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Local input-to-state stability: Characterizations and counterexamples

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ABSTRACT

We show that a nonlinear locally uniformly asymptotically stable infinite-dimensional system is automatically locally input-to-state stable (LISS) provided the nonlinearity possesses some sort of uniform continuity with respect to external inputs. Also we prove that LISS is equivalent to existence of a LISS Lyapunov function. We show by means of a counterexample that if this uniformity is not present, then the equivalence of local asymptotic stability and local ISS does not hold anymore. Using a modification of this counterexample we show that in infinite dimensions a uniformly globally asymptotically stable at zero, globally stable and locally ISS system possessing an asymptotic gain property does not have to be ISS (in contrast to finite dimensional case).

1. Introduction

Input-to-state stability (ISS) theory of ordinary differential equations (ODEs) is nowadays a developed theory with a firm theoretical basis, with a variety of powerful tools for investigation of ISS and with a multitude of applications in the nonlinear control theory, in particular to robust stabilization of nonlinear systems [1], design of nonlinear observers, analysis of large-scale networks [2–4], etc.

Among the most important results in ISS theory for ODE systems are the characterizations of ISS in terms of Lyapunov functions and other stability properties [5,6]. These theorems have shown that ISS is a central notion in stability theory of control systems and at the same time these results played an important role in the proofs of other important results, e.g. small-gain theorems in a trajectory formulation [3]. In contrast to global ISS property, according to author's knowledge, there are no characterizations of local ISS property available in the literature. This is quite surprising, since such characterizations are useful from theoretical as well from the practical point of view, and at the same time closely related results for other kinds of robustness are well-known (see e.g. [7, Corollary 4.2.3]). The most known result in ISS context is [6, Lemma 1] telling that global asymptotic stability for a zero input (0-GAS) implies LISS for ODE systems.

One of our aims in this paper is to obtain the characterization of the local ISS property. We do not restrict ourselves to consideration of ODE systems, since the questions of robust stabilization, control and observation of infinite-dimensional systems are of central importance in control theory and we believe that ISS is a right tool to handle these questions. Thus, we study (L)ISS of general infinite-dimensional systems of the form

$$\dot{x}(t) = Ax(t) + f(x(t), u(t)), \quad x(t) \in X,\ u(t) \in U, \tag{1}$$

where X is a Banach space, U is a linear normed space, A is the generator of a C_0-semigroup $\{T(t),\ \ t \geq 0\}$ and $f : X \times U \to X$. Many classes of evolution equations, such as parabolic and hyperbolic partial differential equations, can be written in the form (1): [7–9].

In the last years ISS of infinite-dimensional systems (1) as well as of partial differential equations has been studied in a number of papers, see [10–17] to cite a few. But in most of these works the attention has been devoted to construction of Lyapunov functions for ISS systems and to design of robust stabilizing controllers for unstable systems. At the same time the problem of characterizations of local and global ISS for systems (1) is still open.

In this paper we make a step towards its solution. There are two contributions in this paper: a 'positive' and a 'negative' one. Our positive result is that under some sort of uniform continuity of f with respect to external inputs, local uniform asymptotic stability of (1) is equivalent to local ISS of (1) and to existence of a Lipschitz LISS Lyapunov function for it. Thus, our findings imply the result [6, Lemma 1] as a very special case. In the proof of this result we use a technique, used to prove a closely related robustness result for infinite-dimensional systems, see [7, Corollary 4.2.3]. We show by means of a counterexample, that if the nonlinearity f is continuous at the neighborhood of zero, but without additional uniformity, then the main result does not hold.

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Note also that for time-delay systems a related result is available, namely that local exponential stability of the undisturbed systems plus some additional properties of the nonlinearity imply ISS of the disturbed system [18]. However, there are important differences between our paper and [18], in particular, we do not require local exponential stability of (1) and instead we consider merely locally uniformly asymptotically stable systems.

The 'negative' contribution of this paper is an example of a system (obtained by a modification of the previous counterexample), which is locally ISS, uniformly globally asymptotically stable at zero (0-UGAS), globally stable (GS) and possessing an asymptotic gain property (AG), but which is not ISS. This shows that restatements of global ISS property proved in [5,6] for ODE systems (such as ISS = AG + GS = AG + 0-UGAS = AG + LISS), do not hold for systems (1) in general.

We believe that the results obtained in this paper will be useful in applications as well as for the characterization of the global ISS property for abstract systems (1).

2. Preliminaries

Let $\mathbb{R}_+ := [0, \infty)$ and $B_r := \{x \in X : \|x\|_X \le r\}$. In all the pages below we assume that the set of input values U is a normed linear space with the norm $\|\cdot\|_U$ and that the input functions $u : \mathbb{R}_+ \to U$ belong to the space $\mathcal{U} := PC(\mathbb{R}_+, U)$ of bounded piecewise continuous functions, which are right continuous. The norm of $u \in \mathcal{U}$ we denote as $\|u\|_{\mathcal{U}} := \sup_{t\ge 0} \|u(s)\|_U$.

