C)$ and $\mathscr{R}(C) := \mathscr{R}_U(C)$.

We start with some basic definitions. Without loss of generality we restrict our analysis to fixed points of the form $(0,0) \in X \times \mathscr{U}$, so that we tacitly assume that the zero input is an element of $\mathscr{U}$.

Definition 2.2: Consider a system $\Sigma = (X, \mathscr{U}, \phi)$ with equilibrium point $0 \in X$. We say that ϕ is continuous at the equilibrium if for every $\varepsilon > 0$ and for any $h > 0$ there exists a $\delta = \delta(\varepsilon, h) > 0$, so that

$$t \in [0,h],\ \|x\| \leq \delta,\ \|u\|_{\mathscr{U}} \leq \delta \ \Rightarrow\ \|\phi(t,x,u)\| \leq \varepsilon. \tag{2}$$

In this case we will also say that Σ has the CEP property.

Even nonuniformly globally asymptotically stable systems do not always have uniform bounds for their reachability sets on finite intervals (see [18]). Systems exhibiting such bounds deserve a special name.

Definition 2.3: We say that $\Sigma = (X, \mathscr{U}, \phi)$ has bounded reachability sets (BRS), if for any $C > 0$ and any $\tau > 0$ it holds that

$$\sup\left\{\|\phi(t,x,u)\| : \|x\| \leq C,\ \|u\|_{\mathscr{U}} \leq C,\ t \in [0,\tau]\right\} < \infty.$$

Let us define the stability notions for control systems

Definition 2.4: System $\Sigma = (X, \mathscr{U}, \phi)$ is called

- *uniformly locally stable (ULS)*, if there exist $\sigma, \gamma \in \mathscr{K}_\infty$ and $r > 0$ such that for all $x \in \overline{B_r}$ and all $u \in \overline{B_{r,\mathscr{U}}}$:

$$\|\phi(t,x,u)\| \leq \sigma(\|x\|) + \gamma(\|u\|_{\mathscr{U}}) \quad \forall t \geq 0. \tag{3}$$

- *uniformly globally stable (UGS)*, if there exist $\sigma, \gamma \in \mathscr{K}_\infty$ such that for all $x \in X, u \in \mathscr{U}$ the estimate (3) holds.

Next we define the attractivity properties for systems with inputs.

Definition 2.5: System $\Sigma = (X, \mathscr{U}, \phi)$ has the

- *asymptotic gain (AG) property*, if there is a $\gamma \in \mathscr{K}_\infty \cup \{0\}$ such that for all $\varepsilon > 0$, for all $x \in X$ and for all $u \in \mathscr{U}$ there exists a $\tau = \tau(\varepsilon, x, u) < \infty$ such that

$$t \geq \tau \quad \Rightarrow \quad \|\phi(t,x,u)\| \leq \varepsilon + \gamma(\|u\|_{\mathscr{U}}). \tag{4}$$

- *strong asymptotic gain (sAG) property*, if there is a $\gamma \in \mathcal{K}_\infty \cup \{0\}$ such that for all $x \in X$ and for all $\varepsilon > 0$ there exists a $\tau = \tau(\varepsilon, x) < \infty$ such that for all $u \in \mathcal{U}$

$$t \geq \tau \quad \Rightarrow \quad \|\phi(t,x,u)\| \leq \varepsilon + \gamma(\|u\|_{\mathcal{U}}). \tag{5}$$

- *uniform asymptotic gain (UAG) property*, if there exists a $\gamma \in \mathcal{K}_\infty \cup \{0\}$ such that for all $\varepsilon, r > 0$ there is a $\tau = \tau(\varepsilon, r) < \infty$ such that for all $u \in \mathcal{U}$ and all $x \in B_r$

$$t \geq \tau \quad \Rightarrow \quad \|\phi(t,x,u)\| \leq \varepsilon + \gamma(\|u\|_{\mathcal{U}}). \tag{6}$$

All three properties AG, sAG and UAG imply that all trajectories converge to the ball of radius $\gamma(\|u\|_{\mathcal{U}})$ around the origin as $t \to \infty$. The difference between AG, sAG and UAG is in the kind of dependence of τ on the states and inputs. In UAG systems this time depends (besides ε) only on the norm of the state, in sAG systems it depends on the state x (and may vary for different states with the same norm), but it does not depend on u. In AG systems τ depends both on x and on u.

