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Existence of non-coercive Lyapunov functions is equivalent to integral uniform global asymptotic stability

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Abstract In this paper a class of abstract dynamical systems is considered which encompasses a wide range of nonlinear finite- and infinite-dimensional systems. We show that the existence of a non-coercive Lyapunov function without any further requirements on the flow of the forward complete system ensures an integral version of uniform global asymptotic stability. We prove that also the converse statement holds without any further requirements on regularity of the system.

Furthermore, we give a characterization of uniform global asymptotic stability in terms of the integral stability properties and analyze which stability properties can be ensured by the existence of a non-coercive Lyapunov function, provided either the flow has a kind of uniform continuity near the equilibrium or the system is robustly forward complete.

Keywords nonlinear control systems · infinite-dimensional systems · Lyapunov methods · global asymptotic stability

1 Introduction

The theory of Lyapunov functions is one of the cornerstones in the dynamical and control systems theory. Numerous applications of Lyapunov theory include characterization of stability properties of fixed points and more complex attractors [28, 5, 14, 11], conditions for forward completeness of trajectories [1], criteria for the existence

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of a bounded absorbing ball [2, Theorem 2.1.2] etc. Some of these uses extend from finite-dimensional applications to the infinite-dimensional case, while others rely on distinct finite-dimensional arguments.

On the other hand numerous converse results have been obtained which prove the existence of certain types of Lyapunov functions characterizing different stability notions. Indeed, before starting to look for a Lyapunov function it is highly desirable to know in advance that such a Lyapunov function for a given class of systems exists. The first results guaranteeing existence of Lyapunov functions for asymptotically stable systems appeared in the works of Kurzweil [13] and Massera [16]. These have been generalized in different directions, see [11, 19] for references.

The standard definition of a Lyapunov function V, found in many textbooks on finite-dimensional dynamical systems, is that it should be a continuous (or more regular) positive definite and proper function, i.e. a function for which there exist $\mathcal{K}_\infty$[1] functions ψ_1, ψ_2, α such that

$$\psi_1(\|x\|) \leq V(x) \leq \psi_2(\|x\|) \quad \forall x \in X, \tag{1}$$

and such that

$$\dot{V}(x) < -\alpha(\|x\|) \quad \forall x \in X, \tag{2}$$

where $\dot{V}(x)$ is some sort of generalized derivative of V along the trajectories of the system.

If V is as above with the exception that instead of (1), V satisfies the weaker property

$$0 < V(x) \leq \psi_2(\|x\|), \quad x \neq 0, \tag{3}$$

then V is called a non-coercive Lyapunov function.

Noncoercive Lyapunov functions are frequently used in the linear infinite-dimensional systems theory. There are at least two reasons for this. On the one hand, using the generalized Datko lemma [4, 15] one can show that the existence of noncoercive Lyapunov functions already proves exponential stability of a linear system (and thus it is not necessary to look for coercive Lyapunov functions). On the other hand, noncoercive Lyapunov functions are in a certain sense even more natural than coercive ones. For example, a classic type of Lyapunov functions for linear exponentially stable systems over Hilbert spaces are quadratic Lyapunov functions constructed by solving the operator Lyapunov equation [3, Theorem 5.1.3]. However, solutions of this equation are not coercive in general, and hence the corresponding Lyapunov functions are not coercive as well.

In spite of these advantages, the usage of non-coercive Lyapunov functions was limited to linear infinite-dimensional systems and to nonlinear time-delay systems, for which the efficient method of Lyapunov-Krasovskii functionals is widely used [6, 21] (Lyapunov-Krasovskii functionals have, however, a different type of noncoercivity, see [18] for a comparison and discussion). Recently the situation has changed: in [19] the authors have shown that for a broad class of forward complete *nonlinear* infinite-dimensional systems existence of a non-coercive Lyapunov function ensures

[1] An increasing, unbounded, continuous, positive definite function from $\mathbb{R}_+$ to itself that maps 0 to 0.

uniform global asymptotic stability (UGAS) of a system, provided the flow of the system has a certain uniform continuity at the origin and finite-time reachability sets of the system are bounded. On the other hand, it was demonstrated in [19] that without these additional assumptions uniform global asymptotic stability cannot be guaranteed, even for systems of ordinary differential equations (ODEs). In particular, the existence of a non-coercive Lyapunov function alone does not ensure forward completeness of the system (in contrast to coercive Lyapunov functions). Hence, although non-coercive Lyapunov functions provide more flexibility for the stability analysis of dynamical systems, further conditions have to be verified separately. Another result of [19] is a construction of a Lipschitz continuous non-coercive Lyapunov function by means of an integration of the solution along trajectories.

In this paper, we continue the investigations initiated in [19]. In *our first main result* (Theorem 3), we show that forward complete systems possessing non-coercive Lyapunov functions (even if they do not satisfy any further assumptions) enjoy an "integral version" of uniform global asymptotic stability (iUGAS), which is a weaker notion than UGAS. *Our second result* (Theorem 4) is a converse non-coercive Lyapunov theorem for the iUGAS property. Since iUGAS is weaker than UGAS, a coercive Lyapunov function does not exist for such systems in general. However, we show (without requiring any further regularity of the flow!) that we can construct a non-coercive Lyapunov function for this system. The construction is motivated by [19] and based upon classic converse theorems and Yoshizawa's method [28, Theorem 19.3], [7, Theorem 4.2.1]. A key tool for achieving our main results are the characterizations of the iUGAS property in terms of weaker stability notions, developed in Theorem 1, which is a third notable result in this work. In Figure 1 we provide a graphical overview of the results obtained in this paper, in particular, the relationship between the introduced stability notions.

Relations between integral and "classic" stability notions have been studied in a number of papers. In particular, in [27] uniform global asymptotic stability of finite-dimensional differential inclusions has been characterized via "integral" uniform attractivity. A natural extension of the iUGAS notion to the case of systems with inputs leads to the nonlinear counterparts of L_2-stability (which was originally introduced in the context of linear systems in the seminal work [29], see also [23]). In [24,12] it was shown that these extensions are equivalent to input-to-state stability for the systems of ordinary differential equations with Lipschitz continuous nonlinearities.

1.1 Notation

The following notation will be used throughout. By $\mathbb{R}_+$ we denote the set of nonnegative real numbers. For an arbitrary set S and $n \in \mathbb{N}$ the n-fold Cartesian product is $S^n := S \times \ldots \times S$. The open ball in a normed linear space X endowed with the norm $\|\cdot\|_X$ with radius r and center in $y \in X$ is denoted by $B_r(y) := \{x \in X \mid \|x-y\|_X < r\}$ (the space X in which the ball is taken, will always be clear from the context). For short, we denote $B_r := B_r(0)$. The (norm)-closure of a set $S \subset X$ will be denoted by $\overline{S}$.

For the formulation of stability properties the following classes of comparison functions are useful, see [5,10]. The set $\mathcal{K}$ is the set of functions $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ that are continuous, strictly increasing and with $\gamma(0) = 0$; $\mathcal{K}_\infty$ is the set of unbounded $\gamma \in \mathcal{K}$; $\mathcal{KL}$ is the set of continuous $\beta : \mathbb{R}_+^2 \to \mathbb{R}_+$, such that $\beta(\cdot, t) \in \mathcal{K}$, for all $t \geq 0$ and $\beta(r, \cdot)$ is decreasing to 0 for all $r > 0$.

2 Problem statement

We consider abstract axiomatically defined time-invariant and forward complete systems on the state space X which are subject to a shift-invariant set of disturbances $\mathcal{D}$.

Definition 1 Consider the triple $\Sigma = (X, \mathcal{D}, \phi)$, consisting of:

(i) A normed linear space $(X, \|\cdot\|_X)$, called the state space, endowed with the norm $\|\cdot\|_X$.
(ii) A set of disturbance values D, which is a nonempty subset of a certain normed linear space.
(iii) A space of disturbances $\mathcal{D} \subset \{d : \mathbb{R}_+ \to D\}$ satisfying the following two axioms.
The axiom of shift invariance states that for all $d \in \mathcal{D}$ and all $\tau \geq 0$ the time shift $d(\cdot + \tau)$ is in $\mathcal{D}$.
The axiom of concatenation is defined by the requirement that for all $d_1, d_2 \in \mathcal{D}$ and for all $t > 0$ the concatenation of d_1 and d_2 at time t

$$d(\tau) := \begin{cases} d_1(\tau), & \text{if } \tau \in [0,t], \\ d_2(\tau - t), & \text{otherwise}, \end{cases} \tag{4}$$

belongs to $\mathcal{D}$.
(iv) A map $\phi : \mathbb{R}_+ \times X \times \mathcal{D} \to X$, called the *transition map*.