We are going to study weak solutions of (1), i.e. the solutions of the integral equation

$$x(t) = T(t)x(0) + \int_0^t T(t-s)f(x(s), u(s))ds \quad \forall t \in [0, \tau], \tag{2}$$

belonging to $C([0, \tau], X)$ for some $\tau > 0$.

We assume that $f(0, 0) = 0$, i.e., $x \equiv 0$ is an equilibrium point of (1).

Assumption 1. We suppose throughout the paper that the nonlinearity f satisfies the following properties:

(i) $f : X \times U \to X$ is Lipschitz continuous on bounded subsets of X, uniformly with respect to the second argument, i.e. $\forall C > 0\ \exists L_f(C) > 0$, such that $\forall x, y : \|x\|_X \le C,\ \|y\|_X \le C, \forall v \in U$, it holds that

$$\|f(y, v) - f(x, v)\|_X \le L_f(C)\|y - x\|_X. \tag{3}$$

(ii) $f(x, \cdot)$ is continuous for all $x \in X$.

Since $\mathcal{U} = PC(\mathbb{R}_+, U)$, Assumption 1 ensures that the weak solution of (1) exists and is unique, according to a variation of the classical existence and uniqueness theorem [8, Proposition 4.3.3]. We denote by $\phi(t, x, u)$ this solution at moment $t \in \mathbb{R}_+$ associated with an initial condition $x \in X$ at $t = 0$, and input $u \in \mathcal{U}$.

Also we assume that the solution ϕ depends continuously on initial states and external inputs at the neighborhood of the origin, namely:

Assumption 2. For any $\varepsilon > 0$ and for any $\tau > 0$ there exists $\delta > 0$ so that for any $x \in X : \|x\|_X \le \delta$ and for any $u \in \mathcal{U} : \|u\|_{\mathcal{U}} \le \delta$ it follows that $\|\phi(t, x, u)\|_X \le \varepsilon$, for all $t \in [0, \tau]$.

For the formulation of stability properties we use the comparison functions formalism:

$$\mathcal{K} := \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous, strictly increasing and } \gamma(0) = 0\}$$

$$\mathcal{K}_\infty := \{\gamma \in \mathcal{K} \mid \gamma \text{ is unbounded}\}$$

$$\mathcal{L} := \Big\{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous and strictly decreasing with } \lim_{t\to\infty} \gamma(t) = 0\Big\}$$

$$\mathcal{KL} := \{\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \mid \beta \text{ is continuous}, \beta(\cdot, t) \in \mathcal{K},\ \beta(r, \cdot) \in \mathcal{L},\ \forall t \ge 0,\ \forall r > 0\}.$$

The main notions of this paper are:

Definition 1. System (1) is called

- input-to-state stable (ISS), if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that $\forall x \in X, \forall u \in \mathcal{U}$ and $\forall t \ge 0$ the following holds

$$\|\phi(t, x, u)\|_X \le \beta(\|x\|_X, t) + \gamma(\|u\|_{\mathcal{U}}). \tag{4}$$

- locally input-to-state stable (LISS), if there exist $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$ and $r > 0$ such that the inequality (4) holds $\forall x \in B_r$, $\forall u \in \mathcal{U} : \|u\|_{\mathcal{U}} \le r$ and $\forall t \ge 0$.

In order to understand the essence of (L)ISS we introduce several other properties:

Definition 2. System (1)

- is globally asymptotically stable at zero uniformly with respect to state (0-UGASs), if $\exists \beta \in \mathcal{KL}$, such that $\forall x \in X, \forall t \ge 0$ the following inequality holds

$$\|\phi(t, x, 0)\|_X \le \beta(\|x\|_X, t). \tag{5}$$

- is locally asymptotically stable at zero uniformly with respect to state (0-UASs), if for certain $\beta \in \mathcal{KL}$ the estimate (5) holds for all $x \in B_r$ with $r > 0$ small enough.
- is globally stable (GS), if $\exists \sigma \in \mathcal{K}_\infty,\ \gamma \in \mathcal{K}_\infty \cup \{0\}$ such that $\forall x \in X, \forall u \in \mathcal{U}, \forall t \ge 0$ we have

$$\|\phi(t, x, u)\|_X \le \sigma(\|x\|_X) + \gamma(\|u\|_{\mathcal{U}}). \tag{6}$$

- has asymptotic gain (AG) property, if $\exists \gamma \in \mathcal{K}_\infty \cup \{0\}$ such that for all $\varepsilon > 0$, for all $x \in X$ and for all $u \in \mathcal{U}$ there exists $\tau_a = \tau_a(\varepsilon, x, u) < \infty$:

$$\forall t \ge \tau_a \quad \Rightarrow \quad \|\phi(t, x, u)\|_X \le \varepsilon + \gamma(\|u\|_{\mathcal{U}}). \tag{7}$$

- has strong asymptotic gain (sAG) property, if $\exists \gamma \in \mathcal{K}_\infty \cup \{0\}$ such that for all $x \in X$ and for all $\varepsilon > 0$ there exists $\tau_a = \tau_a(\varepsilon, x) < \infty$:

$$\forall t \ge \tau_a, \quad \forall u \in \mathcal{U} \Rightarrow \|\phi(t, x, u)\|_X \le \varepsilon + \gamma(\|u\|_{\mathcal{U}}). \tag{8}$$

Both AG and sAG imply that all trajectories converge to the ball of radius $\gamma(\|u\|_{\mathcal{U}})$ around the origin as soon as $t \to \infty$. The difference between AG and sAG is in a kind of dependence of τ_a on states and inputs. In sAG systems this time depends on the state x (and may vary for the states with the same norm), but it does not depend on u. In AG systems τ_a depends both on x and on u.