Next we define properties, similar to AG, sAG and UAG, which formalize reachability of the ε-neighborhood of the ball $B_{\gamma(\|u\|_{\mathcal{U}})}$ by trajectories of Σ.

Definition 2.6: We say that $\Sigma = (X, \mathcal{U}, \phi)$ has the

(i) *limit property (LIM)* if there exists $\gamma \in \mathcal{K}$ such that for all $x \in X$, $u \in \mathcal{U}$ and $\varepsilon > 0$ there is a $t = t(x,u,\varepsilon)$:

$$\|\phi(t,x,u)\| \leq \varepsilon + \gamma(\|u\|_{\mathcal{U}}).$$

(ii) *strong limit property (sLIM)*, if there exists $\gamma \in \mathcal{K}$ so that for every $\varepsilon > 0$ and for every $x \in X$ there exists $\tau = \tau(\varepsilon, x)$ such that for all $u \in \mathcal{U}$ there is a $t \leq \tau$:

$$\|\phi(t,x,u)\| \leq \varepsilon + \gamma(\|u\|_{\mathcal{U}}). \tag{7}$$

(iii) *uniform limit property (ULIM)*, if there exists $\gamma \in \mathcal{K}$ so that for every $\varepsilon > 0$ and for every $r > 0$ there exists a $\tau = \tau(\varepsilon, r)$ such that for all x with $\|x\| \leq r$ and all $u \in \mathcal{U}$ there is a $t \leq \tau$ such that

$$\|\phi(t,x,u)\| \leq \varepsilon + \gamma(\|u\|_{\mathcal{U}}). \tag{8}$$

Now we proceed to the main concept of this paper:

Definition 2.7: System $\Sigma = (X, \mathcal{U}, \phi)$ is called *(uniformly) input-to-state stable (ISS)*, if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for all $x \in X$, $u \in \mathcal{U}$ and $t \geq 0$ it holds that

$$\|\phi(t,x,u)\| \leq \beta(\|x\|, t) + \gamma(\|u\|_{\mathcal{U}}). \tag{9}$$

The following theorem provides the desired characterizations of ISS:

Theorem 2.8: Let $\Sigma = (X, \mathcal{U}, \phi)$ be a forward complete system satisfying the BRS and the CEP property. Then the relations depicted in Figure 1 hold.

The proof is omitted due to space limitations, and can be found in [18]. Instead we are going to focus in this note on time-delay systems and show how the characterizations, achieved in Theorem 2.8 can be refined for this important special class of control systems.

III. Characterizations of ISS for time-delay systems

We consider the retarded differential equations of the form

$$\dot{x}(t) = f(x_t, u), \tag{10}$$

where $x_t \in X := C([-T_d, 0], \mathbb{R}^n)$, $n \in \mathbb{N}$, $T_d > 0$ is the fixed (maximal) time-delay and $x_t(s) = x(t+s)$, $s \in [-T_d, 0]$.

We assume here that $U := \mathbb{R}^m$ and u belongs to the space $\mathcal{U} := L^\infty(\mathbb{R}_+, U)$ of globally essentially bounded, measurable functions $u : \mathbb{R}_+ \to U$. The norm of $u \in \mathcal{U}$ is given by $\|u\|_{\mathcal{U}} := \operatorname{ess.sup}_{t \geq 0} \|u(t)\|_U$.

For system (10), we use the following assumption concerning the nonlinearity f.

Assumption 1: We assume that:

(i) $f : X \times U \to \mathbb{R}^n$ is Lipschitz continuous in x on bounded subsets of X and U, uniformly with respect to the second argument, i.e. for all $C > 0$, there exists a $L_f(C) > 0$, such that for all $x, y \in B_C$ and for all $v \in U, \|v\|_U \leq C$, it holds that

$$|f(x,v) - f(y,v)| \leq L_f(C)\|x - y\|. \tag{11}$$

(ii) $f(x, \cdot)$ is continuous for all $x \in X$.