The triple Σ is called a (forward complete) system, if the following properties hold:

(Σ1) forward completeness: for every $(x,d) \in X \times \mathcal{D}$ and for all $t \geq 0$ the value $\phi(t,x,d) \in X$ is well-defined.
(Σ2) The identity property: for every $(x,d) \in X \times \mathcal{D}$ it holds that $\phi(0,x,d) = x$.
(Σ3) Causality: for every $(t,x,d) \in \mathbb{R}_+ \times X \times \mathcal{D}$, for every $\tilde{d} \in \mathcal{D}$, such that $d(s) = \tilde{d}(s)$, $s \in [0,t]$ it holds that $\phi(t,x,d) = \phi(t,x,\tilde{d})$.
(Σ4) Continuity: for each $(x,d) \in X \times \mathcal{D}$ the map $t \mapsto \phi(t,x,d)$ is continuous.
(Σ5) The cocycle property: for all $t, h \geq 0$, for all $x \in X$, $d \in \mathcal{D}$ we have $\phi(h, \phi(t,x,d), d(t + \cdot)) = \phi(t+h, x, d)$.

Here $\phi(t,x,d)$ denotes the state of the system at the moment $t \in \mathbb{R}_+$ corresponding to the initial condition $x \in X$ and the disturbance $d \in \mathcal{D}$.

We require a stronger version of forward completeness.

Definition 2 The system $\Sigma = (X, \mathcal{D}, \phi)$ is called robustly forward complete (RFC) if for any $C > 0$ and any $\tau > 0$ it holds that

$$\sup\left\{ \|\phi(t,x,d)\|_X \mid \|x\|_X \leq C,\ t \in [0,\tau],\ d \in \mathcal{D} \right\} < \infty.$$

In other words, a system Σ is RFC iff its finite-time reachability sets (emanating from the bounded sets) are bounded.

The condition of robust forward completeness is satisfied by large classes of infinite-dimensional systems.

Definition 3 We call $0 \in X$ an equilibrium point of the system $\Sigma = (X, \mathscr{D}, \phi)$, if $\phi(t,0,d) = 0$ for all $t \geq 0$, $d \in \mathscr{D}$.

Note that according to the above definition disturbances cannot move the system out of the equilibrium position.

Definition 4 We call $0 \in X$ a robust equilibrium point (REP) of the system $\Sigma = (X, \mathscr{D}, \phi)$, if it is an equilibrium point such that for every $\varepsilon > 0$ and for any $h > 0$ there exists $\delta = \delta(\varepsilon, h) > 0$, satisfying

$$t \in [0,h],\ \|x\|_X \leq \delta,\ d \in \mathscr{D} \quad \Rightarrow \quad \|\phi(t,x,d)\|_X \leq \varepsilon. \tag{5}$$

In this paper we investigate the following stability properties of equilibria of abstract systems.

Definition 5 Consider a system $\Sigma = (X, \mathscr{D}, \phi)$ with a fixed point 0. The equilibrium position 0 is called

(i) uniformly locally stable (ULS), if for every $\varepsilon > 0$ there is a $\delta > 0$ so that

$$\|x\|_X \leq \delta,\ d \in \mathscr{D},\ t \geq 0 \quad \Rightarrow \quad \|\phi(t,x,d)\|_X \leq \varepsilon. \tag{6}$$

(ii) uniformly globally asymptotically stable (UGAS) if there exists a $\beta \in \mathscr{KL}$ such that

$$x \in X,\ d \in \mathscr{D},\ t \geq 0 \Rightarrow \|\phi(t,x,d)\|_X \leq \beta(\|x\|_X, t). \tag{7}$$

(iii) uniformly (locally) asymptotically stable (UAS) if there exist a $\beta \in \mathscr{KL}$ and an $r > 0$ such that

$$\|x\|_X \leq r,\ d \in \mathscr{D},\ t \geq 0 \Rightarrow \|\phi(t,x,d)\|_X \leq \beta(\|x\|_X, t).$$

(iv) uniformly globally weakly attractive (UGWA), if for every $\varepsilon > 0$ and for every $r > 0$ there exists a $\tau = \tau(\varepsilon, r)$ such that for all $\|x\|_X \leq r$, $d \in \mathscr{D}$

$$\exists t = t(x,d,\varepsilon) \leq \tau :\ \|\phi(t,x,d)\|_X \leq \varepsilon.$$

(v) uniformly globally attractive (UGATT), if for any $r, \varepsilon > 0$ there exists $\tau = \tau(r, \varepsilon)$ so that

$$\|x\|_X \leq r,\ d \in \mathscr{D},\ t \geq \tau(r,\varepsilon) \quad \Rightarrow \quad \|\phi(t,x,d)\|_X \leq \varepsilon.$$

It is clear, that UGAS of 0 implies UGATT of 0, which in turn implies UGWA of 0.

As we will see, in the study of non-coercive Lyapunov functions one arrives very naturally at “integral” versions of the notions stated above:

Definition 6 We call $0 \in X$ an integrally robust equilibrium point (iREP) of the system $\Sigma = (X, \mathscr{D}, \phi)$, if it is an equilibrium point and there is $\alpha \in \mathscr{K}$ such that for every $\varepsilon > 0$ and for any $h > 0$ there exists $\delta = \delta(\varepsilon, h) > 0$, satisfying

$$\|x\|_X \leq \delta,\ d \in \mathscr{D} \quad \Rightarrow \quad \int_0^h \alpha(\|\phi(s,x,d)\|_X) ds \leq \varepsilon. \tag{8}$$

Definition 7 For a given $\alpha \in \mathscr{K}$, system $\Sigma = (X, \mathscr{D}, \phi)$ is called α-integrally robustly forward complete (α-iRFC), if for any $C > 0$ and any $\tau > 0$ it holds that

$$\sup_{x \in \overline{B_C},\ d \in \mathscr{D}} \int_0^\tau \alpha\big(\|\phi(t,x,d)\|_X\big) dt < \infty.$$

Remark 1 Note that every forward-complete system is automatically α-iRFC for any bounded $\alpha \in \mathscr{K}$, since $\int_0^\tau \alpha\big(\|\phi(t,x,d)\|_X\big) dt < \tau \sup_s \alpha(s)$. On the other hand, if Σ is RFC, then Σ is also α-iRFC for any $\alpha \in \mathscr{K}$.

Definition 8 Consider a forward complete system $\Sigma = (X, \mathscr{D}, \phi)$ with a fixed point at 0. The equilibrium position 0 is called

(i) integrally uniformly locally stable (iULS) provided there are $\alpha \in \mathscr{K}$, $\psi \in \mathscr{K}_\infty$ and $r > 0$ so that

$$\|x\|_X \leq r,\ d \in \mathscr{D} \Rightarrow \int_0^\infty \alpha(\|\phi(s,x,d)\|_X) ds \leq \psi(\|x\|_X). \tag{9}$$

(ii) integrally uniformly globally stable (iUGS) provided there are $\alpha \in \mathscr{K}$, $\psi \in \mathscr{K}_\infty$ so that (9) is valid for $r := \infty$.

(iii) integrally uniformly globally attractive (iUGATT) provided there is $\alpha \in \mathscr{K}$ so that

$$\forall\, r > 0 : \lim_{t \to \infty} \sup_{x \in \overline{B_r},\ d \in \mathscr{D}} \int_t^\infty \alpha(\|\phi(s,x,d)\|_X) ds = 0. \tag{10}$$

(iv) integrally uniformly globally asymptotically stable (iUGAS) provided there are $\alpha \in \mathscr{K}$ and $\beta \in \mathscr{KL}$ so that for all $x \in X,\ d \in \mathscr{D},\ t \geq 0$ we have

$$\int_t^\infty \alpha(\|\phi(s,x,d)\|_X) ds \leq \beta(\|x\|_X, t). \tag{11}$$

Properties (9) and (10) resemble a kind of uniform attractivity. This similarity becomes even more apparent if we rewrite the definition of UGATT in an equivalent form:

Lemma 1 *Let $\Sigma = (X, \mathscr{D}, \phi)$ be a forward complete system with fixed point 0. Then 0 is UGATT iff there is $\alpha \in \mathscr{K}$ so that*

$$\lim_{t \to \infty} \sup_{x \in \overline{B_r},\ d \in \mathscr{D}} \alpha\Big(\big\|\phi(\cdot + t, x, d)\big\|_{C(X)}\Big) = 0 \quad \forall r > 0, \tag{12}$$

where $\|\phi(\cdot + t, x, d)\|_{C(X)}$ is the sup*-norm of the "tail" of the trajectory ϕ after the time t.*

Proof If 0 is UGATT, then for any $\alpha \in \mathscr{K}_\infty$ and any $r, \varepsilon > 0$ there exists $\tau = \tau(r,\varepsilon)$ so that

$$\|x\|_X \le r,\ d \in \mathscr{D},\ t \ge \tau(r,\varepsilon) \quad \Rightarrow \quad \|\phi(t,x,d)\|_X \le \alpha^{-1}(\varepsilon).$$

Equivalently, the left hand side implies $\alpha\big(\|\phi(t,x,d)\|_X\big) \le \varepsilon$ and taking the limit $\varepsilon \to +0$ we arrive at (12). The proof of the converse implication is analogous.