A powerful tool to investigate ISS and LISS of control systems is an ISS/LISS Lyapunov function.

Definition 3. A continuous function $V : D \to \mathbb{R}_+$, $0 \in \text{int}(D) \subset X$ is called a LISS Lyapunov function, if there exist $r > 0$ $\psi_1, \psi_2 \in \mathcal{K}_\infty$, $\alpha \in \mathcal{K}_\infty$ and $\sigma \in \mathcal{K}$ such that $B_r \subset D$ and

$$\psi_1(\|x\|_X) \le V(x) \le \psi_2(\|x\|_X), \quad \forall x \in B_r \tag{9}$$

and Lie derivative

$$\dot{V}_u(x) := \overline{\lim_{t\to+0}} \frac{1}{t}(V(\phi(t, x, u)) - V(x))$$

of V along the trajectories of the system (1) satisfies

$$\dot{V}_u(x) \le -\alpha(\|x\|_X) + \sigma(\|u(0)\|_U) \tag{10}$$

for all $x \in B_r$ and $u \in \mathcal{U} : \|u\|_{\mathcal{U}} \le r$.

In [10, Theorem 1] it was shown that

Proposition 1. *Existence of a LISS Lyapunov function implies LISS of (1).*

Now we are in a position to formulate the aims and structure of the paper in a strict way. It has been shown in [5,6] that for ODE systems it holds that:

- ISS ⇔ AG + GS ⇔ AG + 0-UGASs ⇔ AG + LISS.
- 0-UGASs ⇒ LISS.

In this paper we prove that for infinite-dimensional systems (1) the following statements are true

- if the nonlinearity f in (1) has some sort of uniform continuity w.r.t. the inputs, then:
  0-UAS ⇔ LISS ⇔ there exists a LISS-Lyapunov function for (1).
- without this additional uniformity the characterization of LISS does not hold.
- AG with zero gain + sAG + GS + LISS + 0-UGASs ⇏ ISS. Hence in infinite dimensions ISS (a uniform notion) is not equivalent to combinations of nonuniform global notions like AG + GS.

We discuss these questions in Sections 3, 4, 5 respectively.

3. Characterization of local input-to-state stability

In this section we derive a characterization of LISS property. We rely upon the converse Lyapunov theorem for undisturbed systems (1). To this end define:

Definition 4. A continuous function $V : D \to \mathbb{R}_+$, $0 \in \text{int}(D) \subset X$ is called a 0-UAS Lyapunov function, if there exist $r > 0$, $\psi_1, \psi_2 \in \mathcal{K}_\infty$ and $\alpha \in \mathcal{K}_\infty$ such that $\{x \in X : \|x\|_X \le r\} \in D$ and $\forall x \in X : \|x\|_X \le r$ the following estimate holds:

$$\psi_1(\|x\|_X) \le V(x) \le \psi_2(\|x\|_X). \tag{11}$$

Moreover, Lie derivative of V along the trajectories of the system (1) with $u = 0$ satisfies

$$\dot{V}_0(x) \le -\alpha(\|x\|_X) \tag{12}$$

for all $x \in X : \|x\|_X \le r$.

The classical converse Lyapunov theorem for undisturbed systems (1) is formulated as follows, see e.g. [7, Theorem 4.2.1]:

Proposition 2. *Let $f(\cdot, 0)$ be locally Lipschitz continuous in a certain neighborhood of $x = 0$. If (1) is 0-UASs, then there exists a Lipschitz continuous 0-UAS Lyapunov function for (1).*

Remark 1. Note that in [7, Theorem 4.2.1] an analyticity of a semigroup is assumed. However, this assumption has not been used in the proof of [7, Theorem 4.2.1] and was made just because the book is devoted to analytic semigroups. Thus, [7, Theorem 4.2.1] still is true for merely strongly continuous semigroups.

Next result (which is closely related to the known fact about robustness of the 0-UAS property [7, Corollary 4.2.3]) shows, that a Lipschitz continuous 0-UAS Lyapunov function for (1) is, under a certain assumption on the nonlinearity f, also a LISS Lyapunov function for (1).

Proposition 3. *Let Assumptions 1, 2 hold and let there exist $\sigma \in \mathcal{K}$ and $\rho > 0$ so that for all $v \in U : \|v\|_U \le \rho$ and all $x \in X : \|x\|_X \le \rho$ we have*

$$\|f(x, v) - f(x, 0)\|_X \le \sigma(\|v\|_U). \tag{13}$$

Let V be a Lipschitz continuous 0-UAS Lyapunov function for (1). Then V is also a LISS Lyapunov function for (1).

Proof. Let $V : D \to \mathbb{R}_+$, $D \subset X$, with $0 \in \text{int}(D)$ be a Lipschitz continuous (0-UAS) Lyapunov function for (1), which satisfies (12) for $x \in X : \|x\|_X \le r$.

