Criteria for input-to-state practical stability

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Abstract—For a broad class of infinite-dimensional systems, we characterize input-to-state practical stability (ISpS) using the uniform limit property and in terms of input-to-state stability. We specialize our results to systems with Lipschitz continuous flows and evolution equations in Banach spaces. The characterization of ISpS in terms of the limit property is novel already in the special case of ODE systems.

Index Terms—input-to-state stability, nonlinear systems, practical stability, infinite-dimensional systems.

I. INTRODUCTION

The concept of input-to-state stability (ISS), introduced in [1], has become indispensable for various branches of nonlinear control theory, such as robust stabilization of nonlinear systems [2], design of nonlinear observers [3], analysis of large-scale networks [4], [5], etc.

However, in many cases it is impossible (as in quantized control) or too costly to construct a feedback, ensuring ISS behavior of the closed loop system. To address such applications, a relaxation of the ISS concept has been proposed in [4], called input-to-state practical stability (ISpS, practical ISS). This concept is extremely useful for stabilization of stochastic control systems [6], control under quantization errors [7], [8], sample-data control [9], study of interconnections of nonlinear systems by means of small-gain theorems [4], [10], etc. Practical ISS extends the earlier concept of practical asymptotic stability of dynamical systems [11]. In some works practical ISS is called ISS with bias, see e.g. [8], [12].

Criteria for ISS in terms of other stability properties are among foundational theoretical results in ISS of ordinary differential equations (ODEs). In [13] Sontag and Wang have shown that ISS is equivalent to the existence of a smooth ISS Lyapunov function and in [14] the same authors proved an ISS superposition theorem, saying that ISS is equivalent to the limit property combined with a local stability. Characterizations of ISS greatly simplify the proofs of other important results, such as small-gain theorems for ODEs [5] and hybrid systems [15], [16], Lyapunov-Razumikhin theory for time-delay systems [17], [18], non-coercive ISS Lyapunov theorems [19], relations between ISS and nonlinear $L_2 \to L_2$ stability [20], to name a few examples.

Characterizations of ISS for ODEs in [14] heavily exploit the topological structure of an underlying state space $\mathbb{R}^n$, as well as a special type of dynamics (ODEs). Trying to generalize these criteria to infinite-dimensional systems, we face fundamental difficulties: closed bounded balls are never

compact in infinite-dimensional normed linear spaces, nonuniformly globally asymptotically stable nonlinear systems do not necessarily have bounded reachability sets, and even if they do, this still does not guarantee uniform global stability [19]. These difficulties have been overcome in a recent work [19], where characterizations of ISS have been developed for a general class of control systems, encompassing evolution PDEs, differential equations in Banach spaces, time-delay systems, switched systems, ODEs, etc. Characterizations of local ISS of infinite-dimensional systems were obtained slightly earlier in [21]. The results in [19] naturally extend criteria for ISS of ODEs developed in [14]. New notions and results obtained in [19] establish a solid background for a solution of further problems. The concept of a uniform limit has been useful in the theory of non-coercive Lyapunov functions [19], [22], [23] and for characterization of practical uniform asymptotic stability [23].

Despite a great importance of practical ISS for nonlinear control theory, much less is known about characterizations of practical ISS. There are two complementary directions, in which one could characterize ISpS property: in terms of weaker properties as e.g. limit or asymptotic gain properties and in terms of stronger properties as ISS. The first kind of results is of virtue for the verification of ISpS property, and the second type of results is important if we know that the system is ISpS and would like to get some additional insights about the properties of this system. In ODE case, characterizations of ISpS via (stronger) ISS property have been shown by Sontag and Wang in [14, Proposition VI.3], which states (in conjunction with [14, Theorem 1]) that an ODE system is ISpS if and only if there is a compact set $\mathscr{A} \subset \mathbb{R}^n$ so that the system is ISS w.r.t. $\mathscr{A}$. On the other hand, characterization of ISpS in terms of weaker properties (as limit or asymptotic gain property) remained unexplored even for ODE systems.

In this paper we develop criteria for practical ISS in terms of both stronger and weaker properties for a broad class of infinite-dimensional systems. The understanding of the nature of practical ISS will be beneficial for the development of quantized and sample-data controllers for infinite-dimensional systems and will give further insights into the ISS theory of infinite-dimensional systems, which is currently a hot topic [24], [25], [26], [27], [28], [21], [19], [29], [30]. Since characterizations for ISS have played a key role in the proof of general small-gain theorems for couplings of n ODE systems [5], it is natural to expect that the characterizations of ISpS, obtained in this paper will be helpful for the proof of such general small-gain theorems for networks of ISpS systems.

We prove in Section III that *a nonlinear infinite-dimensional control system Σ possessing bounded reachability sets is practically ISS if and only if there is a bounded subset $\mathscr{A}$ of the state space so that Σ has a uniform limit property (ULIM)*

w.r.t. $\mathscr{A}$. This criterion can be used to prove ISpS of control systems. On the other hand, we show that any ISpS control system has a so-called complete uniform asymptotic gain property (CUAG), which is stronger than uniform asymptotic gain property (UAG) as defined in [14], [19].

An important difference of this criterion of ISpS to the criteria of ISS proved in [14], [19] is that it does not involve any kind of stability w.r.t. the set $\mathscr{A}$ (which is necessary for ISS), which significantly simplifies verification of the ISpS property.

We base ourselves on machinery developed in [19] for characterization of ISS of general infinite-dimensional systems, in particular, we use the notion of the uniform limit and results from [19] related to this property. Additionally, we develop two further technical results which are of independent interest.

Firstly, *we introduce a CUAG property and show in Proposition III.4 that a control system possesses this property if and only if it has a UAG property and its finite-time reachability sets are bounded.*

Secondly, using this CUAG characterization we show in Proposition III.7 that *if a system has the uniform limit property w.r.t. a certain bounded set $\mathscr{A}$ of a state space X and if this system has bounded finite-time reachability sets, then there is a set $\mathscr{B} \supset \mathscr{A}$ so that Σ has a (much stronger than ULIM) CUAG property w.r.t. $\mathscr{B}$.* In our proof we construct a family of such sets.

For systems with Lipschitz continuous flows, we improve some of our criteria in Section IV by showing that Σ is ISpS if and only if there is a bounded invariant (under uniformly bounded controls) set $\mathscr{A}$ so that Σ is ISS w.r.t. $\mathscr{A}$.

Even specialized to ODE systems our main results are novel. In Section IV-C we show that *an ODE system Σ is practically ISS $\Leftrightarrow$ there is a bounded set $\mathscr{A}$ so that Σ has a (non-uniform) limit property w.r.t. $\mathscr{A}$.*

A. Notation

The following notation will be used throughout these notes. Denote $\mathbb{R}_+ := [0, +\infty)$. For an arbitrary set S and $n \in \mathbb{N}$ the n-fold Cartesian product is $S^n := S \times \ldots \times S$.

Let X be a normed linear space with a norm $\|\cdot\|$ and let $\mathscr{A}$ be a nonempty set in X. For any $x \in X$ we define a distance from $x \in X$ to $\mathscr{A}$ by $|x|_{\mathscr{A}} := \inf_{y \in \mathscr{A}} \|x - y\|$. Define also $\|\mathscr{A}\| := \sup_{x \in \mathscr{A}} \|x\|$. The open ball in a normed linear space X with radius r around $\mathscr{A} \subset X$ is denoted by $B_r(\mathscr{A}) := \{x \in X : |x|_{\mathscr{A}} < r\}$. For short, we denote $B_r := B_r(\{0\})$. Similarly, $B_{r,\mathscr{U}} := \{u \in \mathscr{U} : \|u\|_{\mathscr{U}} < r\}$. The closure of a set $S \subset X$ w.r.t. norm $\|\cdot\|$ is denoted by $\overline{S}$. With a slight abuse of notation we define $\overline{B_0(\mathscr{A})} := \mathscr{A}$ and $\overline{B_{0,\mathscr{U}}} = \{0\}$.