Remark 2 Note that merely choosing a positive definite α in (12) (i.e. $\alpha \in C(\mathbb{R}_+,\mathbb{R}_+)$: $\alpha(0) = 0$ and $\alpha(r) > 0$ for $r > 0$) we do not arrive at any kind of attractivity, since the trajectory may grow to infinity, and $\alpha(\|\phi(t,x,d)\|_X)$ may converge to zero at the same time. E.g. consider $\dot{x}(t) = x(t)$, $x(t) \in \mathbb{R}$, $\alpha(r) := \frac{r}{r^2+1}$.

Analogously, it is possible to restate the UGS property. In Theorem 2 we will show that UGAS implies iUGAS.

Finally, it is easy to see that

Lemma 2 *Let $\Sigma = (X,\mathscr{D},\phi)$ be a system. If 0 is a REP, then 0 is an iREP with arbitrary $\alpha \in \mathscr{K}_\infty$.*

Proof Fix $\alpha \in \mathscr{K}_\infty$. Since 0 is a REP of $\Sigma = (X,\mathscr{D},\phi)$, for every $\varepsilon > 0$, $h > 0$ there is $\delta = \delta(\varepsilon,h) > 0$ such that

$$\|x\|_X \le \delta,\ d \in \mathscr{D} \quad \Rightarrow \quad \sup_{t\in[0,h]} \|\phi(s,x,d)\|_X \le \alpha^{-1}\Big(\frac{\varepsilon}{h}\Big).$$

Hence it holds that

$$\|x\|_X \le \delta,\ d \in \mathscr{D} \quad \Rightarrow \quad \int_0^h \alpha\big(\|\phi(s,x,d)\|_X\big)ds \le \varepsilon,$$

which shows 0 is an iREP with the above α.

We now introduce Lyapunov functions which will help in characterizing the UGAS and iUGAS concepts. To this end we first recall the notion of Dini derivative. For $h : \mathbb{R} \to \mathbb{R}$ the right-hand lower Dini derivative D_+ and the right-hand upper Dini derivative D^+ at a point $t \in \mathbb{R}$ are defined by, see [26],

$$\begin{aligned} D_+h(t) &:= \underline{\lim}_{\tau\to+0} \frac{1}{\tau}\big(h(t+\tau) - h(t)\big), \\ D^+h(t) &:= \overline{\lim}_{\tau\to+0} \frac{1}{\tau}\big(h(t+\tau) - h(t)\big). \end{aligned} \tag{13}$$

Consider a system $\Sigma = (X,\mathscr{D},\phi)$ and let $V : X \to \mathbb{R}$ be a map. Given $x \in X, d \in \mathscr{D}$, we consider the (right-hand lower) Dini derivative of the function $t \mapsto V(\phi(t,x,d))$ at $t = 0$ denoted by:

$$\dot{V}_d(x) := \underline{\lim}_{t\to+0} \frac{1}{t}\Big(V\big(\phi(t,x,d)\big) - V(x)\Big). \tag{14}$$

We call this the Dini derivative of V along the trajectories of Σ. We stress that at this point no continuity assumption has been placed on V.

Having introduced the main stability properties, we introduce now a predominant tool for their study, which is a Lyapunov function.

Definition 9 Consider a system $\Sigma = (X, \mathscr{D}, \phi)$ and a function $V : X \to \mathbb{R}_+$, satisfying for each $y \in X$, each $s > 0$ and each $d \in \mathscr{D}$ the inequalities

$$\underline{\lim}_{h\to+0} V\big(\phi(s-h,y,d)\big) \geq V\big(\phi(s,y,d)\big) \geq \underline{\lim}_{h\to+0} V\big(\phi(s+h,y,d)\big).$$

Assume also that the right inequality in (15) is satisfied for $s := 0$ as well. The map V is called:

(i) a *non-coercive Lyapunov function* for the system $\Sigma = (X, \mathscr{D}, \phi)$, if $V(0) = 0$ and if there exist $\psi_2 \in \mathscr{K}_\infty$ and $\alpha \in \mathscr{K}$ such that

$$0 < V(x) \leq \psi_2(\|x\|_X) \quad \forall x \in X \backslash \{0\}. \tag{15}$$

holds and the Dini derivative of V along the trajectories of Σ satisfies

$$\dot{V}_d(x) \leq -\alpha(\|x\|_X) \tag{16}$$

for all $x \in X$ and all $d \in \mathscr{D}$.

(ii) a *(coercive) Lyapunov function* if in addition there is $\psi_1 \in \mathscr{K}_\infty$ satisfying $\psi_1(\|x\|_X) \leq V(x)$ for all $x \in X$.

The inequalities (15) say that if a Lyapunov function is not continuous along a trajectory at some point, then its value jumps down at this point.

The following result is known:

Proposition 1 *Let $\Sigma = (X, \mathscr{D}, \phi)$ be a system. Then:*

(i) If there exists a coercive continuous Lyapunov function for Σ, then 0 is UGAS.

(ii) If there exists a non-coercive continuous Lyapunov function for Σ, and if Σ is RFC and 0 is a robust equilibrium, then 0 is UGAS.

Proposition 1 (i) is a classic result, and item (ii) has been shown in [19], where the concept of a non-coercive Lyapunov function for nonlinear systems has been introduced and analyzed. *There is an apparent distinction in the results (i) and (ii)*, in that in item (ii) the existence of a non-coercive Lyapunov function implies UGAS, provided that REP and RFC hold. In case that either REP or RFC do not hold, non-coercive Lyapunov functions do not imply UGAS, as demonstrated by examples in [19].

This difference in the formulations of items (i) and (ii) of Proposition 1 motivates the first question:

What are the stability properties, which can be inferred from the existence of a non-coercive Lyapunov function, without requiring any further assumptions on Σ?

On the other hand, it is well-known, that UGAS implies existence of a coercive Lyapunov function, at least under certain regularity assumptions on the flow of Σ. This leads to the second problem which we analyze in this paper:

What property, which is weaker than UGAS, implies existence of a non-coercive Lyapunov function (and at the same time does not imply the existence of a coercive Lyapunov function)?

In Section 5 we resolve both these questions by showing that *existence of a non-coercive Lyapunov function is equivalent to the iUGAS property*. Moreover, in Section 3 we show several useful criteria for iUGAS and iUGATT, we give "atomic decompositions" of the UGAS property in Section 4. Furthermore, in Section 5 we analyze which stability properties can be ensured by the existence of a non-coercive Lyapunov function, if it is only assumed that either 0 is a REP or that the RFC property of Σ holds.

In Figure 1 a reader can find a graphical overview of the results obtained in this paper, in particular, the relationship between the introduced stability notions.

3 Criteria for iUGATT and iUGAS

In this section we study "integral" stability properties starting with criteria for integral UGATT and then for iUGAS.

3.1 Criteria for integral UGATT

First we would like to give a criterion for iUGATT in terms of UGWA. To this end we need one more concept:

Definition 10 Let Σ be a forward complete system. We say that the fixed point 0 is ultimately (locally) integrally stable (ultimately iULS) if there is $\alpha \in \mathcal{K}$ so that for any $\varepsilon > 0$ there exist $T = T(\varepsilon) > 0$ and $\delta = \delta(\varepsilon) > 0$ so that

$$\|x\|_X \leq \delta,\ d \in \mathcal{D} \quad \Rightarrow \quad \int_{T(\varepsilon)}^{\infty} \alpha\big(\|\phi(t,x,d)\|_X\big) ds \leq \varepsilon. \tag{17}$$

Now we are in a position to characterize iUGATT.

Proposition 2 *Consider a forward complete system $\Sigma = (X, \mathcal{D}, \phi)$ with fixed point 0. Then 0 is iUGATT with some $\alpha \in \mathcal{K}$ if and only if 0 is UGWA and ultimately iULS with the same α.*

Proof $\Rightarrow$. Assume 0 is iUGATT for a given $\alpha \in \mathcal{K}$. Ultimate iULS of 0 (with the same weight function α) easily follows from iUGATT. Let us show that 0 is UGWA.

Pick any $R > 0$ and any $\varepsilon > 0$. Since 0 is integrally UGATT, there is a time $\tilde{\tau} = \tilde{\tau}(R, \varepsilon)$ so that

$$\sup_{x \in \overline{B_r},\, d \in \mathcal{D}} \int_{\tilde{\tau}(R,\varepsilon)}^{\infty} \alpha(\|\phi(s,x,d)\|_X) ds \leq \frac{1}{2}\alpha(\varepsilon).$$

Assume that for some $x \in \overline{B_r}$, some $d \in \mathcal{D}$ and all $s \in [\tilde{\tau}(R,\varepsilon), \tilde{\tau}(R,\varepsilon)+1]$ it holds that $\|\phi(s,x,d)\|_X \geq \varepsilon$. Then

$$\frac{1}{2}\alpha(\varepsilon) \geq \int_{\tilde{\tau}(R,\varepsilon)}^{\infty} \alpha(\|\phi(s,x,d)\|_X) ds \geq \alpha(\varepsilon),$$

a contradiction. This shows that 0 is uniformly globally weakly attractive with $\tau(R,\varepsilon) := \tilde{\tau}(R,\varepsilon)+1$.