Let Assumption 1 hold and pick ρ given in the formulation of the proposition so that $B_\rho \subset D$.

Due to Assumption 2 there exist $r_1 \in (0, r)$, $r_2 \in (0, \rho)$ and $t^* > 0$ so that for all $x \in X : \|x\|_X \le r_1$ and all $u \in \mathcal{U} : \|u\|_{\mathcal{U}} \le r_2$ the solution $\phi(s, x, u)$ exists for $s \in [0, t^*]$ and $\|\phi(s, x, u)\|_X \le \rho$ for all $s \in [0, t^*]$.

We are going to prove that V is a LISS Lyapunov function for (1). To this end we derive a dissipative estimate for $\dot{V}_u(x)$ for all x, u: $\|x\|_X \le r_1$ and $\|u\|_{\mathcal{U}} \le r_2$. We have:

$$\begin{aligned}\dot{V}_u(x) &= \overline{\lim_{t\to+0}} \frac{1}{t}\Big(V\big(\phi(t,x,u)\big) - V(x)\Big) \\ &= \overline{\lim_{t\to+0}} \frac{1}{t}\Big(V\big(\phi(t,x,0)\big) - V(x) \\ &\quad + V\big(\phi(t,x,u)\big) - V\big(\phi(t,x,0)\big)\Big) \\ &= \dot{V}_0(x) + \overline{\lim_{t\to+0}} \frac{1}{t}\Big(V\big(\phi(t,x,u)\big) - V\big(\phi(t,x,0)\big)\Big).\end{aligned}$$

Since V is a 0-UAS Lyapunov function for (1), due to (12) it holds for a certain $\alpha \in \mathcal{K}_\infty$ that

$$\dot{V}_u(x) \le -\alpha(\|x\|_X) + \overline{\lim_{t\to+0}} \frac{1}{t}\Big|V\big(\phi(t,x,u)\big) - V\big(\phi(t,x,0)\big)\Big|.$$

Since $\phi(t, x, u) \in D$ for all $x, u : \|x\|_X \le r_1$ and $\|u\|_{\mathcal{U}} \le r_2$, and since V is Lipschitz continuous on D, there exists $L > 0$ so that

$$\dot{V}_u(x) \le -\alpha(\|x\|_X) + L\, \overline{\lim_{t\to+0}} \frac{1}{t}\|\phi(t,x,u) - \phi(t,x,0)\|_X. \tag{14}$$

Now we are going to obtain an estimate for $\|\phi(t, x, u) - \phi(t, x, 0)\|_X$ for $t \in [0, t^*]$. The variation of constants formula implies:

$$\begin{aligned}&\|\phi(t,x,u) - \phi(t,x,0)\|_X \\ &\quad = \Big\| \int_0^t T(t-s)\big(f(\phi(s,x,u),u(s)) - f(\phi(s,x,0),0)\big)ds \Big\|_X \\ &\quad \le \int_0^t \|T(t-s)\| \, \big\|f(\phi(s,x,u),u(s)) - f(\phi(s,x,0),0)\big\|_X ds.\end{aligned}$$

Denote $M := \sup_{0 \le s \le t^*} \|T(s)\| < \infty$. With this notation we proceed to

$$\begin{aligned}&\|\phi(t,x,u) - \phi(t,x,0)\|_X \\ &\quad \le \int_0^t M\Big(\|f(\phi(s,x,u),u(s)) - f(\phi(s,x,0),u(s))\|_X \\ &\qquad + \|f(\phi(s,x,0),u(s)) - f(\phi(s,x,0),0)\|_X\Big)ds.\end{aligned}$$

Recalling that $\|\phi(t, x, u)\|_X \le \rho$ for all $t \in [0, t^*]$, using the inequality (13) and due to Lipschitz continuity of f w.r.t. the first argument, there exists $L_2 > 0$:

$$\begin{aligned}&\|\phi(t,x,u) - \phi(t,x,0)\|_X \\ &\quad \le ML_2 \int_0^t \big\|\phi(s,x,u) - \phi(s,x,0)\big\|_X ds + Mt\sigma\big(\sup_{0\le s\le t}\|u(s)\|_U\big) \\ &\quad \le ML_2 t \sup_{0\le s\le t}\|\phi(s,x,u) - \phi(s,x,0)\|_X + Mt\sigma\big(\sup_{0\le s\le t}\|u(s)\|_U\big).\end{aligned}$$

The right hand side of the above inequality is nondecreasing in t and consequently it holds that

$$\begin{aligned}&\sup_{0\le s\le t} \|\phi(s,x,u) - \phi(s,x,0)\|_X \\ &\quad \le ML_2 t \sup_{0\le s\le t}\|\phi(s,x,u) - \phi(s,x,0)\|_X + Mt\sigma\big(\sup_{0\le s\le t}\|u(s)\|_U\big).\end{aligned}$$

Pick $t < t^*$ small enough so that $1 - ML_2 t > 0$. Then

$$\sup_{0\le s\le t} \|\phi(s,x,u) - \phi(s,x,0)\|_X \le \frac{Mt}{1-ML_2t}\sigma\big(\sup_{0\le s\le t} \|u(s)\|_U\big).$$