The validity of Assumption 1 together with forward completeness of (10) ensures that (10) is a control system as defined in Definition 2.1. This means that Theorem 2.8 can be applied to the systems (10) and it gives criteria for ISS of (10). However, since (10) is a special class of control systems, it would be desirable to obtain more precise characterizations of ISS for (10). On the subsequent pages we are going to treat this question is some detail.

A. Continuity at the equilibrium for time-delay systems

In this section we show that for system (10) satisfying Assumption 1, boundedness of reachability sets implies the CEP property. To this end we investigate Lipschitz continuity properties of the flow of the time-delay system.

Definition 3.1: The flow of (10) is called Lipschitz continuous on compact intervals (for uniformly bounded inputs), if for any $\tau > 0$ and any $R > 0$ there exists $L > 0$ s.t. for any $x, y \in \overline{B_R}$, for all $u \in B_{R,\mathcal{U}}$ and all $t \in [0, \tau]$ and it holds that

$$\|\phi(t,x,u) - \phi(t,y,u)\| \leq L\|x - y\|. \tag{12}$$

We have the following:

Lemma 3.2: Let Assumption 1 hold and assume that (10) is BRS. Then (10) has a flow which is Lipschitz continuous on compact intervals for uniformly bounded inputs.

Proof: Pick any $R > 0$, any $x, y \in \overline{B_R}$ and any $u \in B_{R,\mathcal{U}}$. Let $x_t := \phi(t,x,u)$, $y_t := \phi(t,y,u)$ be the solutions of (10) defined on the whole nonnegative time axis and $x(t) = x_t(0)$, $y(t) = y_t(0)$.

Pick any $\tau > 0$ and set

$$K(R, \tau) := \sup_{\|x\| \leq R,\ \|u\|_{\mathcal{U}} \leq R,\ t \in [0,\tau]} \|\phi(t,x,u)\|,$$

which is finite since (10) is BRS.

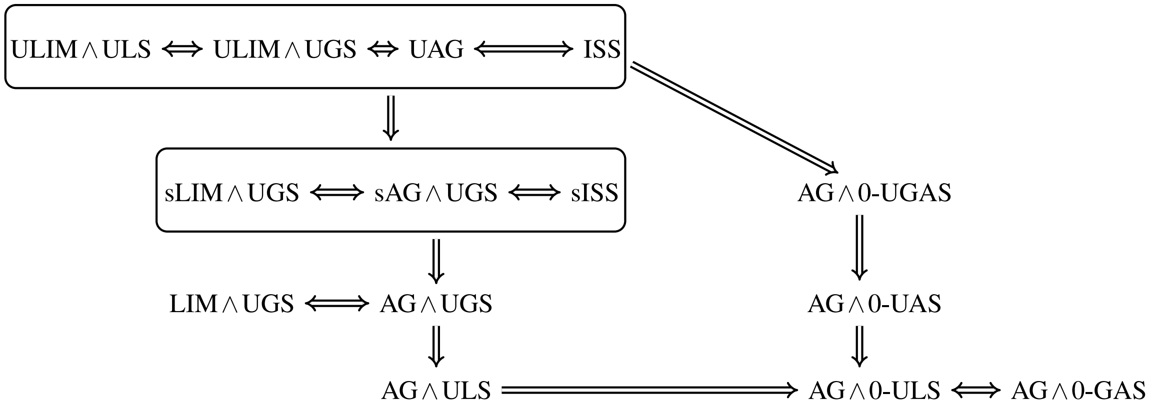

Fig. 1. Relations between stability properties of infinite-dimensional systems, which are continuous at the equilibrium and have bounded reachability sets. Some of the notions depicted here are not defined in this paper; the definitions can be found in [18]

We have for any $t \in [0,\tau]$:

$$|x(t)-y(t)| \le |x(0)-y(0)| + \int_0^t |f(x_s,u(s)) - f(y_s,u(s))| ds$$
$$\le \|x-y\| + L(K(R,t)) \int_0^t \|x_s - y_s\| ds$$
$$\le \|x-y\| + L(K(R,\tau)) \int_0^t \|x_s - y_s\| ds.$$

Taking suprema over $[t-T_d,t]$ of both sides, we obtain:

$$\|x_t - y_t\| \le \|x-y\| + L(K(R,\tau)) \int_0^t \|x_s - y_s\| ds.$$

According to Grönwall's inequality we obtain for $t \in [0,\tau]$:

$$\|x_t - y_t\| \le \|x-y\| e^{L(K(R,\tau))t} \le \|x-y\| e^{L(K(R,\tau))\tau},$$

which proves the lemma. ■

Now we show that (10) has the CEP property.