$\Leftarrow$. Since 0 is ultimately iULS, there exists $\alpha \in \mathcal{K}$ so that for all $\varepsilon > 0$ there are $\delta(\varepsilon) > 0$ and $T(\varepsilon) > 0$ so that (17) holds.

Pick any $\varepsilon > 0$ and $r > 0$. Since 0 is uniformly globally weakly attractive, there is a time $\tilde{\tau} = \tilde{\tau}(r,\varepsilon)$ so that for any $x \in B_r$ and any $d \in \mathscr{D}$ there is a time $\bar{t} \in [0, \tilde{\tau}(r,\varepsilon))$ so that $\|\phi(\bar{t},x,d)\|_X \leq \delta(\varepsilon)$.

In view of the ultimate iULS property we have that

$$t \geq T(\varepsilon) \Rightarrow \int_t^\infty \alpha\big(\|\phi(s,\phi(\bar{t},x,d),d(\bar{t}+\cdot))\|_X\big)ds \leq \varepsilon.$$

Due to the cocycle property it holds that

$$\int_t^\infty \alpha\big(\|\phi(s,\phi(\bar{t},x,d),d(\bar{t}+\cdot))\|_X\big)ds = \int_t^\infty \alpha\big(\|\phi(s+\bar{t},x,d)\|_X\big)ds = \int_{t+\bar{t}}^\infty \alpha\big(\|\phi(s,x,d)\|_X\big)ds.$$

Considering $t \geq \bar{t} + T(\varepsilon)$, it is now easy to see that 0 is iUGATT (with the same α).

Analogously to Proposition 2 one can characterize the UGATT property. We define

Definition 11 Consider a forward complete system $\Sigma = (X, \mathscr{D}, \phi)$ with fixed point 0. The fixed point 0 is called ultimately uniformly stable if for any $\varepsilon > 0$ there exist $T = T(\varepsilon) > 0$ and $\delta = \delta(\varepsilon) > 0$ so that

$$t \geq T,\ \|x\| \leq \delta,\ d \in \mathscr{D} \quad \Rightarrow \quad \|\phi(t,x,d)\| \leq \varepsilon. \tag{18}$$

Proposition 3 *Consider a forward complete system $\Sigma = (X, \mathscr{D}, \phi)$ with fixed point 0. Then 0 is UGATT if and only if 0 is ultimately uniformly stable and UGWA.*

Proof "$\Rightarrow$". Clear.

"$\Leftarrow$". Fix $\varepsilon > 0$. Since 0 is ultimately uniformly stable, there are positive $\delta(\varepsilon)$ and $T(\varepsilon)$ so that (18) holds. Now pick any $r > 0$. By uniform global weak attractivity of 0 there is a time $\tilde{\tau} = \tilde{\tau}(r,\varepsilon)$ so that for any $x \in B_r(0)$ and any $d \in \mathscr{D}$ there is a time $\bar{t} \in [0, \tilde{\tau}(r,\varepsilon))$ so that $\|\phi(\bar{t},x,d)\| \leq \delta(\varepsilon)$.

Due to the cocycle property

$$\phi(t+\bar{t},x,d) = \phi(t,\phi(\bar{t},x,d),d(\bar{t}+\cdot)),$$

and in view of ultimate uniform stability we have that

$$t \geq \bar{t} + T(\varepsilon),\ x \in B_r(0),\ d \in \mathscr{D} \Rightarrow \|\phi(t,x,d)\| \leq \varepsilon,$$

Specializing this to $t \geq \tilde{\tau}(r,\varepsilon) + T(\varepsilon)$, we see that 0 is UGATT.

3.2 Characterization of iUGAS

In [19, Proposition 3.7] the following result has been obtained (the statement in [19] was somewhat different, but the proof is exactly the same):

Proposition 4 *Consider a forward complete system $\Sigma = (X, \mathscr{D}, \phi)$ with fixed point* 0*. If* 0 *is iUGS, then* 0 *is UGWA.*

We also note:

Lemma 3 *Consider a forward complete system $\Sigma = (X, \mathscr{D}, \phi)$ with fixed point* 0*. Then* 0 *is iULS if and only if* 0 *is an iREP and ultimately iULS.*

Proof $\Rightarrow$. This is clear.

$\Leftarrow$. Since 0 is ultimately iULS, there is $\alpha_1 \in \mathscr{K}$ so that for any $\varepsilon > 0$ there are $r = r(\varepsilon) > 0$ and a time $\tau = \tau(\varepsilon) > 0$ satisfying

$$\|x\|_X \le r(\varepsilon),\ d \in \mathscr{D} \Rightarrow \int_\tau^\infty \alpha_1\big(\|\phi(s,x,d)\|_X\big)ds \le \frac{\varepsilon}{2}.$$

Now since 0 is an iREP, there is $\alpha_2 \in \mathscr{K}$ so that for these ε, τ there is a $0 < \tilde{\delta} = \tilde{\delta}(\varepsilon) \le r(\varepsilon)$ so that

$$\|x\|_X \le \tilde{\delta},\ d \in \mathscr{D} \Rightarrow \int_0^\tau \alpha_2\big(\|\phi(s,x,d)\|_X\big)ds \le \frac{\varepsilon}{2}.$$

Define $\alpha(s) := \min\{\alpha_1(s), \alpha_2(s)\}$, $s \ge 0$. Clearly, $\alpha \in \mathscr{K}$ and it holds that

$$\|x\|_X \le \tilde{\delta},\ d \in \mathscr{D} \Rightarrow \int_0^{+\infty} \alpha\big(\|\phi(s,x,d)\|_X\big)ds \le \varepsilon.$$

Without loss of generality we can assume that $\tilde{\delta}$ is non-decreasing as a function of ε. Furthermore, by construction it holds that $\tilde{\delta}$ can be continuously extended by $\tilde{\delta}(0) = 0$. Then it can be lowerbounded by a certain $\delta \in \mathscr{K}$.

Now iULS of 0 follows by choosing $\psi(s) := \delta^{-1}(s)$, $s \in [0, \lim_{s\to\infty} \delta(s))$.

The main result in this section is the characterization of the notion of iUGAS:

Theorem 1 *Consider a forward complete system $\Sigma = (X, \mathscr{D}, \phi)$. Then the following statements are equivalent:*

(i) 0 *is iUGAS.*
(ii) 0 *is iUGS.*
(iii) 0 *is iULS (with a certain $\alpha \in \mathscr{K}$) and* 0 *is UGWA.*
(iv) 0 *is an iREP and* 0 *is iUGATT.*
(v) 0 *is iULS and* 0 *is iUGATT.*
(vi) 0 *is iUGS and* 0 *is iUGATT.*

Moreover, in item (iv) the function α can be chosen to be equal to α from item (iii).

Proof **(i) ⇒ (ii).** Evident.

(ii) ⇒ (iii). Follows by Proposition 4.

(iii) ⇒ (iv). As 0 is iULS it follows that it is an iREP. Furthermore, since 0 is UGWA and ultimately iULS with $\alpha \in \mathcal{K}$, then, by means of Proposition 2, 0 is iUGATT with the same α.

(iv) ⇒ (v). This follows directly from Lemma 3 and Proposition 2.

(v) ⇒ (vi). Note that if 0 is iULS or iUGATT for a certain $\alpha \in \mathcal{K}$, then the same is true for any $\tilde{\alpha} \in \mathcal{K}$ with $\tilde{\alpha} \le \alpha$ (here $\le$ is the pointwise ordering). Thus without loss of generality 0 is both iULS and iUGATT for same $\alpha \in \mathcal{K} \setminus \mathcal{K}_\infty$.

As 0 is iUGATT we know that for each $R > 0$ and $\varepsilon > 0$ there is $\tau := \tau(R, \varepsilon)$ so that

$$\sup_{x \in \overline{B_R},\, d \in \mathcal{D}} \int_\tau^\infty \alpha(\|\phi(s,x,d)\|_X) ds \le \varepsilon.$$

Denoting $\alpha(\infty) := \lim_{s \to \infty} \alpha(s) < \infty$ we see that

$$\sup_{x \in \overline{B_R},\, d \in \mathcal{D}} \int_0^\tau \alpha(\|\phi(s,x,d)\|_X) ds \le \tau \alpha(\infty),$$

and hence, for all $R > 0$,

$$\tilde{\sigma}(R) := \sup_{x \in \overline{B_R},\, d \in \mathcal{D}} \int_0^\infty \alpha(\|\phi(s,x,d)\|_X) ds < \infty.$$

Clearly, $\tilde{\sigma}$ is a non-decreasing function of R, and so there exist $\overline{\sigma} \in \mathcal{K}_\infty$ and $c > 0$ so that $\tilde{\sigma}(r) \le \overline{\sigma}(r) + c$ for all $r \in \mathbb{R}_+$. Consequently, for any $x \in X$ and any $d \in \mathcal{D}$ we obtain

$$\int_0^\infty \alpha(\|\phi(s,x,d)\|_X) ds \le \overline{\sigma}(\|x\|_X) + c. \tag{19}$$

As 0 is iULS, there exist $\psi \in \mathcal{K}_\infty$ and $r > 0$ so that

$$\|x\|_X \le r,\ d \in \mathcal{D} \Rightarrow \int_0^\infty \alpha(\|\phi(s,x,d)\|_X) ds \le \psi(\|x\|_X). \tag{20}$$

Using (19), (20) and standard manipulations of $\mathcal{K}_\infty$-functions (see e.g. [25, proof of Lemma I.2, p. 1287]), it may be seen that there is a $\sigma \in \mathcal{K}_\infty$ so that

$$x \in X,\ d \in \mathcal{D} \Rightarrow \int_0^\infty \alpha(\|\phi(s,x,d)\|_X) ds \le \sigma(\|x\|_X).$$

This shows that 0 is iUGS with $\alpha \in \mathcal{K} \setminus \mathcal{K}_\infty$.