For the formulation of stability properties the following classes of comparison functions are useful:

$$
\begin{aligned}
\mathscr{K} \quad &:= \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \,|\, \gamma \text{ is continuous, strictly} \\
&\qquad\qquad\qquad\qquad \text{increasing and } \gamma(0) = 0\}, \\
\mathscr{K}_\infty \quad &:= \{\gamma \in \mathscr{K} \,|\, \gamma \text{ is unbounded}\}, \\
\mathscr{L} \quad &:= \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \,|\, \gamma \text{ is continuous and strictly} \\
&\qquad\qquad\qquad \text{decreasing with } \lim_{t \to \infty} \gamma(t) = 0\}, \\
\mathscr{K}\mathscr{L} \quad &:= \{\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \,|\, \beta \text{ is continuous,} \\
&\qquad \beta(\cdot, t) \in \mathscr{K}, \; \beta(r, \cdot) \in \mathscr{L}, \; \forall t \geq 0, \; \forall r > 0\}.
\end{aligned}
$$

II. PRELIMINARIES

In this paper, we consider abstract axiomatically defined time-invariant and forward complete systems on the state space X which are subject to a shift-invariant set of inputs $\mathscr{U}$.

Definition II.1. Consider the triple $\Sigma = (X, \mathscr{U}, \phi)$ consisting of

(i) A normed linear space $(X, \|\cdot\|)$, called the state space, endowed with the norm $\|\cdot\|$.

(ii) A set of input values U, which is a nonempty subset of a certain normed linear space.

(iii) A space of inputs $\mathscr{U} \subset \{f : \mathbb{R}_+ \to U\}$, $0 \in \mathscr{U}$ endowed with a norm $\|\cdot\|_{\mathscr{U}}$ satisfying two axioms:
The axiom of shift invariance states that for all $u \in \mathscr{U}$ and all $\tau \geq 0$ the time shift $u(\cdot + \tau)$ belongs to $\mathscr{U}$ with $\|u\|_{\mathscr{U}} \geq \|u(\cdot + \tau)\|_{\mathscr{U}}$.
The axiom of concatenation is defined by the requirement that for all $u_1, u_2 \in \mathscr{U}$ and for all $t > 0$ the concatenation of u_1 and u_2 at time t

$$
u(\tau) := \begin{cases} u_1(\tau), & \text{if } \tau \in [0,t], \\ u_2(\tau - t), & \text{otherwise}, \end{cases} \tag{1}
$$

belongs to $\mathscr{U}$. Furthermore, if $u_2 \equiv 0$, then $\|u\|_{\mathscr{U}} \leq \|u_1\|_{\mathscr{U}}$.

(iv) A transition map $\phi : \mathbb{R}_+ \times X \times \mathscr{U} \to X$.

The triple Σ is called a (forward complete) control system, if the following properties hold:

(Σ1) Forward completeness: for every $(x, u) \in X \times \mathscr{U}$ and for all $t \geq 0$ the value $\phi(t, x, u) \in X$ is well-defined.

(Σ2) The identity property: for every $(x, u) \in X \times \mathscr{U}$ it holds that $\phi(0, x, u) = x$.

(Σ3) Causality: for every $(t, x, u) \in \mathbb{R}_+ \times X \times \mathscr{U}$, for every $\tilde{u} \in \mathscr{U}$, such that $u(s) = \tilde{u}(s)$, $s \in [0,t]$ it holds that $\phi(t, x, u) = \phi(t, x, \tilde{u})$.

(Σ4) Continuity: for each $(x, u) \in X \times \mathscr{U}$ the map $t \mapsto \phi(t, x, u)$ is continuous.

(Σ5) The cocycle property: for all $t, h \geq 0$, for all $x \in X$, $u \in \mathscr{U}$ we have $\phi(h, \phi(t, x, u), u(t + \cdot)) = \phi(t + h, x, u)$.

Remark II.2. By contrast to the paper [19], upon which this note is based, we impose here an additional requirement on the space $\mathscr{U}$, that the concatenation of any input u with a zero input has the norm which is not larger than $\|u\|_{\mathscr{U}}$. This condition is satisfied by most of the "natural" input spaces.

Definition II.3. Let a control system $\Sigma = (X, \mathscr{U}, \phi)$, a real number $s \geq 0$ and $\mathscr{A} \subset X$, $\mathscr{A} \neq \emptyset$ be given.

- $\mathscr{A}$ is called s-invariant if $\phi(t, x, u) \in \mathscr{A}$ for all $t \geq 0$, $x \in \mathscr{A}$ and $u \in \overline{B_{s,\mathscr{U}}}$.

- An s-invariant set $\mathscr{A} \subset X$ is called robustly s-invariant if for every $\varepsilon > 0$ and any $h > 0$ there is a $\delta = \delta(\varepsilon, h) > 0$, so that

$$
t \in [0,h], \; |x|_{\mathscr{A}} \leq \delta, \; \|u\|_{\mathscr{U}} \leq \delta \; \Rightarrow \; |\phi(t, x, u)|_{\mathscr{A}} \leq \varepsilon. \tag{2}
$$

Remark II.4. If $\mathscr{A} = \{0\}$, then robust 0-invariance of such $\mathscr{A}$ is precisely the continuity of ϕ w.r.t. states and inputs at the trivial equilibrium, see [19]. The concept of s-invariance seems to be less standard, but it helps a lot to establish the relation between ISS and ISpS in Theorem IV.6, due to the

fact that it is much easier to show robustness of s-invariant sets with $s > 0$ than with $s = 0$, see Lemma IV.3.

The central notion of this paper is:

Definition II.5. A control system $\Sigma = (X, \mathscr{U}, \phi)$ is called *(uniformly) input-to-state practically stable (ISpS)* w.r.t. a nonempty set $\mathscr{A} \subset X$, if there exist $\beta \in \mathscr{KL}$, $\gamma \in \mathscr{K}_\infty$ and $c > 0$ such that for all $x \in X$, $u \in \mathscr{U}$ and $t \geq 0$ the following holds:

$$|\phi(t, x, u)|_{\mathscr{A}} \leq \beta(|x|_{\mathscr{A}}, t) + \gamma(\|u\|_{\mathscr{U}}) + c. \qquad (3)$$

If ISpS property w.r.t. $\mathscr{A}$ holds with $c := 0$, then Σ is called *input-to-state stable (ISS)* w.r.t $\mathscr{A}$.

In what follows we assume that the set, w.r.t. which the stability property is considered (usually denoted by $\mathscr{A}$) is always nonempty.