Lemma 3.3: Let 0 be an equilibrium of (10) and let Assumption 1 hold. Assume also that (10) is BRS. Then (10) has the CEP property.

Proof: Pick any $\varepsilon > 0$, $\tau \ge 0$, $\delta > 0$ and choose any $x \in X$ with $\|x\| \le \delta$ as well as any $u \in B_{\delta,\mathcal{U}}$. It holds that

$$\|\phi(t,x,u)\| \quad \le \quad \|\phi(t,x,u) - \phi(t,0,u)\| + \|\phi(t,0,u)\|.$$

By Lemma 3.2, the flow of (10) is Lipschitz continuous on compact time intervals. Hence there exists a $L(\tau,\delta)$ so that for $t \in [0,\tau]$

$$\|\phi(t,x,u) - \phi(t,0,u)\| \le L(\tau,\delta)\|x\| \le L(\tau,\delta)\delta.$$

Let us estimate $\|\phi(t,0,u)\|$. We have:

$$\|\phi(t,0,u)\| = \sup_{s\in[-T_d,0]} |\phi(t+s,0,u)|$$
$$\le \int_0^t \big|f(\phi(s,0,u),u(s))\big| ds$$
$$\le \int_0^t \Big(\big|f(\phi(s,0,u),u(s)) - f(0,u(s))\big| + |f(0,u(s))| \Big) ds.$$

Since $f(0,\cdot)$ is continuous, for any $\varepsilon_2 > 0$ there exists $\delta_2 < \delta$ so that $u(s) \in B_{\delta_2}$ implies that $|f(0,u(s)) - f(0,0)| \le \varepsilon_2$. Since 0 is an equilibrium of (10), $f(0,0) = 0$ and for the above u we have $|f(0,u(s))| \le \varepsilon_2$.

Due to the BRS property, there exists $K(\tau,\delta_2)$ with $\|\phi(s,0,u)\| \le K(\tau,\delta_2)$ for any $u \in B_{\delta_2,\mathcal{U}}$ and $s \in [0,\tau]$. Now, Lipschitz continuity of f shows that

$$\|\phi(t,0,u)\| \quad \le \quad \int_0^t L(K(\tau,\delta))\|\phi(s,0,u)\| + \varepsilon_2 ds$$
$$\le \quad L(K(\tau,\delta)) \int_0^t \|\phi(s,0,u)\| ds + \tau\varepsilon_2.$$

Now Grönwall's Lemma implies that

$$\|\phi(t,0,u)\| \quad \le \quad \tau\varepsilon_2 e^{L(K(\tau,\delta_2))\tau} \le \tau\varepsilon_2 e^{L(K(\tau,\delta_2))\tau}.$$

To finish the proof choose ε_2 and δ_2 small enough to ensure that $\tau\varepsilon_2 e^{L(K(\tau,\delta_2))\tau} \le \varepsilon$. ■

The developments of this section together with Theorem 2.8 lead to the following result:

Theorem 3.4: Consider a forward complete system (10) satisfying Assumption 1. Let also 0 be an equilibrium point (i.e. $f(0,0) = 0$) and (10) possess bounded reachability sets. Then the relations depicted in Figure 1 hold.

Proof: Apply Theorem 2.8 and Lemmas 3.3, 3.2. ■

We hasten to add that additional investigations are needed in order to understand whether Theorem 3.4 can be further strengthened. In particular, it is of interest to know whether forward completeness of (10) implies boundedness of reachability sets for the same system; and whether the LIM and ULIM properties coincide for (10). In the next sections we throw some light onto these problems.

B. On Boundedness of reachability sets of time-delay systems

Before we proceed to time-delay systems, let us take a quick look onto the properties of reachability sets of ODE systems.