(vi) ⇒ (i). As in the previous step, without loss of generality we may assume that the function α is the same in the definitions of iUGS and iUGATT. We now consider a fixed, suitable α.

Since 0 is a iUGS fixed point, there exists $\psi \in \mathcal{K}_\infty$ so that for all $t \ge 0$, $\delta \ge 0$, $\|x\|_X \le \delta$, $d \in \mathcal{D}$ we have

$$\int_0^\infty \alpha(\|\phi(t,x,d)\|_X) dt \le \psi(\delta). \tag{21}$$

For a fixed $\delta \geq 0$, define $\varepsilon_n := \frac{1}{2^n}\psi(\delta)$, $n \in \mathbb{N}$. Let $\tau_0 := 0$. As 0 is iUGATT there exist times $\tau_n := \tau(\varepsilon_n, \delta)$, $n \geq 1$, which we assume without loss of generality to be strictly increasing, such that

$$t \geq \tau_n,\ \|x\|_X \leq \delta,\ d \in \mathscr{D} \Rightarrow \int_t^\infty \alpha(\|\phi(s,x,d)\|_X)ds \leq \varepsilon_n.$$

Define $\omega(\delta,0) := 2\psi(\delta)$ and $\omega(\delta,\tau_n) := \varepsilon_{n-1}$, for $n \in \mathbb{N}$, $n \neq 0$. Extend the function $\omega(\delta,\cdot)$ to $\mathbb{R}_+$ so that $\omega(\delta,\cdot) \in \mathscr{L}$. Note that for any $n \in \mathbb{N}$ and for all $t \in (\tau_n, \tau_{n+1})$ it holds that

$$\int_t^\infty \alpha(\|\phi(s,x,d)\|_X)ds \leq \varepsilon_n < \omega(\delta,t). \tag{22}$$

Doing this for all $\delta \in \mathbb{R}_+$ we obtain a function $\omega : \mathbb{R}_+^2 \to \mathbb{R}_+$.

Now define $\tilde{\beta}(r,t) := \sup_{0 \leq s \leq r} \omega(s,t) \geq \omega(r,t)$. Obviously, $\tilde{\beta}$ is non-decreasing in the first argument and decreasing in the second. Moreover, for each fixed $t \geq 0$, $\tilde{\beta}(r,t) \leq \sup_{0 \leq s \leq r} \omega(s,0) = 2\psi(r)$, which implies that $\tilde{\beta}$ is continuous in the first argument at $r = 0$ for any fixed $t \geq 0$. Now Proposition 9 implies that $(r,t) \mapsto \tilde{\beta}(r,t) + |r|e^{-t}$ may be upper bounded by $\beta \in \mathscr{KL}$ and the estimate (11) is satisfied with such a β.

Remark 3 Note that in all the integral notions we have assumed that the corresponding function α belongs to the class $\mathscr{K}$. If we require in the definitions that α must belong to the class $\mathscr{K}_\infty$, we obtain stronger versions of the corresponding concepts. The difference is that every forward-complete system is automatically α-integrally RFC with $\alpha \in \mathscr{K} \backslash \mathscr{K}_\infty$ (see Remark 1), but it need not be α-integrally RFC for all $\alpha \in \mathscr{K}_\infty$.

For the stronger concepts the proof of (iv) $\Rightarrow$ (i) in Theorem 1 does not work as described. In order to close the gap in the proof the stronger version of this implication, we need to strengthen the assumptions in items (iii), (iv) by assuming in addition that the system is α-iRFC with a certain $\alpha \in \mathscr{K}_\infty$. Then after some minor modifications we recover the characterization of iUGAS with $\alpha \in \mathscr{K}_\infty$.

We leave the details to the reader.

4 "Integral" characterization of the UGAS property

Until now we have worked nearly completely on the level of the "integral" notions, which is almost parallel to the world of classic notions of stability. Now we are going to relate "integral" and "classic" worlds.

The next proposition shows that classic stability properties can be recovered from the "integral" version combined with the REP property.

Proposition 5 *Consider a forward complete system $\Sigma = (X, \mathscr{D}, \phi)$. Then the following holds:*

(i) If 0 is a REP and iULS, then 0 is ULS.
(ii) If 0 is a REP and iUGATT, then 0 is UGATT and UAS.

Remark 4 We note that in [17, Theorem 3.1] it is shown that if Σ is RFC and 0 is iUGATT or iUGS or iUGAS, then Σ satisfies a property that is termed practically UGAS in [17] and which amounts to saying that not the fixed point 0 but a certain neighborhood of it has a stability property.

Proof (of Proposition 5).
(i). Seeking a contradiction, assume that Σ is not uniformly stable in $x^* = 0$. Then there exist an $\varepsilon > 0$ and sequences $\{x_k\}_{k\in\mathbb{N}}$ in X, $\{d_k\}_{k\in\mathbb{N}}$ in $\mathscr{D}$, and $t_k \geq 0$ such that $x_k \to 0$ as $k \to \infty$ and

$$\|\phi(t_k, x_k, d_k)\|_X = \varepsilon \quad \forall k \geq 1.$$

Since 0 is iULS, there are $\alpha \in \mathscr{K}$ and $\psi \in \mathscr{K}_\infty$ so that for the above ε there is a $\delta_1 = \delta_1(\varepsilon) > 0$ satisfying

$$\|x\|_X \leq \delta_1,\ d \in \mathscr{D} \Rightarrow \int_0^\infty \alpha\big(\|\phi(s,x,d\|_X\big)ds \leq \psi(\|x\|_X). \tag{23}$$

Without loss of generality we assume that $\|x_k\|_X \leq \delta_1$ for all $k \in \mathbb{N}$ (otherwise we can pick a subsequence of $\{x_k\}$ with this property).

Since 0 is a REP, for the above ε there is a $\delta = \delta(\varepsilon, 1)$ so that

$$\|x\|_X \leq \delta,\ t \in [0,1],\ d \in \mathscr{D} \quad \Rightarrow \quad \|\phi(t,x,d)\|_X \leq \frac{\varepsilon}{2}. \tag{24}$$

Define for this δ the following quantities:

$$\tilde{t}_k := \sup\{t \in [0,t_k] : \|\phi(t,x_k,d_k)\|_X \leq \delta\},$$

provided the supremum is taken over a nonempty set, and $\tilde{t}_k := 0$ otherwise. Denote also $\eta_k := t_k - \tilde{t}_k$, $k \in \mathbb{N}$. There are two possibilities.

First assume that $\{\eta_k\}_{k\in\mathbb{N}}$ does not converge to 0 as $k \to \infty$. Then there is a $\eta^* > 0$ and a subsequence of $\{\eta_{k_m}\}$ so that $\eta_{k_m} \geq \eta^*$ for all $m \geq 1$.

Using (23) for $x := x_{k_m}$, $d := d_{k_m}$ and $t := t_{k_m}$, we see that

$$\eta^* \alpha(\delta) \leq \eta_{k_m} \alpha(\delta) \leq \psi(\|x_{k_m}\|_X).$$

Since $\psi(\|x_{k_m}\|_X) \to 0$ as $m \to \infty$, we obtain a contradiction.

Now assume that $\eta_k \to 0$ as $k \to \infty$. Then there is a $k_1 > 0$ so that $\eta_{k_1} < 1$. In view of a cocycle property, we have that

$$\phi(t_k, x_k, d_k) = \phi(\eta_k, \phi(\tilde{t}_k, x_k, d_k), d_k(\cdot + \tilde{t}_k)).$$

Since $\|\phi(\tilde{t}_k, x_k, d_k)\|_X \leq \delta$, by (24) we obtain $\|\phi(t_k, x_k, d_k)\|_X \leq \frac{\varepsilon}{2}$, which contradicts to the assumption that $\|\phi(t_k, x_k, d_k)\|_X = \varepsilon$. This shows uniform stability of 0.

(ii). It is easy to see that iUGATT implies ultimate iULS. According to Lemma 2, 0 is an iREP. Using Lemma 3 and Proposition 5 (i) we have that 0 is ULS.

Furthermore, by Proposition 2 the equilibrium point 0 is UGWA, and Proposition 3 shows that 0 is UGATT. Finally, since 0 is UGATT and ULS, then 0 is UAS as well.