We are interested in practical ISS w.r.t. bounded subsets of X. The following simple result holds:

Proposition II.6. *Let Σ be a control system as in Definition II.1. If Σ is ISpS w.r.t. a certain bounded set $\mathscr{A}_1 \subset X$, then Σ is ISpS w.r.t. any bounded subset of X.*

Proof. Let $\mathscr{A}_1$ be a bounded subset of X and let $\Sigma = (X, \mathscr{U}, \phi)$ be ISpS w.r.t. $\mathscr{A}_1$. Using a simple inequality

$$\|x\| - \|\mathscr{A}_1\| \leq |x|_{\mathscr{A}_1} \leq \|x\| + \|\mathscr{A}_1\|, \qquad (4)$$

which holds for all $x \in X$, as well as the fact that $\beta(a+b, t) \leq \beta(2a, t) + \beta(2b, t)$ for all $a, b, t \geq 0$, we arrive at

$$\begin{aligned}
\|\phi(t, x, u)\| - \|\mathscr{A}_1\| &\leq |\phi(t, x, u)|_{\mathscr{A}_1} \\
&\leq \beta(\|x\| + \|\mathscr{A}_1\|, t) + \gamma(\|u\|_{\mathscr{U}}) + c \\
&\leq \beta(2\|x\|, t) + \beta(2\|\mathscr{A}_1\|, t) + \gamma(\|u\|_{\mathscr{U}}) + c.
\end{aligned}$$

Defining $\tilde{c} := \beta(2\|\mathscr{A}_1\|, 0) + \|\mathscr{A}_1\| + c$ we see that

$$\|\phi(t, x, u)\| \leq \beta(2\|x\|, t) + \gamma(\|u\|_{\mathscr{U}}) + \tilde{c}. \qquad (5)$$

Using (5) and (4) once again, we obtain that Σ is ISpS w.r.t. any bounded $\mathscr{A} \subset X$. $\qquad\square$

Proposition II.6 motivates the following definition:

Definition II.7. A control system $\Sigma = (X, \mathscr{U}, \phi)$ is called *ISpS*, if there is a bounded set $\mathscr{A} \subset X$ so that Σ is ISpS w.r.t. $\mathscr{A}$.

Our aim is to prove criteria for practical ISS in terms of more basic stability properties. Next we enlist the most important of such notions.

Definition II.8. A control system $\Sigma = (X, \mathscr{U}, \phi)$

- has *bounded reachability sets (BRS)*, if for any $C > 0$ and any $\tau > 0$ it holds that

$$\sup_{\|x\| \leq C, \ \|u\|_{\mathscr{U}} \leq C, \ t \in [0, \tau]} \|\phi(t, x, u)\| < \infty.$$

- is called *uniformly globally bounded (UGB)*, if there exist a bounded set $\mathscr{A} \subset X$, functions $\sigma, \gamma \in \mathscr{K}_\infty$ and $c > 0$ such that for all $x \in X$, and all $u \in \mathscr{U}$ it holds that

$$|\phi(t, x, u)|_{\mathscr{A}} \leq \sigma(|x|_{\mathscr{A}}) + \gamma(\|u\|_{\mathscr{U}}) + c \quad \forall t \geq 0. \qquad (6)$$

- has the *uniform asymptotic gain (UAG) property* w.r.t. $\mathscr{A} \subset X$, if there exists a $\gamma \in \mathscr{K}_\infty$ such that for all $\varepsilon, r > 0$ there is a $\tau = \tau(\varepsilon, r) < \infty$ s.t. for all $u \in \mathscr{U}$ and all $x \in \overline{B_r(\mathscr{A})}$

$$t \geq \tau \quad \Rightarrow \quad |\phi(t, x, u)|_{\mathscr{A}} \leq \varepsilon + \gamma(\|u\|_{\mathscr{U}}). \qquad (7)$$

- has the *limit property (LIM)* w.r.t. $\mathscr{A} \subset X$ if there is a $\gamma \in \mathscr{K}_\infty$: for all $x \in X$, $u \in \mathscr{U}$ and $\varepsilon > 0$ there is a $t = t(x, u, \varepsilon)$:

$$|\phi(t, x, u)|_{\mathscr{A}} \leq \varepsilon + \gamma(\|u\|_{\mathscr{U}}).$$

- has the *uniform limit property (ULIM)* w.r.t. $\mathscr{A} \subset X$, if there exists $\gamma \in \mathscr{K}_\infty$ so that for all $\varepsilon > 0$ and all $r > 0$ there is a $\tau = \tau(\varepsilon, r)$ s.t. for all $u \in \mathscr{U}$:

$$|x|_{\mathscr{A}} \leq r \Rightarrow \exists t \leq \tau(\varepsilon, r) : |\phi(t, x, u)|_{\mathscr{A}} \leq \varepsilon + \gamma(\|u\|_{\mathscr{U}}). \qquad (8)$$

Remark II.9. For ODEs forward completeness implies BRS property, see [31, Proposition 5.1]. For ∞-dimensional systems this is not always the case (see [19, Example 2]).

Note that trajectories of ULIM systems do not only approach the ball $B_{\gamma(\|u\|_{\mathscr{U}})}(\mathscr{A})$ (as trajectories of LIM systems do), but they do it uniformly. Indeed, the time of approachability τ depends only on the norm of the state and ε and does not depend on the state itself.

Uniform asymptotic gain property assures that the trajectories possess a uniform convergence rate. However, UAG property per se does not guarantee that the solutions possess any kind of uniform global bounds (one can construct examples of control systems, illustrating this fact, using ideas from [19, Example 2]). Since it is often desirable both to have uniform attraction rates as well as uniform bounds on solutions, we introduce (motivated by [32, Definition 4.1.3], where a similar concept with $\gamma = 0$ has been employed) a new notion:

Definition II.10. We say that a control system Σ satisfies the *completely uniform asymptotic gain property (CUAG)* w.r.t. $\mathscr{A} \subset X$, if there are $\beta \in \mathscr{KL}$, $\gamma \in \mathscr{K}_\infty$ and $C > 0$ s.t. for all $x \in X$, $u \in \mathscr{U}$, $t \geq 0$ it holds that:

$$|\phi(t, x, u)|_{\mathscr{A}} \leq \beta(|x|_{\mathscr{A}} + C, t) + \gamma(\|u\|_{\mathscr{U}}). \qquad (9)$$

In Section III-A we show that CUAG is equivalent to a combination of BRS and UAG properties.

III. CHARACTERIZATIONS OF ISpS

In this section we prove the following characterization of ISpS:

Theorem III.1. *Let Σ be a control system as in Definition II.1. The following statements are equivalent:*

(i) Σ is ISpS
(ii) There is a bounded 0-invariant set $\mathscr{A} \subset X$ so that Σ is CUAG w.r.t. $\mathscr{A}$.
(iii) Σ is BRS and there is a bounded set $\mathscr{A} \subset X$ so that Σ is ULIM w.r.t. $\mathscr{A}$.

Theorem III.1 can be used in two ways. On the one hand, to prove ISpS of a system, we can merely check the conditions in item (iii) of Theorem III.1. On the other hand, if we know that a certain system is ISpS, item (ii) shows that it enjoys also a CUAG property w.r.t. a certain bounded 0-invariant set.

The proof will be divided into several lemmas which will be subdivided into two subsections. We start with a characterization of the CUAG property.

A. Characterization of CUAG property

We start with a preliminary result.

Proposition III.2. *Let Σ be a control system. If Σ is BRS and ULIM w.r.t. certain bounded set $\mathscr{A} \subset X$, then Σ is UGB.*

Proof. This claim was proved for $\mathscr{A} = \{0\}$ in [19, Proposition 7]. The proof for general bounded $\mathscr{A}$ is analogous. $\quad\square$

Now we provide a simple restatement of the UGB property.