1) Reachability sets of ODE systems: Let $X = \mathbb{R}^n$, $U = \mathbb{R}^m$ and $\mathcal{U} := L_\infty(\mathbb{R}_+,U)$ (the space of globally essentially bounded functions endowed with the essential supremum norm). For $f : X \times U \to X$ consider the system

$$\dot{x} = f(x,u). \tag{13}$$

Assuming that f is continuous and locally Lipschitz continuous in x uniformly in u and that (13) is forward complete, classical Carathéodory theory implies ($\Sigma 2$)-($\Sigma 5$). We will sometimes briefly speak of ODE systems, when referring to (13).

The following result due to [15, Proposition 5.1] shows boundedness of reachability sets of forward complete systems.

Proposition 3.5: Let (13) be forward-complete. For each bounded $\mathscr{S} \subset \mathbb{R}^m$, each $T \geq 0$, and each bounded subset $C \subseteq \mathbb{R}^n$ the reachability set $\mathscr{R}^T_{\mathscr{S}}(C)$ is bounded.

One of the technical results on the way to Proposition 3.5 is [15, Lemma 5.2], formulated next:

Lemma 3.6: Let K be a closed bounded subset of $\mathbb{R}^n$ and $\mathscr{S} \subset \mathbb{R}^m$ be closed and bounded and let $T > 0$ be any real number. Then $\overline{\mathscr{R}^T_{\mathscr{S}}(K)}$ is bounded if and only if $\overline{\mathscr{R}^T_{\mathscr{S}}(x)}$ is bounded for any $x \in K$.

In the terminology of [7, p. 59] Lemma 3.6 states that the solution map ϕ is bounded (for uniformly bounded inputs).

The next example shows that the relation between forward completeness and the BRS property changes dramatically for infinite-dimensional systems.

2) Example: Reachability sets of general infinite-dimensional systems: In [18] the following example was presented to show that there are systems which are forward complete, globally (nonuniformly) asymptotically stable and uniformly locally asymptotically stable, but which do not have bounded reachability sets.

$$\Sigma^1 : \left\{ \begin{array}{l} \Sigma^1_k : \left\{ \begin{array}{l} \dot{x}_k = -x_k + x_k^2 y_k - \frac{1}{k^2} x_k^3, \\ \dot{y}_k = -y_k. \end{array} \right. \\ k = 1, 2, \ldots, \end{array} \right. \tag{14}$$

with the state space X given by

$$l_2 = \left\{ (z_k)_{k=1}^{\infty} : \sum_{k=1}^{\infty} |z_k|^2 < \infty, \quad z_k = (x_k, y_k) \in \mathbb{R}^2 \right\}. \tag{15}$$

3) Reachability sets of time-delay systems: Let us consider again time-delay systems (10). In [7, Property 1.3, p. 59] Hale shows that a counterpart of Lemma 3.6 does not hold for time-delay systems with locally Lipschitz nonlinearities. This makes it impossible to mimic the proof of Proposition 3.5 for time-delay systems. At the same time the following result holds, showing local boundedness (in the terminology of [7, p. 59]) of the solution map ϕ for uniformly bounded inputs.

Lemma 3.7: Let $\mathscr{S} \subset U$ be bounded and let $T > 0$ be any real number. If $\overline{\mathscr{R}^T_{\mathscr{S}}(x)}$ is bounded for some $x \in X$, then there is a neighborhood N of x so that $\overline{\mathscr{R}^T_{\mathscr{S}}(N)}$ is bounded.

Proof: Let $\mathscr{S} \subset U$ be bounded and let $T > 0$ be any real number. Denote the distance between $z \in X$ and $Z \subset X$ by $\rho(z, Z) := \inf\{\|y - z\| : y \in Z\}$.

Pick any $x \in X$ and assume that $\overline{\mathscr{R}^T_{\mathscr{S}}(x)}$ is bounded. Then also the set $M := \{y \in X : \rho(y, \overline{\mathscr{R}^T_{\mathscr{S}}(x)}) \leq 1\}$ is bounded. Denote $C := \max\{\sup\{\|y\| : y \in M\},\ \sup\{\|u\|_U : u \in \mathscr{S}\}\}$.