Using this estimate in (14), taking the limit $t \to +0$ and recalling that $u(\cdot)$ is a right-continuous function, we obtain for all $x : \|x\|_X \le r_1$ and all $u \in \mathcal{U} : \|u\|_{\mathcal{U}} \le r_2$ the LISS estimate

$$\dot{V}_u(x) \le -\alpha(\|x\|_X) + LM\sigma\big(\|u(0)\|_U\big). \quad (15)$$

This shows that V is a LISS Lyapunov function for (1). □

As a corollary of the previous proposition we obtain our main result:

Theorem 4. *Let Assumptions 1, 2 hold and let there exist $\sigma \in \mathcal{K}$ and $\rho > 0$ so that for all $v \in U : \|v\|_U \le \rho$ and all $x \in X : \|x\|_X \le \rho$ we have*

$$\|f(x,v) - f(x,0)\|_X \le \sigma(\|v\|_U).$$

Then for the system (1) the following properties are equivalent:

(i) 0-*UAS.*
(ii) *Existence of a Lipschitz* 0*-UAS Lyapunov function.*
(iii) *Existence of a Lipschitz LISS Lyapunov function.*
(iv) *LISS.*

Proof. The claim of the theorem is a consequence of Propositions 2, 3, 1 and of an obvious fact that LISS implies 0-UAS. □

Remark 2. If $X = \mathbb{R}^n$ and $U = \mathbb{R}^m$ for natural m, n, and if f is continuous w.r.t. the second argument, then (13) holds with

$$\sigma(r) := \sup_{v\in\mathcal{U}:\|v\|_U\le r}\ \sup_{x\in X:\|x\|_X\le\rho} \|f(x,v) - f(x,0)\|_X + r$$

due to compactness of closed balls in $\mathbb{R}^m$. Thus, Theorem 4 implies that 0-UGAS implies LISS for ODE systems (which was shown in finite-dimensional context in [6, Lemma I.1]).

4. Necessity of uniformity w.r.t. inputs

In this section we show by means of an example that the additional assumption in Theorem 4 cannot be dropped in infinite dimensions. We show even more: *a system, which satisfies Assumptions 1, 2 and is 0-UGASs, sAG, AG with zero gain and GS with zero gain may still be not LISS.*

Consider a system Σ with the state space $X = l_1 := \{(x_k)_{k=1}^\infty : \sum_{k=1}^\infty |x_k| < \infty\}$ and with the input space $\mathcal{U} := PC(\mathbb{R}_+, \mathbb{R})$.

Let the dynamics of the kth mode of Σ be given by

$$\dot{x}_k(t) = -\frac{1}{1+k|u(t)|}x_k(t). \quad (16)$$

We use the notation $\phi_k(t, x_k, u)$ for the state of the kth mode of (16). Then $\phi(t,x,u) = (\phi_k(t,x_k,u))_{k=1}^\infty$ (we indicate here that the dynamics of different modes are independent on each other).

Clearly, Σ is 0-UGASs, since for $u \equiv 0$ its dynamics are given by $\dot{x} = -x$. At the same time the inequality $\|\phi(t,x,u)\|_X \le \|x\|_X$ holds for all $t \ge 0$, $x \in X$ and $u \in \mathcal{U}$, and thus Σ is GS with zero gain.

Next we show step-by-step that:

(i) Σ satisfies Assumptions 1 and 2 (and thus belongs to the class of systems which we consider).
(ii) Σ is AG with zero gain.
(iii) Σ is sAG with arbitrarily small (but necessarily nonzero) linear gain. In other words, one must pay for uniformity, but this payment can be made arbitrarily small.
(iv) Finally, Σ is not LISS.

(i) Assumption 2 is automatically satisfied, since as we mentioned above the system Σ is GS with zero gain. Next we show that Assumption 1 also holds.

Note that Σ is globally Lipschitz, since for any $x, y \in X$ and any $v \in U$

$$\|f(x,v) - f(y,v)\|_X = \sum_{k=1}^\infty \frac{1}{1+k|v|}|x_k - y_k| \le \sum_{k=1}^\infty |x_k - y_k| = \|x-y\|_X.$$

Now pick any $x \in X$ and let us show that $f(x,\cdot)$ is continuous at any $v \ne 0$. Consider

$$\begin{aligned}\|f(x,v) - f(x,v_1)\|_X &= \sum_{k=0}^\infty \left|\frac{1}{1+k|v_1|} - \frac{1}{1+k|v|}\right| |x_k| \\ &= \sum_{k=0}^\infty \frac{k\big||v_1| - |v|\big|}{(1+k|v_1|)(1+k|v|)}|x_k| \\ &\le |v_1 - v| \sup_{k\in\mathbb{N}} \frac{k}{(1+k|v_1|)(1+k|v|)} \sum_{k=0}^\infty |x_k|.\end{aligned}$$