Lemma III.3. *A control system Σ is UGB w.r.t. $\mathscr{A} \subset X$ if and only if there are $\sigma_1, \gamma \in \mathscr{K}_\infty$ and $c > 0$ so that*

$$|\phi(t,x,u)|_{\mathscr{A}} \leq \sigma_1(|x|_{\mathscr{A}} + c) + \gamma(\|u\|_{\mathscr{U}}). \qquad (10)$$

Proof. Let Σ be UGB. Then there are $\sigma, \gamma \in \mathscr{K}_\infty$ and $c > 0$ so that for any $x \in X$, any $u \in \mathscr{U}$ and any $t \geq 0$ it holds that

$$
\begin{aligned}
|\phi(t,x,u)|_{\mathscr{A}} &\leq \sigma(|x|_{\mathscr{A}}) + \gamma(\|u\|_{\mathscr{U}}) + c \\
&\leq \sigma(|x|_{\mathscr{A}} + c) + |x|_{\mathscr{A}} + c + \gamma(\|u\|_{\mathscr{U}}) \\
&=: \sigma_1(|x|_{\mathscr{A}} + c) + \gamma(\|u\|_{\mathscr{U}})
\end{aligned}
$$

for $\sigma_1(r) := \sigma(r) + r$, $r \geq 0$. Clearly, $\sigma_1 \in \mathscr{K}_\infty$ and hence (10) holds.

Conversely, let (10) hold. Then there are $\sigma_1, \gamma \in \mathscr{K}_\infty$ and $c > 0$ s.t. for any $x \in X$, any $u \in \mathscr{U}$ and any $t \geq 0$ it holds that

$$
\begin{aligned}
|\phi(t,x,u)|_{\mathscr{A}} &\leq \sigma_1(|x|_{\mathscr{A}} + c) + \gamma(\|u\|_{\mathscr{U}}) \\
&\leq \sigma_1(2|x|_{\mathscr{A}}) + \gamma(\|u\|_{\mathscr{U}}) + \sigma_1(2c),
\end{aligned}
$$

and thus Σ is UGB. $\quad\square$

The following proposition gives a useful characterization of CUAG.

Proposition III.4. *Let $\mathscr{A} \subset X$ be a bounded set. A control system Σ is CUAG w.r.t. $\mathscr{A} \Leftrightarrow \Sigma$ is BRS and UAG w.r.t. $\mathscr{A}$.*

Proof. "$\Rightarrow$". Let Σ be a CUAG control system with the corresponding β, γ, C as in (9). Then Σ is BRS. Now for every $\varepsilon > 0$ and any $r > 0$ pick $\tau(\varepsilon, r)$ as a solution of the equation $\beta(C + r, \tau) = \varepsilon$ (if it exists), or set $\tau(\varepsilon, r) := 0$, if the equation has no solution. This shows UAG of Σ w.r.t. $\mathscr{A}$ with this τ.

"$\Leftarrow$". Let a control system Σ be UAG w.r.t. $\mathscr{A}$ and BRS. According to Proposition III.2, the system Σ is UGB and in view of Lemma III.3 there are $\sigma_1, \gamma \in \mathscr{K}_\infty$ and $c > 0$ so that (10) holds for all $x \in X$, $t \geq 0$ and $u \in \mathscr{U}$.

Without loss of generality, we may assume that γ in (7) is the same as in (10) (otherwise pick γ as a maximum of both of them).

Fix arbitrary $r \in \mathbb{R}_+$ and define $\varepsilon_n := 2^{-n}\sigma_1(r + c)$, for $n \in \mathbb{N}$. The UAG property implies that there exists a sequence of times $\tau_n := \tau(\varepsilon_n, r)$, $n \in \mathbb{N}$ which we may without loss of generality assume to be strictly increasing, such that for all $x \in \overline{B_r(\mathscr{A})}$ and all $u \in \mathscr{U}$

$$|\phi(t,x,u)|_{\mathscr{A}} \leq \varepsilon_n + \gamma(\|u\|_{\mathscr{U}}) \quad \forall t \geq \tau_n.$$

From Lemma III.3 we see that the previous inequality is valid also for $n = 0$, if we set $\tau_0 := 0$. Define $\tilde{\omega}(r, \tau_n) := \varepsilon_{n-1} = 2^{-n+1}\sigma_1(r + c)$, for $n \in \mathbb{N}$, $n \neq 0$ and $\tilde{\omega}(r, 0) := 2\varepsilon_0 = 2\sigma_1(r + c)$. Here we assume $r \in [-c, +\infty)$.

Now extend the definition of $\tilde{\omega}$ to a function $\tilde{\omega}(r, \cdot) \in \mathscr{L}$, $r \in [-c, +\infty)$ (see [19, Lemma 7] for details) so that

$$|\phi(t,x,u)|_{\mathscr{A}} \leq \tilde{\omega}(|x|_{\mathscr{A}}, t) + \gamma(\|u\|_{\mathscr{U}}). \qquad (11)$$

Set $\omega(r, t) := \tilde{\omega}(r - c, t)$, $r \geq 0$, $t \geq 0$.

As in the proof of [19, Lemma 7], there is $\beta \in \mathscr{KL}$ so that $\omega(r, t) \leq \beta(r, t)$ for all $r, t \geq 0$. From (11) we obtain

$$|\phi(t,x,u)|_{\mathscr{A}} \leq \beta(|x|_{\mathscr{A}} + c, t) + \gamma(\|u\|_{\mathscr{U}}),$$

which shows CUAG of Σ. $\quad\square$

Proposition III.5. *If there is a bounded $\mathscr{A} \subset X$ so that a control system $\Sigma = (X, \mathscr{U}, \phi)$ is CUAG w.r.t. $\mathscr{A}$, then Σ is ISpS.*

Proof. Let $\Sigma = (X, \mathscr{U}, \phi)$ be a control system which is CUAG w.r.t. certain bounded $\mathscr{A} \subset X$. Then there are $\beta \in \mathscr{KL}$ and $\gamma \in \mathscr{K}_\infty$ so that for all $x \in X$, $u \in \mathscr{U}$, $t \geq 0$ we have

$$
\begin{aligned}
|\phi(t,x,u)|_{\mathscr{A}} &\leq \beta(|x|_{\mathscr{A}} + C, t) + \gamma(\|u\|_{\mathscr{U}}) \\
&\leq \beta(2|x|_{\mathscr{A}}, t) + \beta(2C, t) + \gamma(\|u\|_{\mathscr{U}}) \\
&\leq \beta(2|x|_{\mathscr{A}}, t) + \beta(2C, 0) + \gamma(\|u\|_{\mathscr{U}}),
\end{aligned}
$$

which shows ISpS of Σ. $\quad\square$

B. Improving stability properties via enlarging of the attracting set

Assume that a control system Σ is ULIM w.r.t a certain set $\mathscr{A} \subset X$. Then it is clear that Σ is ULIM w.r.t. any $\mathscr{B} \supset \mathscr{A}$. However, it may exist certain subsets of X w.r.t. which Σ has better properties than merely ULIM. In this section we show that this is indeed the case provided Σ is BRS. We exploit a simple lemma:

Lemma III.6. *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be strictly increasing. Then $g : t \mapsto \frac{1}{t} \int_0^t f(s)\,ds$ is again strictly increasing.*

Assume that $\Sigma = (X, \mathscr{U}, \phi)$ is given. For any $\mathscr{A} \subset X$, any $\varepsilon > 0$ and any $\gamma \in \mathscr{K}_\infty$ define

$$\mathscr{A}_{\varepsilon, \gamma} := \{\phi(t,x,u) : t \in \mathbb{R}_+, \ x \in B_\varepsilon(\mathscr{A}), \ \|u\|_{\mathscr{U}} \leq \gamma^{-1}(\tfrac{\varepsilon}{2})\}. \quad (12)$$

Note that from the identity axiom ($\Sigma 2$), for each $\varepsilon > 0$ and any $\gamma \in \mathscr{K}_\infty$ it holds that $\mathscr{A} \subset B_\varepsilon(\mathscr{A}) \subset \mathscr{A}_{\varepsilon, \gamma}$. The construction of the sets $\mathscr{A}_{\varepsilon, \gamma}$ is motivated by the notion of the positive prolongation of a set, see [33].