Since f is Lipschitz continuous on bounded balls, there is $L > 0$: for all $y_1, y_2 \in X$: $\|y_1\| \leq C$, $\|y_2\| \leq C$ and for all $u \in U$: $\|u\|_U \leq C$ it holds that

$$\|f(y_1, u) - f(y_2, u)\| \leq L\|y_1 - y_2\|.$$

Set $r := e^{-LT}$ and pick any $y \in B_r(x)$ and any $u \in \mathscr{U}_{\mathscr{S}}$. Let $t_m = t_m(y, u) := \inf\{t \geq 0 : \phi(t, y, u) \notin M\}$. Assume that $t_m < T$. Then for every $t \in [0, t_m]$ it holds that

$$\begin{aligned} &|\phi(t, x, u)(0) - \phi(t, y, u)(0)| \\ &\leq |x(0) - y(0)| + \int_0^t \left| f(\phi(s, x, u), u) - f(\phi(s, y, u), u) \right| ds. \end{aligned}$$

Hence

$$\begin{aligned} &\|\phi(t, x, u) - \phi(t, y, u)\| = \sup_{s \in [-T_d, 0]} |\phi(t, x, u)(s) - \phi(t, y, u)(s)| \\ &= \sup_{s \in [-T_d, 0]} |\phi(t + s, x, u)(0) - \phi(t + s, y, u)(0)| \\ &\leq \max\Big\{ \|x - y\|, \\ &\qquad |x(0) - y(0)| + \int_0^t \left| f(\phi(s, x, u), u) - f(\phi(s, y, u), u) \right| ds \Big\} \\ &\leq \|x - y\| + \int_0^t \left| f(\phi(s, x, u), u) - f(\phi(s, y, u), u) \right| ds \\ &\leq \|x - y\| + L \int_0^t \left\| \phi(s, x, u) - \phi(s, y, u) \right\| ds. \end{aligned}$$

By Grönwall's inequality we proceed to

$$\|\phi(t, x, u) - \phi(t, y, u)\| \leq \|x - y\| e^{Lt} \leq r e^{Lt_m} = e^{L(t_m - T)} < 1.$$

Thus, $\phi(t, y, u)$ is in the interior of M for any $t \in [0, t_m]$, which contradicts to the definition of t_m. Hence $t_m \geq T$ and herewith $\phi(t, y, u)$ stays within M for all $t \in [0, T]$. ■

It is an interesting and important problem, whether in spite of the failure of Lemma 3.6 for time-delay systems, the counterpart of Proposition 3.5 still holds for the systems (10) satisfying Assumption 1.

IV. The uniform limit property

The following Proposition 4.1 shows that all versions of the limit property coincide in the finite dimensional case.

Proposition 4.1: Assume the finite-dimensional system (13) is forward-complete. Then (13) is LIM if and only if it is ULIM.

Proof: It is clear that ULIM implies LIM. For the converse statement we will make use of [23, Corollary III.3]. The result may be applied as follows. Assume (13) is LIM and let $\gamma \in \mathscr{K}_\infty$ be the corresponding gain. Fix $\varepsilon > 0$, $r > 0$ and $R > 0$. By the LIM property, for all $x_0 \in \mathbb{R}^n$ and all $u \in \mathscr{U}$ with $\|u\|_\infty \leq R$ there is a time $t \geq 0$ such that $|\phi(t, x, u)| \leq \frac{\varepsilon}{2} + \gamma(R)$. Then [23, Corollary III.3] states that there is a $\tau = \tau(\varepsilon, r, R)$ such that for all $x \in \overline{B_r}$, $u \in B_{R, \mathscr{U}}$ there exists a $t \leq \tau(\varepsilon, r, R)$ such that

$$|\phi(t, x, u)| \leq \varepsilon + \gamma(R). \tag{16}$$

With this argument at hand, we now proceed to show ULIM. Fix $\varepsilon > 0$ and $r > 0$. We are going to find a $\tau = \tau(\varepsilon, r)$ for which the ULIM condition is satisfied. To this end define $R_1 := \gamma^{-1}(\max\{r - \varepsilon, 0\})$. Then for each $u \in \mathscr{U}$: $\|u\|_\infty \geq R_1$ and each $x \in \overline{B_r}$ it holds that

$$|\phi(0, x, u)| = |x| \leq r \leq \varepsilon + \gamma(R_1) \leq \varepsilon + \gamma(\|u\|_\infty),$$

and the time $t(\varepsilon,r,u)$ in the definition of ULIM can be chosen for such u as $t:=0$.