Next we show criteria for UGAS in terms of integral stability notions. To this end we need two technical results. The first one is Sontag's well-known $\mathcal{KL}$-lemma [24, Proposition 7]:

Lemma 4 *For all $\beta \in \mathcal{KL}$ there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ with*

$$\beta(r,t) \leq \alpha_2(\alpha_1(r)e^{-t}) \quad \forall r \geq 0,\ \forall t \geq 0. \tag{25}$$

The second one is a characterization of UGAS in terms of the UGATT property from [9, Theorem 2.2]:

Proposition 6 *Consider $\Sigma = (X, \mathcal{D}, \phi)$. Then 0 is UGAS if and only if Σ is robustly forward complete and 0 is a UGATT robust equilibrium point for Σ.*

The main result of this section is:

Theorem 2 *Consider a forward complete system $\Sigma = (X, \mathcal{D}, \phi)$. Then the following statements are equivalent:*

(i) 0 *is UGAS.*
(ii) Σ *is RFC and* 0 *is a REP* $\wedge$ *iUGAS.*
(iii) Σ *is RFC and* 0 *is a REP* $\wedge$ *iUGATT.*
(iv) Σ *is RFC and* 0 *is a REP* $\wedge$ *UGWA* $\wedge$ *ultimately iULS.*
(v) Σ *is RFC and* 0 *is a REP* $\wedge$ *UGWA* $\wedge$ *ultimately ULS.*

Proof **(i) $\Rightarrow$ (ii).** Since 0 is UGAS, there is a $\beta \in \mathcal{KL}$ so that (7) holds. In view of Lemma 4 there are $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ so that (25) holds. Set $\alpha := \alpha_2^{-1}$. Then we have for any $r > 0$ and any $t > 0$ it holds that:

$$\sup_{x \in \overline{B_r},\ d \in \mathcal{D}} \int_t^\infty \alpha(\|\phi(s,x,d)\|_X)ds \leq \sup_{x \in \overline{B_r}} \int_t^\infty \alpha(\beta(\|x\|_X, s))ds$$
$$\leq \int_t^\infty \alpha_1(r)e^{-s}ds = \alpha_1(r)e^{-t}$$

and 0 is iUGAS with $\psi := \alpha_1 \in \mathcal{K}_\infty$ and $\alpha \in \mathcal{K}_\infty$.

(ii) $\Rightarrow$ (iii). Clear.
(iii) $\Leftrightarrow$ (iv). Follows from Proposition 2
(iii) $\Rightarrow$ (v). Follows from Proposition 5, item (ii).
(v) $\Rightarrow$ (i). Follows from Propositions 3 and 6.

Remark 5 ("Atomic decompositions") Items (iv) and (v) of Theorem 2 give a decomposition of UGAS into elementary stability notions. In some sense the notions of UGWA, REP, RFC and ultimate ULS and their integral counterparts iREP, ultimate iULS and (possibly) iRFC are the "atoms" by combinations of which the other stability notions can be constructed.

Comparing items (iv) and (v) of Theorem 2 to the analogous "atomic" decompositions of iUGAS shown in Theorem 1, we see that the notion of UGWA plays a remarkable role in such characterizations, supported by the integral variants of REP and ultimate ULS. Uniform global weak attractivity is the common point of the worlds of classic and integral notions, which are otherwise largely parallel.

Remark 6 It is worth mentioning that for the special case of linear systems over Banach spaces without disturbances the notions of UGAS, iUGAS and UGWA coincide, as can be seen from [17, Proposition 5.1].

5 Non-coercive Lyapunov theorems

In this section we relate the existence of noncoercive Lyapunov functions to the integral stability concepts we have introduced. It is shown that for forward complete systems the existence of noncoercive Lyapunov functions implies iUGAS. In the next step we treat a converse result.

5.1 Direct Lyapunov theorems

For the proof of direct Lyapunov theorems we need the generalized Newton-Leibniz formula (see [22, Theorem 7.3, p. 204-205] and the comments directly after that result):

Proposition 7 *Suppose that $F : \mathbb{R} \to \mathbb{R}$ is a function[2] such that for all $x \in \mathbb{R}$ we have*

$$\overline{\lim_{h\to+0}} F(x-h) \le F(x) \le \overline{\lim_{h\to+0}} F(x+h). \tag{26}$$

Let g be a Perron-integrable[3] function of a real variable satisfying $D^+F(x) \ge g(x)$ for all $x \in I$. Then for all $a,b > 0$: $a < b$ it holds that

$$F(b) - F(a) \ge (P)\int_a^b g(x)dx. \tag{27}$$

Using Theorem 1 and Proposition 7, we can show that the existence of a non-coercive Lyapunov function implies iUGAS without any further requirements on the flow of the system. If we additionally assume either the REP or the RFC property, we obtain additional stability properties.

Theorem 3 *Consider a forward complete system $\Sigma = (X, \mathscr{D}, \phi)$. Assume that V is a non-coercive Lyapunov function for Σ with corresponding functions $\alpha \in \mathscr{K}$ and $\psi_2 \in \mathscr{K}_\infty$. Then:*

- *(i) The following statements hold:*
  - *(i-a) 0 is iUGS with this α and with $\psi := \psi_2$.*
  - *(i-b) 0 is iUGATT with this α.*
  - *(i-c) 0 is iUGAS.*
- *(ii) If additionally 0 is a REP, then 0 is UGATT and UAS.*
- *(iii) If additionally 0 is a REP and Σ is RFC, then 0 is UGAS.*

Proof **(i-a).** Since V is a non-coercive Lyapunov function (with a corresponding $\alpha \in \mathscr{K}$), we have the decay estimate (16). Pick any $x \in X$ and any $d \in \mathscr{D}$ and define $\xi : \mathbb{R}_+ \to \mathbb{R}$ via $\xi(t) := -V(\phi(t,x,d))$.

Along the trajectory ϕ of Σ we have the inequality

$$\dot{V}_{d(t+\cdot)}(\phi(t,x,d)) \le -\alpha(\|\phi(t,x,d)\|_X), \quad \forall t \ge 0. \tag{28}$$

[2] In the formulation of [22, Theorem 7.3, p. 204-205] the terminology that F is a finite function is used, which means that $F(x) \in \mathbb{R}$ for any $x \in \mathbb{R}$ (see [22, p. 6]).

[3] For a definition of Perron integrability see e.g. [22, p. 201]. The (P) in front of the integral in (27) indicates that this is a Perron integral.

Due to the cocycle property we have

$$\begin{aligned}
&\dot{V}_{d(t+\cdot)}(\phi(t,x,d))\\
&\quad= \underline{\lim}_{h\to+0}\frac{1}{h}\big(V(h,\phi(t,x,d),d(t+\cdot))-V(\phi(t,x,d))\big)\\
&\quad= \underline{\lim}_{h\to+0}\frac{1}{h}\big(V(\phi(t+h,x,d))-V(\phi(t,x,d))\big)\\
&\quad= \underline{\lim}_{h\to+0}\frac{1}{h}\big(-\xi(t+h)+\xi(t)\big)=D_+(-\xi(t))=-D^+\xi(t).
\end{aligned}$$

With this new notation, equation (28) can be rewritten as

$$D^+\xi(t)\geq\alpha(\|\phi(t,x,d)\|_X),\quad \forall t\geq 0. \tag{29}$$

In view of (15) and since $\underline{\lim}_{h\to+0} -\xi(t+h) = -\overline{\lim}_{h\to+0}\xi(t+h)$, we see that the inequality

$$\overline{\lim}_{h\to+0}\xi(t-h)\leq\xi(t)\leq\overline{\lim}_{h\to+0}\xi(t+h) \tag{30}$$

is satisfied for all $t>0$, and the right inequality is satisfied for $t=0$ as well.

Now we can apply Proposition 7 to the above inequality. Since $t\mapsto\phi(t,x,d)$ is continuous due to the continuity axiom $\Sigma 4$, the function $g: t\mapsto\alpha(\|\phi(t,x,d)\|_X)$ is continuous as well, and thus it is Riemann integrable on any compact interval in $\mathbb{R}_+$. As g is a positive function, the Riemann and the Perron integral coincide (see [22, p. 203]). Thus in our case the Perron integral in the formula (27) is merely a Riemann integral.

Applying Proposition 7, we obtain:

$$\xi(t)-\xi(0)\geq\int_0^t\alpha(\|\phi(s,x,d)\|_X)ds,\quad \forall t\geq 0. \tag{31}$$

Since $\xi(0)=-V(\phi(0,x,d))=-V(x)$ due to the identity axiom of Σ, the above inequality immediately implies that

$$V(\phi(t,x,d))-V(x)\leq-\int_0^t\alpha(\|\phi(s,x,d)\|_X)ds,$$

which in turn shows that for all $t\geq 0$ we have

$$\int_0^t\alpha(\|\phi(s,x,d)\|_X)ds\leq V(x)\leq\psi_2(\|x\|_X). \tag{32}$$

Taking the limit $t\to\infty$, we see that 0 is iUGS.