Next we show the central technical result in this paper. It may be viewed as a strengthening of [14, Lemma VI.2].

Proposition III.7. *Assume that Σ is a BRS control system and Σ has the ULIM property w.r.t. a bounded (not necessarily 0-invariant) set $\mathscr{A} \subset X$, with $\gamma \in \mathscr{K}_\infty$ as in (8). Then for any $\varepsilon > 0$ the set $\mathscr{A}_{\varepsilon, \gamma}$ is bounded, 0-invariant and Σ is CUAG w.r.t. $\mathscr{A}_{\varepsilon, \gamma}$.*

Proof. We divide the proof into four parts.

I. Boundedness of $\mathscr{A}_{\varepsilon,\gamma}$. Pick any $\varepsilon > 0$, any $R > 0$ and fix them. Since Σ is ULIM w.r.t. $\mathscr{A}$, there is a $\tilde{\tau} = \tilde{\tau}(\varepsilon, R) > 0$ so that for any $u \in \mathscr{U}$

$$|x|_{\mathscr{A}} \leq R \;\Rightarrow\; \exists \bar{t} \leq \tilde{\tau}: \; |\phi(\bar{t},x,u)|_{\mathscr{A}} \leq \frac{\varepsilon}{2} + \gamma(\|u\|_{\mathscr{U}}). \quad (13)$$

Without loss of generality we can assume that $\tilde{\tau}$ is decreasing w.r.t. ε and increasing w.r.t. R. Then $\tilde{\tau}$ is integrable. Define

$$\tau(\varepsilon, R) := \frac{2}{\varepsilon R} \int_R^{2R} \int_{\varepsilon/2}^{\varepsilon} \tilde{\tau}(\varepsilon_1, R_1) d\varepsilon_1 dR_1.$$

Since $\tilde{\tau}$ is strictly increasing w.r.t. the second argument and strictly decreasing w.r.t. the first one, for any $\varepsilon, R > 0$ we have that

$$\min\{\tilde{\tau}(\varepsilon_1, R_1): \; \varepsilon_1 \in [\varepsilon/2, \varepsilon], \; R_1 \in [R, 2R]\} = \tilde{\tau}(\varepsilon, R)$$

and thus $\tau(\varepsilon, R) \geq \tilde{\tau}(\varepsilon, R)$.

By standard lemmas from analysis, τ is continuous.

Let us show that τ is increasing w.r.t. R. Pick any $R_1, R_2 > 0$ so that $R_2 > R_1$. We have for $i = 1, 2$:

$$\tau(\varepsilon, R_i) = \frac{2}{\varepsilon} \int_{\varepsilon/2}^{\varepsilon} \frac{1}{R_i} \int_{R_i}^{2R_i} \tilde{\tau}(\varepsilon_1, R) dR d\varepsilon_1.$$

Since $\tilde{\tau}(\varepsilon_1, \cdot)$ is strictly increasing, and since $R_2 > R_1$, we have:

$$\frac{1}{R_2} \int_{R_2}^{2R_2} \tilde{\tau}(\varepsilon_1, R) dR > \frac{1}{R_2} \int_{R_1}^{R_1+R_2} \tilde{\tau}(\varepsilon_1, R) dR$$

Again since $\tilde{\tau}(\varepsilon_1, \cdot)$ is strictly increasing and applying Lemma III.6 we see that

$$\frac{1}{R_2} \int_{R_1}^{R_1+R_2} \tilde{\tau}(\varepsilon_1, R) dR > \frac{1}{R_1} \int_{R_1}^{2R_1} \tilde{\tau}(\varepsilon_1, R) dR.$$

This shows that $\tau(\varepsilon, R_2) > \tau(\varepsilon, R_1)$, and thus τ is increasing w.r.t. the second argument.

Analogously, τ is decreasing w.r.t. ε. Consequently, for any $u \in \mathscr{U}$

$$|x|_{\mathscr{A}} \leq R \;\Rightarrow\; \exists \bar{t} \leq \tau(\varepsilon, R): \; |\phi(\bar{t},x,u)|_{\mathscr{A}} \leq \frac{\varepsilon}{2} + \gamma(\|u\|_{\mathscr{U}}) \; (14)$$

and in particular

$$|x|_{\mathscr{A}} \leq \frac{\varepsilon}{2} + \gamma(\|u\|_{\mathscr{U}})$$
$$\Rightarrow \; \exists \tilde{t} \in (0, \tau(\varepsilon, \tfrac{\varepsilon}{2} + \gamma(\|u\|_{\mathscr{U}}))): \; |\phi(\tilde{t},x,u)|_{\mathscr{A}} \leq \frac{\varepsilon}{2} + \gamma(\|u\|_{\mathscr{U}}).$$

This means, that trajectories corresponding to the input u, emanating from $B_{\frac{\varepsilon}{2} + \gamma(\|u\|_{\mathscr{U}})}(\mathscr{A})$ return to this ball in time not larger than $\tau(\varepsilon, \frac{\varepsilon}{2} + \gamma(\|u\|_{\mathscr{U}}))$. In particular, for any $\varepsilon > 0$, any $x \in B_{\varepsilon}(\mathscr{A})$ and any $u \in \mathscr{U}: \|u\|_{\mathscr{U}} \leq \gamma^{-1}(\frac{\varepsilon}{2})$ there is $\bar{t} \leq \tau(\varepsilon, \varepsilon)$ so that $\phi(\bar{t},x,u) \in B_{\varepsilon}(\mathscr{A})$.

For any $t > 0$ due to the cocycle property it holds that

$$\phi(t+\bar{t},x,u) = \phi(t, \phi(\bar{t},x,u), u(\bar{t}+\cdot)).$$

The axiom of shift invariance tells us that $\|u(\bar{t}+\cdot)\|_{\mathscr{U}} \leq \|u\|_{\mathscr{U}} \leq \gamma^{-1}(\frac{\varepsilon}{2})$. Hence $\phi(t+\bar{t},x,u) \in \mathscr{A}_{\varepsilon,\gamma}$ for all $t \geq 0$ by definition of $\mathscr{A}_{\varepsilon,\gamma}$ and for our system Σ the set $\mathscr{A}_{\varepsilon,\gamma}$ can be represented as

$$\mathscr{A}_{\varepsilon,\gamma} = \{\phi(t,x,u): t \in [0, \tau(\varepsilon,\varepsilon)], \; x \in B_{\varepsilon}(\mathscr{A}), \; \|u\|_{\mathscr{U}} \leq \gamma^{-1}(\tfrac{\varepsilon}{2})\}.$$

Since Σ is BRS, $\mathscr{A}_{\varepsilon,\gamma}$ is bounded for any $\varepsilon > 0$.