Now set $\tau_1 := \tau(\frac{\varepsilon}{2},r,R_1)$. Then by the argument leading to (16) we have for all $x\in\overline{B_r}$ and $u\in\mathscr{U}:\|u\|_\infty\le R_1$ a time $t\le\tau_1$ such that

$$|\phi(t,x,u)|\le\varepsilon+\gamma(R_1)-\frac{\varepsilon}{2}. \tag{17}$$

Define

$$R_2 := \gamma^{-1}\Big(\max\big\{\gamma(R_1)-\frac{\varepsilon}{2},0\big\}\Big)=\gamma^{-1}\Big(\max\big\{r-\frac{3\varepsilon}{2},0\big\}\Big).$$

From (17) we obtain for all u with $R_2\le\|u\|_\infty\le R_1$ that for the above t

$$|\phi(t,x,u)|\le\varepsilon+\gamma(\|u\|_\infty).$$

For $k\in\mathbb{N}$ define the times $\tau_k := \tau(\frac{\varepsilon}{2},r,R_k)$ and

$$R_k := \gamma^{-1}\Big(\max\big\{\gamma(R_{k-1})-\frac{\varepsilon}{2},0\big\}\Big) = \gamma^{-1}\Big(\max\big\{r-\frac{(k+1)\varepsilon}{2},0\big\}\Big).$$

Repeating the previous argument we see that for all $x\in\overline{B_r}$ and all $u\in\mathscr{U}$ with $R_{k+1}\le\|u\|_\infty\le R_k$ there is a time $t\le\tau_k$ such that $|\phi(t,x,u)|\le\varepsilon+\gamma(\|u\|_\infty)$. The procedure ends after finitely many steps because eventually $r-\frac{(k+1)\varepsilon}{2}$ becomes negative. The claim now follows for $\tau:=\max\big\{\tau_k \mid k=1,\dots,\lfloor\frac{2r}{\varepsilon}\rfloor+1\big\}$, where $\lfloor\cdot\rfloor$ denotes the integer part of a real number. ■

The key argument in the previous proof is (16) which gives a uniform time in which trajectories corresponding to initial conditions bounded by r and inputs bounded by R are below a certain bound. The rest of the proof is a straightforward manipulation of inequalities. We conjecture that similar arguments are possible for time-delay systems.

V. Conclusion

In this note we collected recent results dealing with characterizations of the input-to-state stability for time-delay systems. First we present a broad picture of relations between ISS and other stability notions for general infinite-dimensional systems (much more general than time-delay systems). Next we show that time-delay systems with Lipschitz continuous nonlinearities and a trivial equilibrium are continuous at the equilibrium provided the system has bounded reachability sets. This makes the characterizations of ISS more precise and easier to apply in practice.

Here another interesting question appears naturally: whether boundedness of reachability sets can be inferred from forward completeness. Though we have not solved this problem, we collected several insights throwing some light on essence of this question. Another important problem is whether limit property is equivalent to uniform limit property for nonlinear forward complete time-delay systems. Again, this problem is outside of the scope of this paper, but we show that this result holds for finite-dimensional systems.

References

[1] M. Arcak and P. Kokotović. Nonlinear observers: a circle criterion design and robustness analysis. *Automatica*, 37(12):1923–1930, 2001.

[2] C. Cai and A. Teel. Characterizations of input-to-state stability for hybrid systems. *Systems & Control Letters*, 58(1):47–53, 2009.

[3] S. Dashkovskiy and M. Kosmykov. Input-to-state stability of interconnected hybrid systems. *Automatica*, 49(4):1068–1074, 2013.