(i-b). By Proposition 4 and item (i) we see that 0 is UGWA. Checking the proof of the Theorem 1 (implication (iii) $\Rightarrow$ (iv)) we see that 0 is iUGATT with the same α.

(i-c). Follows from items (i), (ii) and Theorem 1.

(ii). By the item (i-b) of this theorem, 0 is iUGATT. Now Corollary 5 implies that 0 is UGATT and UAS.

(iii). By the item (i-b) of this theorem, 0 is iUGATT. The rest follows from Theorem 2.

Remark 7 Condition (15) means that for each $x \in X$ and $d \in \mathscr{D}$ the map $t \mapsto V(\phi(t,x,d))$ is either continuous at t^* or the function jumps down at t^*, for arbitrary $t^* \in \mathbb{R}_+$.

Remark 8 The crucial difference of Theorem 3 from classic Lyapunov theorems is that we do not assume the coercivity of a Lyapunov function. This makes it impossible to use any kind of a comparison principle to derive the desired UGAS stability property.

On the other hand, in contrast to the non-coercive direct Lyapunov theorem shown in [19] we assume for item (i) of Theorem 3 neither robustness of the trivial equilibrium, nor the RFC property of the system Σ (however, we still assume in advance the forward completeness of system Σ). Even under such mild assumptions (and with very mild regularity assumptions on V) we are able to infer the iUGAS property. We note that it is also possible to show a practical UGAS property if in addition to the existence of V we assume RFC. Item (ii) of Theorem 3 is a variation of [17, Corollary 3.10] and is given here for completeness. Item (iii) of Theorem 3 is slightly stronger than [19, Theorem 4.5], where a more direct proof of this result was given.

5.2 Converse non-coercive Lyapunov theorem

We proceed to the converse Lyapunov theorem.

Theorem 4 *Consider a forward complete system $\Sigma = (X, \mathscr{D}, \phi)$ and let 0 be an equilibrium of Σ. Assume that Σ is iUGS with $\alpha \in \mathscr{K}$ and $\psi \in \mathscr{K}_\infty$. Then for any $\rho \in \mathscr{K} \setminus \mathscr{K}_\infty$ so that $\rho(r) \le \alpha(r)$ for all $r \in \mathbb{R}_+$ it holds that*

$$V(x) := \sup_{d \in \mathscr{D}} \int_0^\infty \rho(\|\phi(s,x,d)\|_X) ds \tag{33}$$

is a (possibly non-coercive) Lyapunov function for Σ, satisfying (15) *with ψ_2 as above and so that* (15) *holds.*

Before we proceed to the proof of Theorem 4, we would like to stress, that in contrast to most of the converse Lyapunov theorems for infinite-dimensional nonlinear systems (as [17], [9, Section 3.4]), we do not impose any additional regularity assumptions on the flow of the system, in particular, we assume neither continuous dependence on data, nor robustness of the equilibrium point, nor the RFC property. Theorems 3 and 4 together show that noncoercive Lyapunov functions are a natural tool for analysis of integral stability properties.

In the proof we follow ideas from [9, Section 3.4], [17, Theorem 5.6].

Proof (of Theorem 4).

Pick any $\rho \in \mathscr{K} \setminus \mathscr{K}_\infty$ so that $\rho(r) \le \alpha(r)$ for all $r \in \mathbb{R}_+$. **(i).** Since 0 is iUGS with α, ψ, it follows that

$$0 \le V(x) \le \sup_{d \in \mathscr{D}} \int_0^\infty \alpha(\|\phi(s,x,d)\|_X) ds \le \psi_2(\|x\|_X).$$

Since 0 is an equilibrium of Σ, $\phi(s,0,d) \equiv 0$ for all $s \ge 0$ and all $d \in \mathscr{D}$, which immediately implies that $V(0) = 0$. If $x \neq 0$, then by continuity of solutions, we have for every $d \in D$ a $T > 0$ such that $\phi(t,x,d) \neq 0$ for all $t \in [0,T]$. This implies that $V(x) > 0$ and in summary (15) is satisfied.

(ii). To compute the Dini derivative of V, fix $x \in X$ and $v \in \mathscr{D}$. In view of the cocycle property we have for any $h > 0$:

$$\begin{aligned} V\big(\phi(h,x,v)\big) &= \sup_{d\in\mathscr{D}} \int_0^\infty \rho(\|\phi(t,\phi(h,x,v),d)\|_X)dt \\ &= \sup_{d\in\mathscr{D}} \int_0^\infty \rho(\|\phi(t+h,x,\tilde{d})\|_X)dt \\ &= \sup_{d\in\mathscr{D}} \int_h^\infty \rho(\|\phi(t,x,\tilde{d})\|_X)dt, \end{aligned}$$

where the disturbance function $\tilde{d}$ is defined as

$$\tilde{d}(t) := \begin{cases} v(t), & \text{if } t \in [0,h] \\ d(t-h) & \text{otherwise.} \end{cases}$$

Note that $\tilde{d} \in \mathscr{D}$ due to the axiom of concatenation. Since $\tilde{d}(t) = v(t)$ for $t \in [0,h]$, it holds that

$$\begin{aligned} &\int_0^h \rho(\|\phi(t,x,v)\|_X)dt + V\big(\phi(h,x,v)\big) \\ &\quad = \sup_{d\in\mathscr{D}} \Big(\int_0^h \rho(\|\phi(t,x,v)\|_X)dt + \int_h^\infty \rho(\|\phi(t,x,\tilde{d})\|_X)dt \Big) \\ &\quad = \sup_{d\in\mathscr{D}} \Big(\int_0^h \rho(\|\phi(t,x,\tilde{d})\|_X)dt + \int_h^\infty \rho(\|\phi(t,x,\tilde{d})\|_X)dt \Big) \\ &\quad = \sup_{d\in\mathscr{D}} \int_0^\infty \rho(\|\phi(t,x,\tilde{d})\|_X)dt. \end{aligned}$$

Since the supremum cannot decrease, if we allow a larger class of disturbances, it may be seen that for all $h > 0$ we have

$$\begin{aligned} &\int_0^h \rho(\|\phi(t,x,v)\|_X)dt + V\big(\phi(h,x,v)\big) \\ &\qquad\qquad \leq \sup_{d\in\mathscr{D}} \int_0^\infty \rho(\|\phi(t,x,d)\|_X)dt = V(x). \end{aligned} \tag{34}$$

The obtained inequality may be interpreted as an instance of Bellman's principle. To compute the Dini derivative of V along trajectories we note that rearranging the inequality (34) we obtain for all $h > 0$ that

$$\frac{1}{h}\big(V\big(\phi(h,x,v)\big) - V(x)\big) \leq -\frac{1}{h}\int_0^h \rho(\|\phi(t,x,v)\|_X)dt. \tag{35}$$

As the map $t \mapsto \rho(\|\phi(t,x,v)\|_X)$ is continuous by the axiom of continuity, it follows that

$$\lim_{h\to+0} \frac{1}{h}\int_0^h \rho(\|\phi(t,x,v)\|_X)dt = \rho(\|x\|_X),$$

and due to (35) we obtain $\dot{V}_v(x) \leq -\rho(\|x\|_X)$ and so (16) holds for the given $\rho \in \mathcal{K}$.

(iii). It remains to show that (15) holds. The right hand side in this inequality holds (even in $s = 0$) by a direct application of (34) to the case $x = \phi(s,y,d)$ and arbitrary $s \geq 0, y \in X, d \in \mathcal{D}$, because the integral on the left hand side of (34) is always nonnegative.

To show the left hand side, fix $d \in \mathcal{D}$, $s > 0$, $y \in X$ and $h \in (0,s)$. Substitute $x := \phi(s-h,y,d)$ and $v := d(s-h+\cdot)$ into Bellman's inequality (34) to obtain

$$\int_0^h \rho\big(\|\phi(t,\phi(s-h,y,d),d(s-h+\cdot))\|_X\big)dt + V\big(\phi(h,\phi(s-h,y,d),d(s-h+\cdot))\big) \leq V(\phi(s-h,y,d)).$$

Using again the cocycle property and rearranging the terms of the above inequality, we conclude for all $h \in (0,s)$ that

$$V(\phi(s-h,y,d)) \geq V\big(\phi(s,y,d)\big) + \int_0^h \rho(\|\phi(t+s-h,y,d)\|_X)dt.$$

Arguing as above, we obtain that:

$$\underline{\lim}_{h\to+0} V\big(\phi(s-h,y,d)\big) \geq V\big(\phi(s,y,d)\big). \tag{36}$$

This shows (15) and the proof is complete.

6 Conclusions

In order to understand the implications of the existence of non-coercive Lyapunov functions we have introduced several integral notions of stability, which do not measure the pointwise distance to the equilibrium but rather a weighted average along trajectories. It has been shown that in a quite general setting noncoercive Lyapunov functions characterize these integral notions. Also the relation to standard stability notions are discussed, see also Figure 1. It will be of interest to investigate how the results obtained here carry over to questions of input-to-state stability (ISS). Some results in this direction have been recently developed in [20, 8].