II. 0-invariance of $\mathscr{A}_{\varepsilon,\gamma}$. Let $y \in \mathscr{A}_{\varepsilon,\gamma}$. Then there are certain $s \geq 0$, $x \in B_{\varepsilon}(\mathscr{A})$ and $u \in \mathscr{U}: \|u\|_{\mathscr{U}} \leq \gamma^{-1}(\frac{\varepsilon}{2})$ so that $y = \phi(s,x,u)$. Due to cocycle property it holds for any $t \geq 0$ that

$$\phi(t,y,0) = \phi(t+s,x,w), \quad w(\tau) := \begin{cases} u(\tau), & \text{if } \tau \in [0,s], \\ 0, & \text{otherwise.} \end{cases}$$

By the axiom of concatenation we have $w \in \mathscr{U}$, $\|w\|_{\mathscr{U}} \leq \|u\|_{\mathscr{U}} \leq \gamma^{-1}(\frac{\varepsilon}{2})$ and hence $\phi(t,y,0) \in \mathscr{A}_{\varepsilon,\gamma}$.

III. Σ is UAG w.r.t. $\mathscr{A}_{\varepsilon,\gamma}$. Let us fix $\varepsilon > 0$ and $R > 0$ and revisit the implication (14), which implies that

$$|x|_{\mathscr{A}} \leq R, \|u\|_{\mathscr{U}} \leq \gamma^{-1}(\tfrac{\varepsilon}{2}) \;\Rightarrow\; \exists \bar{t} \leq \tau: \; |\phi(\bar{t},x,u)|_{\mathscr{A}} \leq \varepsilon, \; (15)$$

where $\tau := \tau(\varepsilon, R)$, and for the same τ we have:

$$|x|_{\mathscr{A}} \leq R, \; \|u\|_{\mathscr{U}} \geq \gamma^{-1}(\tfrac{\varepsilon}{2})$$
$$\Rightarrow \; \exists \bar{t} \leq \tau: \; |\phi(\bar{t},x,u)|_{\mathscr{A}} \leq 2\gamma(\|u\|_{\mathscr{U}}). \quad (16)$$

Taking (15) and (16) and using the reasoning exploited to show boundedness of $\mathscr{A}_{\varepsilon,\gamma}$ we obtain

$$|x|_{\mathscr{A}} \leq R, \; \|u\|_{\mathscr{U}} \leq \gamma^{-1}(\tfrac{\varepsilon}{2}), \; t \geq \tau \;\Rightarrow\; \phi(t,x,u) \in \mathscr{A}_{\varepsilon,\gamma} \quad (17)$$

and

$$|x|_{\mathscr{A}} \leq R, \; \|u\|_{\mathscr{U}} \geq \gamma^{-1}(\tfrac{\varepsilon}{2}), \; t \geq \tau \;\Rightarrow\; \phi(t,x,u) \in \mathscr{A}_{2\gamma(\|u\|_{\mathscr{U}}),\gamma} \quad (18)$$

Thus, for all $u \in \mathscr{U}$ we have

$$|x|_{\mathscr{A}} \leq R, \; t \geq \tau(R,\varepsilon) \;\Rightarrow\; \phi(t,x,u) \in \mathscr{A}_{\varepsilon,\gamma} \cup \mathscr{A}_{2\gamma(\|u\|_{\mathscr{U}}),\gamma}. \; (19)$$

As $\mathscr{A}_{k,\gamma}$ is bounded for any $k > 0$, the following function is well-defined:

$$f_{\varepsilon}: s \mapsto \sup_{x \in \mathscr{A}_{s,\gamma}} \|x\|_{\mathscr{A}_{\varepsilon,\gamma}}, \quad s \geq 0. \quad (20)$$

Since $\mathscr{A}_{k_1,\gamma} \subset \mathscr{A}_{k_2,\gamma}$ for $k_1 < k_2$, f_{ε} is nondecreasing and $f_{\varepsilon}(s) = 0$ for $s \in [0,\varepsilon]$. Hence there exists $\sigma_{\varepsilon} \in \mathscr{K}_{\infty}: f_{\varepsilon}(s) \leq \sigma_{\varepsilon}(s)$ for all $s \geq 0$.

With this notation we can reformulate (19) into

$$|x|_{\mathscr{A}} \leq R, \; u \in \mathscr{U}, \; t \geq \tau(R,\varepsilon)$$
$$\Rightarrow \; |\phi(t,x,u)|_{\mathscr{A}_{\varepsilon,\gamma}} \leq \sigma_{\varepsilon}(2\gamma(\|u\|_{\mathscr{U}})). \quad (21)$$

By (4), there is a $C = C(\varepsilon)$ so that $|x|_{\mathscr{A}} \leq |x|_{\mathscr{A}_{\varepsilon,\gamma}} + C$.

Define $\tilde{\tau}(R,\varepsilon) := \tau(R+C,\varepsilon)$. We have:

$$|x|_{\mathscr{A}_{\varepsilon,\gamma}} \leq R, \; u \in \mathscr{U}, \; t \geq \tilde{\tau}(R,\varepsilon)$$
$$\Rightarrow \; |\phi(t,x,u)|_{\mathscr{A}_{\varepsilon,\gamma}} \leq \sigma_{\varepsilon}(2\gamma(\|u\|_{\mathscr{U}})). \quad (22)$$

This shows that Σ is UAG w.r.t. $\mathscr{A}_{\varepsilon,\gamma}$.

Note that here ε is a design parameter of a set $\mathscr{A}_{\varepsilon,\gamma}$ w.r.t. which Σ is UAG, and it is not connected to the parameter ε in the Definition II.8 of the UAG property.

IV. Σ is CUAG w.r.t. $\mathscr{A}_{\varepsilon,\gamma}$. Follows by Proposition III.4. $\quad\square$

Remark III.8. Proposition III.7 shows under certain assumptions that Σ is UAG w.r.t. $\mathscr{A}_{\varepsilon,\gamma}$ for any $\varepsilon > 0$. It is natural to ask, what is the smallest set w.r.t. which Σ is UAG, in particular, whether Σ is UAG w.r.t. $\bigcap_{\varepsilon > 0} \mathscr{A}_{\varepsilon,\gamma}$. We do not follow this line here, but a reader may consult [23, Section 3.2] for related results. In order to understand additional difficulties which arise on this way, note that if $\varepsilon_1 < \varepsilon_2$, then $f_{\varepsilon_1}(s) \geq f_{\varepsilon_2}(s)$

for all $s \in \mathbb{R}_+$ (this follows from (20)). However, this does not imply that there is a continuous function f_0 with $f_0(0) = 0$ so that $f_0(s) \geq f_\varepsilon(s)$ for any $s \geq 0$ and any $\varepsilon > 0$.

Finally, we can prove the main result of this paper:

Proof. **(of Theorem III.1)**

(i) $\Rightarrow$ (ii). Assume, that Σ is an ISpS control system. Then there are $\beta \in \mathscr{KL}$, $\gamma \in \mathscr{K}_\infty$ and $c > 0$ so that for all $x \in X$, $t \geq 0$ and $u \in \mathscr{U}$ we have

$$\|\phi(t,x,u)\| \leq \beta(\|x\|,t) + \gamma(\|u\|_{\mathscr{U}}) + c. \qquad (23)$$

By (4), for any $y \in X$ and any $c > 0$ it holds that $\|y\| \leq |y|_{\overline{B_c(0)}} + c$. Furthermore, for $y \notin \overline{B_c(0)}$ it holds that:

$$|y|_{\overline{B_c(0)}} = \inf_{z \in B_c(0)} \|y - z\| \leq \|y - \frac{c}{\|y\|}y\| = \|y\| - c.$$

Now we infer from (23) for $\phi(t,x,u) \notin \overline{B_c(0)}$ that

$$|\phi(t,x,u)|_{\overline{B_c(0)}} \leq \beta(|x|_{\overline{B_c(0)}} + c, t) + \gamma(\|u\|_{\mathscr{U}}). \qquad (24)$$

Otherwise, if $\phi(t,x,u) \in \overline{B_c(0)}$, then $|\phi(t,x,u)|_{\overline{B_c(0)}} = 0$ and (24) also holds. Thus Σ is CUAG w.r.t. $\overline{B_c(0)}$ (however, $\overline{B_c(0)}$ does not have to be 0-invariant). According to Proposition III.7 there is a bounded 0-invariant set $\mathscr{A}$ so that Σ is CUAG w.r.t. $\mathscr{A}$.