[4] S. Dashkovskiy, M. Kosmykov, A. Mironchenko, and L. Naujok. Stability of interconnected impulsive systems with and without time delays, using Lyapunov methods. *Nonlinear Analysis: Hybrid Systems*, 6(3):899–915, 2012.

[5] S. Dashkovskiy, B. S. Rüffer, and F. R. Wirth. An ISS small gain theorem for general networks. *Mathematics of Control, Signals, and Systems*, 19(2):93–122, 2007.

[6] R. A. Freeman and P. V. Kokotovic. *Robust Nonlinear Control Design: State-Space and Lyapunov Techniques*. Birkhäuser, Boston, MA, 2008.

[7] J. Hale. *Theory of functional differential equations*. Applied Mathematical Sciences Series. SPRINGER VERLAG GMBH, 1977.

[8] H. Ito, Z.-P. Jiang, and P. Pepe. Construction of Lyapunov–Krasovskii functionals for networks of iISS retarded systems in small-gain formulation. *Automatica*, 49(11):3246–3257, 2013.

[9] H. Ito, P. Pepe, and Z.-P. Jiang. A small-gain condition for iISS of interconnected retarded systems based on Lyapunov–Krasovskii functionals. *Automatica*, 46(10):1646–1656, 2010.

[10] Z.-P. Jiang, A. R. Teel, and L. Praly. Small-gain theorem for ISS systems and applications. *Mathematics of Control, Signals, and Systems*, 7(2):95–120, 1994.

[11] I. Karafyllis. Lyapunov theorems for systems described by retarded functional differential equations. *Nonlinear Analysis: Theory, Methods & Applications*, 64(3):590–617, 2006.

[12] I. Karafyllis and Z.-P. Jiang. A vector small-gain theorem for general non-linear control systems. *IMA Journal of Mathematical Control and Information*, 28:309–344, 2011.

[13] I. Karafyllis and Z.-P. Jiang. *Stability and stabilization of nonlinear systems*. Communications and Control Engineering Series. Springer-Verlag, London, 2011.

[14] I. Karafyllis and Z.-P. Jiang. A new small-gain theorem with an application to the stabilization of the chemostat. *Int. J. Robust. Nonlinear Control*, 22(14):1602–1630, 2012.

[15] Y. Lin, E. D. Sontag, and Y. Wang. A smooth converse Lyapunov theorem for robust stability. *SIAM Journal on Control and Optimization*, 34(1):124–160, 1996.

[16] A. Mironchenko. Local input-to-state stability: Characterizations and counterexamples. *Systems & Control Letters*, 87:23–28, 2016.

[17] A. Mironchenko and F. Wirth. Restatements of input-to-state stability in infinite dimensions: what goes wrong? In *Proc. of the 22th International Symposium on Mathematical Theory of Systems and Networks*, pages 667–674, 2016.

[18] A. Mironchenko and F. Wirth. Characterizations of input-to-state stability for infinite-dimensional systems. *IEEE Transactions on Automatic Control*, 2017. Accepted for publication.

[19] P. Pepe and Z.-P. Jiang. A Lyapunov-Krasovskii methodology for ISS and iISS of time-delay systems. *Systems & Control Letters*, 55(12):1006–1014, 2006.

[20] Z. Qiao and H. Guang-Da. Converse Lyapunov Theorem of Input-to-State Stability for Time-delay Systems. *Acta Automatica Sinica*, 2010.

[21] E. D. Sontag. Smooth stabilization implies coprime factorization. *IEEE Transactions on Automatic Control*, 34(4):435–443, 1989.

[22] E. D. Sontag and Y. Wang. On characterizations of the input-to-state stability property. *Systems & Control Letters*, 24(5):351–359, 1995.

[23] E. D. Sontag and Y. Wang. New characterizations of input-to-state stability. *IEEE Transactions on Automatic Control*, 41(9):1283–1294, 1996.

[24] A. R. Teel. Connections between Razumikhin-type theorems and the ISS nonlinear small-gain theorem. *IEEE Transactions on Automatic Control*, 43(7):960–964, 1998.

[25] S. Tiwari, Y. Wang, and Z.-P. Jiang. Nonlinear small-gain theorems for large-scale time-delay systems. *Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis*, 19(1):27–63, 2012.