We point out that in [17] the relation between uniform weak attractivity and closely related concepts of weak attractivity and recurrence are discussed for systems without inputs.

A Appendix

In this appendix, we show a result providing conditions for the existence of a $\mathcal{KL}$-bound for a function of two arguments that has been used in our proofs. Although the result may not be surprising for the expert, we have not found an explicit reference and so prefer to present the construction here.

We start with an auxiliary statement:

Proposition 8 *For any $z \in C(\mathbb{R}_+,\mathbb{R})$ and for any $\varepsilon > 0$ there is a sequence $\{R_k\}_{k\in\mathbb{Z}} \subset (0,+\infty)$ satisfying the following properties:*

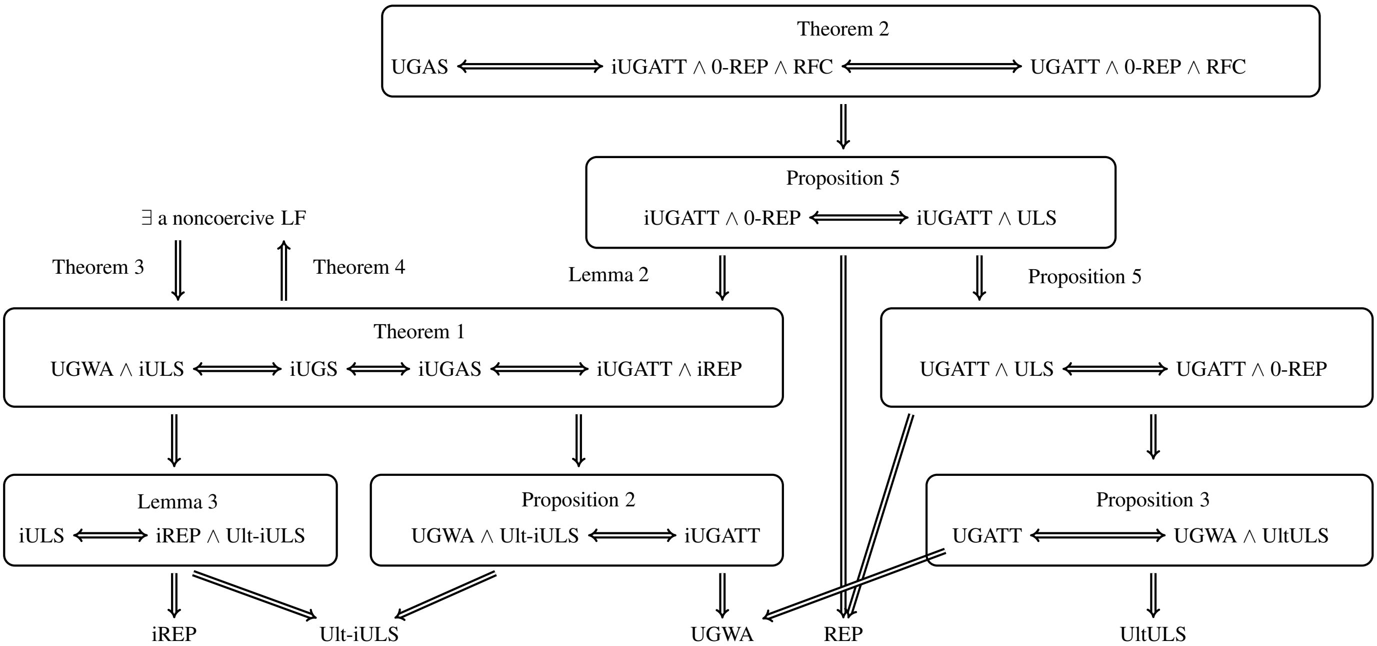

Fig. 1 Relations between stability notions

(i) $R_k \to 0$ *as* $k \to -\infty$.
(ii) $R_k \to +\infty$ *as* $k \to +\infty$.
(iii) $R_k < R_{k+1}$ *for all* $k \in \mathbb{Z}$.
(iv) $\max_{s\in[R_k,R_{k+1}]} z(s) - \min_{s\in[R_k,R_{k+1}]} z(s) < \varepsilon$, *for all* $k \in \mathbb{Z}$.

We stress that by conditions (i), (ii), (iii), zero is the only finite accumulation point of $\{R_k\}_{k\in\mathbb{Z}}$.

Proof It is easy to see that a sequence satisfying the properties (i)–(iii) always exists (and can be chosen independently on z). Pick any such sequence and denote it $\{S_k\}_{k\in\mathbb{Z}}$. Since $z \in C(\mathbb{R}_+,\mathbb{R})$, by Cantor's theorem z is uniformly continuous on $[S_k,S_{k+1}]$ for any $k \in \mathbb{Z}$. Thus, there exists a partition of $[S_k,S_{k+1}]$ into finitely many subintervals with boundary points $S_k = S_{k1} < S_{k2} \ldots < S_{km(k)} = S_{k+1}$ so that $\max_{s\in[S_{ki},S_{ki+1}]} z(s) - \min_{s\in[S_{ki},S_{ki+1}]} z(s) < \varepsilon$, for each $i = 1,\ldots,m(k)-1$.
Now define the desired sequence $\{R_k\}_{k\in\mathbb{Z}}$ by inserting considering the ordered sequence $\{S_{kj}\}_{k\in\mathbb{Z},j=1,\ldots,mk}$. Clearly, $\{R_k\}_{k\in\mathbb{Z}}$ satisfies (i)-(iv).

The following estimation result is useful in our derivations.

Proposition 9 *Let* $\psi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *be any function which is nondecreasing and continuous at 0 in the first argument, nonincreasing in the second argument and so that* $\lim_{t\to\infty} \psi(r,t) = 0$ *for any* $r \geq 0$. *Let also* $\psi(0,t) = 0$ *for any* $t \geq 0$. *Then there exists a* $\beta \in \mathcal{KL}$ *such that*

$$\psi(r,t) \leq \beta(r,t) \quad \forall r,t \geq \mathbb{R}_+.$$

Proof Pick a sequence $R := \{R_k\}_{k\in\mathbb{Z}} \subset (0,+\infty)$ satisfying the properties (i)-(iii) of Proposition 8 and another sequence $\tau := \{\tau_m\}_{m\in\mathbb{N}} \subset [0,+\infty)$ satisfying the properties (ii)-(iii) of Proposition 8 and so that $\tau_0 = 0$. The Cartesian product $R \times \tau$ defines a mesh over $\mathbb{R}_+ \times \mathbb{R}_+$. Let $\omega \in \mathcal{KL}$ be arbitrary so that $\omega(r,t) > 0$ for all $(r,t) \in (0,\infty) \times \mathbb{R}_+$.
For each $k \in \mathbb{Z}$ and $m \in \mathbb{N}$, $m \neq 0$ define

$$\beta(R_k,\tau_m) := \psi(R_{k+1},\tau_{m-1}) + \omega(R_{k+1},\tau_{m-1}).$$

For $m = 0$ define

$$\beta(R_k,0) = \beta(R_k,\tau_0) := 2\psi(R_{k+1},0) + \omega(R_{k+1},0).$$

For each $k \in \mathbb{Z}$ and $m \in \mathbb{N}$ define $\beta(r,t)$ for all (r,t) in the triangles with corner points $((R_{k+1},\tau_m)$, (R_k,τ_{m+1}), $(R_k,\tau_m))$ or $((R_{k+1},\tau_m)$, (R_{k+1},τ_{m+1}), $(R_k,\tau_{m+1}))$ by linear interpolation of the values in the corner points (which have already been defined). This defines the values of β in $(0,+\infty)\times[0,+\infty) = \bigcup_{k\in\mathbb{Z}}\bigcup_{m\in\mathbb{N}}[R_k,R_{k+1}]\times[\tau_k,\tau_{k+1}]$.

Defining $\beta(0,t) := 0, t \geq 0$, we see that β is defined over $\mathbb{R}_+\times\mathbb{R}_+$, is continuous, strictly increasing in the first argument and decreasing in the second argument. Since $\lim_{t\to\infty}\psi(r,t) = 0$ for any $r \geq 0$, it holds also that $\lim_{t\to\infty}\beta(r,t) = 0$ for any $r \geq 0$. Overall, $\beta \in \mathcal{KL}$.

It remains to show that β estimates ψ from above. To see this, pick any $k \in \mathbb{Z}$ and $m \in \mathbb{N}$. For every $(r,s) \in [R_k,R_{k+1}]\times[\tau_m,\tau_{m+1}]$ we have:

$$\begin{aligned}\beta(r,s) - \psi(r,s) &\geq \beta(R_k,\tau_{m+1}) - \psi(R_{k+1},\tau_m)\\ &= \psi(R_{k+1},\tau_m) + \omega(R_{k+1},\tau_m) - \psi(R_{k+1},\tau_m)\\ &= \omega(R_{k+1},\tau_m).\end{aligned}$$

Hence, $\beta(r,s) \geq \psi(r,s)$ for all $r,s \geq 0$.

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