(ii) $\Rightarrow$ (iii). Clear.

(iii) $\Rightarrow$ (i). Proposition III.7 implies that there is a bounded set $\mathscr{A} \subset X$ so that Σ is CUAG w.r.t. $\mathscr{A}$. Proposition III.5 shows ISpS of Σ. □

IV. SPECIAL CLASSES OF SYSTEMS

One of the criteria for ISpS, shown in Theorem III.1, states that ISpS of a control system Σ is equivalent to existence of a 0-invariant set $\mathscr{A}$ so that Σ has a CUAG property w.r.t. $\mathscr{A}$. It is natural to ask whether an ISS property (which is stronger than CUAG) holds w.r.t. this set. This problem can be approached using the following result:

Proposition IV.1. *Let $\mathscr{A} \subset X$ be bounded. A control system Σ is ISS w.r.t. $\mathscr{A}$ if and only if Σ is CUAG w.r.t. $\mathscr{A}$ and $\mathscr{A}$ is a robustly 0-invariant set.*

Proof. This result has been shown in [19, Theorem 5] for $\mathscr{A} = \{0\}$. In view of a characterization of the CUAG property in Proposition III.4. The proof for general bounded sets $\mathscr{A}$ is completely analogous and hence is omitted. □

However, the proof or disproof of robust 0-invariance of the constructed 0-invariant set $\mathscr{A}$ is in general a difficult task. In Section IV-A we show that this is possible for systems with a Lipschitz continuous flow under certain restrictions on the input space. The notion of s-invariance and the corresponding Lemma IV.3 help us heavily on this way. Next, in Section IV-B above criteria will be applied to semilinear evolution equations in Banach spaces. Finally, Section IV-C is devoted to ODE systems. In this case we can strengthen the results even further, due to the fact that ULIM and LIM notions coincide for ODE systems.

A. Systems with Lipschitz continuous flows

Definition IV.2. The flow of a control system Σ is called Lipschitz continuous on compact intervals (for uniformly bounded inputs), if for any $h > 0$ and any $r > 0$ there is $L > 0$ s.t. for any $x, y \in \overline{B_r}$, all $u \in \overline{B_{r,\mathscr{U}}}$ and all $t \in [0,h]$ it holds that

$$\|\phi(t,x,u) - \phi(t,y,u)\| \leq L\|x - y\|. \qquad (25)$$

Many classes of systems possess flows which are Lipschitz continuous on compact intervals. Semilinear evolution equations and ODEs with Lipschitz continuous nonlinearities, which are considered next, are particular of examples of such kind of systems. Lipschitz continuity of the flow helps to prove such significant results as e.g. converse Lyapunov theorems for infinite-dimensional systems [34, Section 3.4], [22].

Lemma IV.3. *Let a control system Σ be given and let $\mathscr{A} \subset X$ be a bounded s-invariant set, for a certain $s > 0$. If the flow of Σ is Lipschitz continuous on compact intervals, then $\mathscr{A}$ is a robustly s-invariant set.*

Proof. Let $\mathscr{A}$ be a bounded s-invariant set, for a certain $s > 0$.

Pick any $\varepsilon > 0$, $h > 0$ and set $r := \|\mathscr{A}\| + 1$. For this r there is a $L = L(2r, h)$ so that for any $x, y \in \overline{B_{2r}}$, $u \in \overline{B_{2r,\mathscr{U}}}$ and $t \in [0,h]$ it holds that

$$\|\phi(t,x,u) - \phi(t,y,u)\| \leq L\|x - y\|.$$

Set $\delta := \min\left\{\frac{\varepsilon}{L}, r\right\}$ and pick any $x \in \overline{B_\delta(\mathscr{A})}$ (hence $\|x\| \leq \|\mathscr{A}\| + \delta < 2r$). Then there is a $y \in \mathscr{A}$: $\|x - y\| \leq \delta$ and the following estimates hold for $t \in [0,h]$ and $u \in \overline{B_{\min\{r,s\},\mathscr{U}}}$ (note that $\phi(t,y,u) \in \mathscr{A}$ due to s-invariance of $\mathscr{A}$):

$$|\phi(t,x,u)|_{\mathscr{A}} = \inf_{z \in \mathscr{A}} \|\phi(t,x,u) - z\| \leq \|\phi(t,x,u) - \phi(t,y,u)\|$$
$$\leq L\|x - y\|$$
$$\leq \varepsilon.$$

This shows robust invariance of $\mathscr{A}$. □

Lemma IV.4. *Let Σ be a control system. Let also for each $u,v \in \mathscr{U}$ and for all $t > 0$ the concatenation*

$$w(s) := \begin{cases} u(s), & \text{if } s \in [0,t], \\ v(s-t), & \text{otherwise}, \end{cases}$$

of u and v at time t satisfy the property

$$\|w\|_{\mathscr{U}} \leq \max\{\|u\|_{\mathscr{U}}, \|v\|_{\mathscr{U}}\}. \qquad (26)$$

Then for any $\varepsilon > 0$ and any $\gamma \in \mathscr{K}_\infty$ the space $\mathscr{A}_{\varepsilon,\gamma}$ is $\gamma^{-1}(\frac{\varepsilon}{2})$-invariant.

Proof. Pick any $y \in \mathscr{A}_{\varepsilon,\gamma}$ and any $v \in \overline{B_{\gamma^{-1}(\frac{\varepsilon}{2}),\mathscr{U}}}$. Then there are certain $s \geq 0$, $x \in B_\varepsilon(\mathscr{A})$ and $u \in \overline{B_{\gamma^{-1}(\frac{\varepsilon}{2}),\mathscr{U}}}$ so that $y = \phi(s,x,u)$. Due to cocycle property it holds for each $t > 0$ that

$$\phi(t,y,v) = \phi(t+s,x,w), \qquad (27)$$

where w is a concatenation of u and v at time s.

In view of the assumption (26) it holds that $\|w\|_{\mathscr{U}} \leq \gamma^{-1}(\frac{\varepsilon}{2})$, and hence $\phi(t+s,x,w) \in \mathscr{A}_{\varepsilon,\gamma}$ due to the definition of $\mathscr{A}_{\varepsilon,\gamma}$. Hence $\phi(t,y,v) \in \mathscr{A}_{\varepsilon,\gamma}$. □

Remark IV.5. An additional assumption on the input space $\mathscr{U}$ in Lemma IV.4 restricts the class of inputs. In particular, the inputs from L_p spaces or from Sobolev spaces do not fulfill it. However, the spaces of continuous, piecewise continuous and L_∞ functions do satisfy it (w.r.t. the natural sup or esssup-norm respectively).

Finally, we are able to characterize ISpS in terms of ISS:

Theorem IV.6. *Let $\Sigma = (X, \mathscr{U}, \phi)$ be a control system, ϕ be Lipschitz continuous on compact time intervals and the input space $\mathscr{U}$ satisfy the assumptions of Lemma IV.4. Then:*

$$\Sigma \text{ is ISpS} \quad \Leftrightarrow \quad \begin{array}{l} \text{For any } s > 0 \text{ there is a bounded} \\ s\text{-invariant set } \mathscr{A} \subset X: \Sigma \text{ is ISS w.r.t. } \mathscr{A} \end{array}$$

Proof. "⇐". Since Σ is CUAG w.r.t. $\mathscr{A}$, Proposition III.5 shows that Σ is ISpS.