Since $v \ne 0$ and we consider v_1 which are close to v we assume that $v_1 \ne 0$ as well. Then

$$\sup_{k\in\mathbb{N}} \frac{k}{(1+k|v_1|)(1+k|v|)} \le \sup_{k\in\mathbb{N}} \frac{k}{1+k(|v_1|+|v|)} = \lim_{k\to\infty} \frac{k}{1+k(|v_1|+|v|)} = \frac{1}{|v_1|+|v|}.$$

And overall we get that

$$\|f(x,v) - f(x,v_1)\|_X \le \frac{|v_1 - v|}{|v_1|+|v|}\|x\|_X.$$

Let $|v_1 - v| \le \delta$ for some $\delta \in (0, \frac{|v|}{2})$. Then $|v_1| \ge |v|/2$ and

$$\|f(x,v) - f(x,v_1)\|_X \le \frac{\delta}{3/2|v|}\|x\|_X.$$

Now for any $\varepsilon > 0$ pick $\delta < \frac{|v|}{2}$ so that $\frac{\delta}{3/2|v|}\|x\|_X < \varepsilon$ (which is always possible since $v \ne 0$). This shows continuity of $f(x,\cdot)$ for $v \ne 0$. Note that the choice of δ depends on $\|x\|_X$, but does not depend on x itself.

Next we show that $f(x,\cdot)$ is continuous at zero as well, but it is no more uniform w.r.t. the first argument of f. Pick again any $x \in X$. For any $\varepsilon > 0$ there exists $N = N(\varepsilon, x) > 0$ so that $\sum_{k=N+1}^\infty |x_k| < \frac{\varepsilon}{2}$. We have for this x:

$$\begin{aligned}\|f(x,0) - f(x,v_1)\|_X &= \sum_{k=0}^\infty \frac{k|v_1|}{1+k|v_1|}|x_k| \\ &= \sum_{k=0}^N \frac{k|v_1|}{1+k|v_1|}|x_k| + \sum_{k=N+1}^\infty \frac{k|v_1|}{1+k|v_1|}|x_k| \\ &\le \sum_{k=0}^N \frac{N|v_1|}{1+N|v_1|}|x_k| + \sum_{k=N+1}^\infty |x_k| \\ &< \frac{N|v_1|}{1+N|v_1|}\|x\|_X + \frac{\varepsilon}{2} \\ &\le N|v_1|\,\|x\|_X + \frac{\varepsilon}{2}.\end{aligned}$$

Pick $\delta := \frac{\varepsilon}{2\|x\|_X N}$. Then for all $v_1 \in U : |v_1| \le \delta$ it follows that $\|f(x,0) - f(x,v_1)\|_X < \varepsilon$. This shows continuity of $f(x,\cdot)$ at zero for any given $x \in X$. However, note that δ depends on x (since N does) and not only on $\|x\|_X$. Overall we have shown that Assumption 1 is fulfilled for Σ.

(ii) Σ **is AG with zero gain**. For any $x \in X$, for any $\varepsilon > 0$ there exists $N = N(x, \varepsilon) \in \mathbb{N}$ so that $\sum_{k=N+1}^{\infty} |x_k(0)| < \frac{\varepsilon}{2}$.

The norm of the state of Σ at time t can be estimated as follows:

$$\begin{aligned}\|\phi(t,x,u)\|_X &= \sum_{k=1}^{N} |\phi_k(t,x_k,u)| + \sum_{k=N+1}^{\infty} |\phi_k(t,x_k,u)| \\ &\le \sum_{k=1}^{N} |\phi_k(t,x_k,u)| + \sum_{k=N+1}^{\infty} |\phi_k(0,x_k,u)| \\ &\le \sum_{k=1}^{N} |\phi_k(t,x_k,u)| + \frac{\varepsilon}{2}.\end{aligned}$$

Now we estimate the state of the kth mode of our system for all $k = 1, \ldots, N$:

$$\begin{aligned}|\phi_k(t,x_k,u)| &= e^{-\int_0^t \frac{1}{1+k|u(s)|}ds} |x_k(0)| \\ &\le e^{-\int_0^t \frac{1}{1+k\|u\|_{\mathcal{U}}}ds} |x_k(0)| \\ &= e^{-\frac{1}{1+k\|u\|_{\mathcal{U}}}t} |x_k(0)| \\ &\le e^{-\frac{1}{1+N\|u\|_{\mathcal{U}}}t} |x_k(0)|, \end{aligned} \tag{17}$$

which holds for any $u \in \mathcal{U}$ and any $x_k(0) \in \mathbb{R}$. Using this estimate we proceed to

$$\|\phi(t,x,u)\|_X \le \sum_{k=1}^{N} e^{-\frac{1}{1+N\|u\|_{\mathcal{U}}}t} |x_k(0)| + \frac{\varepsilon}{2}.$$

Clearly, for any $u \in \mathcal{U}$ there exists $\tau_a = \tau_a(x, \varepsilon, u)$ so that $\sum_{k=1}^{N} e^{-\frac{1}{1+N\|u\|_{\mathcal{U}}}t} |x_k(0)| \le \frac{\varepsilon}{2}$ for $t \ge \tau_a$.