"⇒". According to Theorem III.1 and Proposition III.7, ISpS of Σ with a corresponding gain $\gamma \in \mathscr{K}_\infty$ implies that for each $\varepsilon > 0$ the system Σ is CUAG w.r.t. $\mathscr{A}_{\varepsilon,\gamma}$. Since the assumptions of Lemma IV.4 hold, $\mathscr{A}_{\varepsilon,\gamma}$ is a $\gamma^{-1}(\frac{\varepsilon}{2})$-invariant bounded set. Now Lemma IV.3 shows that $\mathscr{A}_{\varepsilon,\gamma}$ is a robustly $\gamma^{-1}(\frac{\varepsilon}{2})$-invariant bounded set. Finally, Proposition IV.1 proves that Σ is ISS w.r.t. $\mathscr{A}_{\varepsilon,\gamma}$. Since $\varepsilon > 0$ can be chosen arbitrarily, and since $\gamma \in \mathscr{K}_\infty$, then $\gamma^{-1}(\frac{\varepsilon}{2})$ can be made arbitrarily large by choosing sufficiently large ε. □

B. Semilinear evolution equations

Here we specify the results obtained previously to semilinear evolution equations in Banach spaces.

Let X be a Banach space and A be the generator of a strongly continuous semigroup T of bounded linear operators on X and let $f : X \times U \to X$. Consider the system

$$\dot{x}(t) = Ax(t) + f(x(t), u(t)). \tag{28}$$

We study mild solutions of (28), i.e. solutions $x : [0, \tau] \to X$ of the integral equation

$$x(t) = T(t)x(0) + \int_0^t T(t-s)f(x(s), u(s))ds, \tag{29}$$

belonging to the space of continuous functions $C([0, \tau], X)$ for some $\tau > 0$.

We assume that the set of input values U is a normed linear space and that the input functions belong to the space $\mathscr{U} := PC(\mathbb{R}_+, U)$ of globally bounded, piecewise continuous functions $u : \mathbb{R}_+ \to U$, which are right continuous. The norm of $u \in \mathscr{U}$ is given by $\|u\|_{\mathscr{U}} := \sup_{t \geq 0} \|u(t)\|_U$.

We assume that the solution of (28) exists and is unique on $\mathbb{R}_+$ (i.e. (28) is forward complete). ISpS of (28) can be characterized as follows:

Proposition IV.7. *Consider a BRS system (28) satisfying*
- *$f : X \times U \to X$ is Lipschitz continuous on bounded subsets of X.*
- *$f(x, \cdot)$ is continuous for all $x \in X$.*

The following statements are equivalent:

(i) (28) is ISpS
(ii) For any $s > 0$ there is a bounded s-invariant set $\mathscr{A} \subset X$: (28) is ISS w.r.t. $\mathscr{A}$.

(iii) There is a bounded set $\mathscr{A} \subset X$: (28) is ULIM w.r.t. $\mathscr{A}$.

Proof. According to [19, Section VII], a BRS system (28) satisfying assumptions of the proposition has a flow which is Lipschitz continuous on compact intervals (in [19, Section VII] a stronger assumption on f has been imposed, but the assertion which we need here can be shown without these further requirements). The input space $\mathscr{U} := PC(\mathbb{R}_+, U)$ satisfies the assumptions of Lemma IV.4. Now the application of Theorems III.1, IV.6 proves the claim of the proposition. □

C. Ordinary differential equations

Finally, we turn our attention to the ISpS theory of ODEs

$$\dot{x} = f(x, u), \tag{30}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is locally Lipschitz continuous w.r.t. the first argument and inputs u belong to the set $\mathscr{U} := L_\infty(\mathbb{R}_+, \mathbb{R}^m)$ of Lebesgue measurable globally essentially bounded functions with values in $\mathbb{R}^m$.

In [14, Proposition VI.3] Sontag and Wang have shown the following result:

Proposition IV.8. *Let (30) be forward complete. The following statements are equivalent:*

(i) (30) is ISpS
(ii) (30) has compact ISS property, i.e. there is a compact 0-invariant set $\mathscr{A} \subset \mathbb{R}^n$ so that (30) is UAG w.r.t. $\mathscr{A}$.
(iii) There is a compact set $\mathscr{A} \subset \mathbb{R}^n$: (30) is ISS w.r.t. $\mathscr{A}$.

Note that in item (iii) of Proposition IV.8 the set $\mathscr{A}$ is automatically 0-invariant, as (30) is ISS w.r.t. $\mathscr{A}$. The equivalence between items (ii) and (iii) was not explicitly stated in [14, Proposition VI.3], but directly follows from [14, Theorem 1].

The characterizations given in Proposition IV.8 show that an ISpS system possesses a stronger (ISS) property, but relative to a larger set. Next we show that the results developed in the previous sections, provide us with a criterion of ISpS in terms of a weaker limit property.

The following result has been shown in [19] for $\mathscr{A} = \{0\}$ on the basis of [14, Corollary III.3]. The proof for general bounded $\mathscr{A}$ is analogous and thus omitted.

Proposition IV.9. *Consider a system (30) with $\mathscr{U}$ as above. Let $\mathscr{A} \subset \mathbb{R}^n$ be any bounded set. Then Σ is ULIM w.r.t. $\mathscr{A}$ if and only if Σ is LIM w.r.t. $\mathscr{A}$.*

The main result of this section is:

Corollary IV.10. *Let (30) be forward complete. The following statements are equivalent:*

(i) (30) is ISpS
(ii) For any $s > 0$ there is a compact s-invariant set $\mathscr{A} \subset \mathbb{R}^n$: (30) is ISS w.r.t. $\mathscr{A}$.
(iii) There is a bounded set $\mathscr{A} \subset \mathbb{R}^n$: (30) is LIM w.r.t. $\mathscr{A}$.

Proof. According to [31, Proposition 5.1], for (30) forward completeness is equivalent to the BRS property.

(i) ⇒ (ii). By [31, Proposition 5.5], the flow of (30) is Lipschitz continuous on compact subsets. The input space

$\mathscr{U} := L_\infty(\mathbb{R}_+, \mathbb{R}^m)$ satisfies the assumptions of Lemma IV.4. Application of Theorem IV.6 proves that for any $s > 0$ there is a bounded s-invariant set $\mathscr{A} \subset \mathbb{R}^n$: (30) is ISS w.r.t. $\mathscr{A}$.

Since f is Lipschitz continuous w.r.t. the first argument, the solutions of (30) depend continuously on initial data. Hence $\overline{\mathscr{A}}$ is again s-invariant (as a closure of an s-invariant set). Since $\mathscr{A}$ is bounded, $\overline{\mathscr{A}}$ is compact.

(ii) $\Rightarrow$ (iii). Clear.

(iii) $\Rightarrow$ (i). Follows by Proposition IV.9 and Theorem III.1. $\qquad \square$

Remark IV.11. The equivalence between items (i) and (ii) can be seen as a slight strengthening of Proposition IV.8. The equivalence between items (i) and (iii) is novel.

V. Conclusion

For a broad class of infinite-dimensional systems, we have proved criteria for practical ISS in terms of uniform limit property and in terms of ISS. The characterization of ISpS in terms of the limit property is novel already for ODE systems.

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