Overall we see that for any $x \in X$, for any $t \ge 0$ and for all $u \in \mathcal{U}$ there exists $\tau_a = \tau_a(x, \varepsilon, u)$, so that for all $t \ge \tau_a$ it holds that $\|\phi(t,x,u)\|_X \le \varepsilon$. This shows that Σ is AG with $\gamma \equiv 0$. However, the time τ_a depends on u and thus the above argument does not tell us whether the system is strongly AG.

(iii) Next we show that Σ is sAG, but we should pay for this by adding a linear gain. However, this linear gain can be made arbitrarily small.

In (17) we have obtained the following estimate for the state of the kth mode of Σ:

$$|\phi_k(t,x_k,u)| \le e^{-\frac{1}{1+k\|u\|_{\mathcal{U}}}t} |x_k(0)|.$$

This expression can be further estimated as:

$$\begin{aligned}|\phi_k(t,x_k,u)| &\le e^{-\frac{1}{1+k\max\left\{\|u\|_{\mathcal{U}}, \frac{2^k}{r}|x_k(0)|\right\}}t} \\ &\quad \cdot \max\left\{|x_k(0)|, \frac{r}{2^k}\|u\|_{\mathcal{U}}\right\}.\end{aligned} \tag{18}$$

For $|x_k(0)| \ge \frac{r}{2^k}\|u\|_{\mathcal{U}}$ we obtain

$$|\phi_k(t,x_k,u)| \le e^{-\frac{1}{1+k\frac{2^k}{r}|x_k(0)|}t} |x_k(0)|. \tag{19}$$

For $|x_k(0)| \le \frac{r}{2^k}\|u\|_{\mathcal{U}}$ the inequality (18) implies

$$|\phi_k(t,x_k,u)| \le e^{-\frac{1}{1+k\|u\|_{\mathcal{U}}}t} \frac{r}{2^k}\|u\|_{\mathcal{U}} \le \frac{r}{2^k}\|u\|_{\mathcal{U}}. \tag{20}$$

Overall, for any $x_k(0) \in \mathbb{R}$ and any $u \in \mathcal{U}$ we obtain from (19) and (20):

$$|\phi_k(t,x_k,u)| \le e^{-\frac{1}{1+k\frac{2^k}{r}|x_k(0)|}t} |x_k(0)| + \frac{r}{2^k}\|u\|_{\mathcal{U}}. \tag{21}$$

Having this estimate for the state of the kth mode, we proceed to the estimate for the whole state of Σ:

$$\begin{aligned}\|\phi(t,x,u)\|_X &= \sum_{k=1}^{\infty} |\phi_k(t,x_k,u)| \\ &\le \sum_{k=1}^{\infty} e^{-\frac{1}{1+k\frac{2^k}{r}|x_k(0)|}t} |x_k(0)| + \sum_{k=1}^{\infty} \frac{r}{2^k}\|u\|_{\mathcal{U}} \\ &= \sum_{k=1}^{\infty} e^{-\frac{1}{1+k\frac{2^k}{r}|x_k(0)|}t} |x_k(0)| + r\|u\|_{\mathcal{U}}.\end{aligned} \tag{22}$$

This estimate is true for all $t \ge 0$, all $x \in X$, all $u \in \mathcal{U}$ and for any $r > 0$.

Now we apply the trick used above in the proof that the system is AG. For any $x \in X$, for any $\varepsilon > 0$ there exists $N = N(x, \varepsilon) \in \mathbb{N}$ so that $\sum_{k=N+1}^{\infty} |x_k(0)| < \frac{\varepsilon}{2}$. Using this fact we continue estimates from (22):

$$\begin{aligned}\|\phi(t,x,u)\|_X &\le \sum_{k=1}^{N} e^{-\frac{1}{1+k\frac{2^k}{r}|x_k(0)|}t} |x_k(0)| \\ &\quad + \sum_{k=N+1}^{\infty} e^{-\frac{1}{1+k\frac{2^k}{r}|x_k(0)|}t} |x_k(0)| + r\|u\|_{\mathcal{U}}. \\ &\le \sum_{k=1}^{N} e^{-\frac{1}{1+k\frac{2^k}{r}|x_k(0)|}t} |x_k(0)| + \frac{\varepsilon}{2} + r\|u\|_{\mathcal{U}}.\end{aligned}$$

Now for the above ε and x we can find sufficiently large time $\tau_a = \tau_a(\varepsilon, x)$ (clearly, τ_a depends also on N, but N itself depends only on x and ε), so that for all $u \in \mathcal{U}$ and all $t \ge \tau_a$

$$\sum_{k=1}^{N} e^{-\frac{1}{1+k\frac{2^k}{r}|x_k(0)|}t} |x_k(0)| \le \frac{\varepsilon}{2}.$$

Overall, we obtain that for any $r > 0$, for all $\varepsilon > 0$ and for any $x \in X$ there exists $\tau_a = \tau_a(t, x)$:

$$\|\phi(t,x,u)\|_X \le \varepsilon + r\|u\|_{\mathcal{U}},$$

for all $u \in \mathcal{U}$ and all $t \ge \tau_a$. This shows that Σ satisfies strong AG property with the gain $\gamma(s) = rs$.

In order to finish the proof of (iii), we show that Σ is not sAG for the gain $\gamma \equiv 0$.

Indeed, pick $\varepsilon := \frac{1}{2}$ and $x = e_1 = (1, 0, \ldots, 0, \ldots)^T$ and consider the constant inputs $u(\cdot) \equiv c$. The corresponding solution will have only one nonzero component—the first one. The norm of the state equals

$$\|\phi(t,e_1,u)\|_X = |\phi_1(t,1,u)| = e^{-\frac{1}{1+c}t}.$$

If Σ would be sAG with the zero gain, then it would exist a time τ_a, which does not depend on u, so that for all c and all $t \ge \tau_a$ it holds that $e^{-\frac{1}{1+c}t} \le \frac{1}{2}$. But this is false since $e^{-\frac{1}{1+c}\tau_a}$ monotonically increases to 1 as long as $c \to \infty$. Thus, Σ is not sAG with zero gain.

(iv) Next we show that Σ is not LISS. To this end assume that Σ is LISS and hence there exist $r > 0$, $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ so that the inequality (4) holds for all $x \in X : \|x\|_X \le r$ and for all $u \in \mathcal{U} : \|u\|_{\mathcal{U}} \le r$.

Pick a constant input $u(t) = \varepsilon$ for all $t \ge 0$, where $\varepsilon > 0$ is chosen so that $\max\{\varepsilon, 3\gamma(\varepsilon)\} \le r$. This is always possible since $\gamma \in \mathcal{K}_\infty$. Denote $c := \gamma(\varepsilon)$.

LISS property of Σ implies that for all $x \in X$ with the norm $\|x\|_X = 3c \le r$ and for all $t \ge 0$ it holds that

$$\|\phi(t,x,u(\cdot))\|_X \le \beta(3c,t) + c. \tag{23}$$

Since $\beta \in \mathcal{KL}$, there exists t^* (which depends on c, but does not depend on x) so that $\beta(3c, t^*) = c$[1] and thus for all $x \in X$ with $\|x\|_X = 3c$ it must hold

$$\|\phi(t^*, x, u(\cdot))\|_X \le 2c. \quad (24)$$

Now consider the initial states of the form $x_{<k>} = 3c \cdot e_k$, where e_k is the kth standard basis vector of $X = l_1$. Certainly, $\|x_{\langle k \rangle}\|_X = 3c$ for all $k \in \mathbb{N}$. Then

$$\|\phi(t^*, x_{\langle k \rangle}, u(\cdot))\|_X = |\phi_k(t^*, 3c, u(\cdot))| = e^{-\frac{1}{1+k\varepsilon}t^*} 3c.$$

Now, for any $t^* > 0$ there exists k so that $e^{-\frac{1}{1+k\varepsilon}t^*} 3c > 2c$, which contradicts to (24). This shows that Σ is not LISS.

Remark 3. It is possible to check directly, that the second assumption of Theorem 4 does not hold.

Remark 4. The fact that 'AG with zero gain' and 'sAG with zero gain' properties are not equivalent is not the novelty of infinite-dimensional systems. Above arguments show, that these two notions are not equivalent already for a particular one-dimensional mode of the above example:

$$\dot{x}(t) = -\frac{1}{1 + |u(t)|} x(t). \quad (25)$$

This system is AG with zero gain and 0-GAS, but it is not sAG with zero gain.

5. 0-UGASs + sAG + LISS + GS does not ensure ISS

In the previous example we have shown that a system which is 0-UGASs, sAG, AG with zero gain and GS with zero gain, does not have to be LISS. Next we modify this example to show that *if the system in addition to the above list of properties is LISS, this still does not guarantee ISS.*

Consider a system Σ with the state space $X := l_1$ and input space $\mathcal{U} := PC(\mathbb{R}_+, \mathbb{R})$. Let also the dynamics of the kth mode of Σ be given by

$$\dot{x}_k(t) = -\frac{1}{1 + |u(t)|^k} x_k(t). \quad (26)$$

We continue to use the notation $\phi_k(t, x_k, u)$ for the state of the kth mode (26).

As in Section 4 one can prove that this system satisfies Assumptions 1 and 2, is 0-UGASs, GS with zero gain, AG with zero gain, and sAG with a nonzero gain (we skip this proof since it is completely analogous). Moreover, for $u : \|u\|_{\mathcal{U}} \le 1$ and for all $x \in X$ it holds that

$$\|\phi(t, x, u)\|_X \le e^{-\frac{1}{2}t} \|x\|_X \quad (27)$$

and thus Σ is LISS with zero gain and with $r = 1$.

The proof that Σ is not ISS goes along the lines of Section 4, with the change that the norm of the considered inputs should be larger than 1.

6. Conclusions

We have shown that 0-UAS automatically implies LISS for the system (1) if the right hand side f has a kind of uniform continuity w.r.t. inputs. If this uniformity is not present, the equivalence between 0-UAS and LISS may not hold. We have shown that sAG + LISS + GS + 0-UGAS does not imply ISS. The challenging problem for the future is to derive the characterizations of the global ISS property.

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[1] Note that any $\beta \in \mathcal{KL}$ satisfying the LISS estimate automatically satisfies $\beta(r, 0) \ge r$ for all $r > 0$ (consider $t = 0$ and $u \equiv 0$ in LISS estimate).