SMALL GAIN THEOREMS FOR GENERAL NETWORKS OF HETEROGENEOUS INFINITE-DIMENSIONAL SYSTEMS

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Abstract. We prove a small-gain theorem for interconnections of n nonlinear heterogeneous input-to-state stable control systems of a general nature, covering partial, delay and ordinary differential equations. Furthermore, for the same class of control systems we derive small-gain theorems for asymptotic gain, uniform global stability and weak input-to-state stability properties. We show that our technique is applicable for different formulations of ISS property (summation, maximum, semimaximum) and discuss tightness of achieved small-gain theorems. Finally, we introduce variations of asymptotic gain and limit properties, which are particularly useful for small-gain arguments and characterize ISS in terms of these notions.

Key words. infinite-dimensional systems, input-to-state stability, interconnected systems, nonlinear systems.

AMS subject classifications. 93C25, 37C75, 93A15, 93C10.

1. Introduction.

The notion of input-to-state stability (ISS), introduced in [34] for ordinary differential equations (ODEs), has become a backbone for much of nonlinear control theory, and is currently a well developed theory with a firm theoretical basis and such powerful tools for ISS analysis, as Lyapunov and small-gain methods. Broad applications of ISS theory include robust stabilization of nonlinear systems [7], design of nonlinear observers [1], analysis of large-scale networks [13, 6, 5], etc.

The impact of finite-dimensional ISS theory and the need of proper tools for robust stability analysis of distributed parameter systems resulted in generalizations of ISS concepts to broad classes of infinite-dimensional systems, including partial differential equations (PDEs) with distributed and boundary controls, nonlinear evolution equations in Banach spaces with bounded and unbounded input operators, etc. [4, 26, 25, 21, 37, 11, 17, 23, 9, 19, 20, 18].

Techniques developed within infinite-dimensional ISS theory include characterizations of ISS and ISS-like properties in terms of weaker stability concepts [26, 23], [9, 32], constructions of ISS Lyapunov functions for PDEs with distributed and boundary controls [21, 31, 4, 25, 37, 42], non-coercive ISS Lyapunov functions [26, 8], efficient methods for study of boundary control systems [41, 9, 10, 17, 20], etc.

One of the cornerstones of the mathematical control theory is the analysis of interconnected systems. Large-scale nonlinear systems can be very complex, so that a direct stability analysis of such systems is seldom possible. Small-gain theorems, which are one of the pillars on which the ISS theory stands, make it possible to overcome this obstacle. These theorems guarantee input-to-state stability of an interconnected system, provided all subsystems are ISS and the interconnection structure, described by gains, satisfies the small-gain condition.

1.1. Existing ISS small-gain results.

There are two kinds of nonlinear small-gain theorems: theorems in terms of trajectories and in terms of Lyapunov functions. In *small-gain theorems in the trajectory formulation* one assumes that each subsystem is ISS both w.r.t. external inputs and internal inputs from other subsystems, and the so-called internal gains of subsystems characterizing the influence of subsystems

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on each other are known. The small-gain theorem states that the feedback interconnection is ISS provided the gains satisfy the small-gain condition. First small-gain theorems of this type have been developed in [13] for feedback couplings of two ODE systems and in [6] for couplings of n ODE systems.

In *Lyapunov small-gain theorems* it is assumed that all subsystems are ISS w.r.t. external and internal inputs and the ISS Lyapunov functions for subsystems are given together with the corresponding Lyapunov gains. If Lyapunov gains satisfy the small-gain condition, then the whole interconnection is ISS and moreover, an ISS Lyapunov function for the overall system can be constructed. For couplings of 2 systems such theorems have been shown in [12] and this result has been extended to couplings of n nonlinear ODE systems in [5].

As was shown in [4], ISS small-gain theorems in a Lyapunov formulation can be extended to interconnections of n infinite-dimensional systems without radical changes in the formulation and proof technique. Generalization of integral ISS small-gain theorems is more difficult primarily due to the fact that the state spaces for subsystems should be carefully chosen (see [25]), although the formulation of the small-gain theorem itself remains similar to the ODE case. ISS small-gain theorems in terms of vector Lyapunov functions have been reported in [14, 15].

The case of trajectory-based infinite-dimensional small-gain theorems for couplings of $n > 2$ *systems is significantly more complicated* since the proof of such theorems in [6] is based on the fundamental result that ISS of ODE systems is equivalent to uniform global stability (UGS) combined with the asymptotic gain (AG) property shown in [36, 35]. Such a characterization is not valid for infinite-dimensional systems, as argued in [23, 26] which makes the proof of [6] not applicable to infinite-dimensional systems without substantial modifications.

In particular, considerable attention has been devoted to small-gain theorems in terms of trajectories for time-delay systems. To the knowledge of the author, the first attempt to obtain ISS and, more generally, IOS (input-to-output stability) small-gain results for time-delay systems has been made in [30]. In this paper small-gain theorems for couplings of 2 time-delay systems possessing UGS and AG properties have been derived, but no small-gain theorem for ISS property. As AG $\wedge$ UGS is (probably) weaker than ISS for time-delay systems, the ISS small-gain theorem has not been obtained in this work. Small-gain theorems for couplings of n time-delay systems with AG $\wedge$ UGS properties have been obtained in [39] and [29].

The first ISS small-gain theorems, applicable for time-delay systems have been achieved in [14], where the small-gain theorems in terms of trajectories (in maximum formulation) have been shown for couplings of two control systems of a rather general nature, covering in particular time-delay systems.

The obstacle that ISS is (at least potentially) not equivalent to AG $\wedge$ UGS, was overcome in [40] where ISS small-gain theorems for couplings of $n \geq 2$ time-delay systems have been obtained by using a Razumikhin-type argument, motivated by [38]. In this approach the delayed state in the right hand side of a time-delay system is treated as an input to the system, which makes the time-delay system a delay-free system with an additional input. However, the transformation of time-delay systems to the delay-free form is not always straightforward.

Recently in [2] the small-gain theorems for couplings of n input-to-output stable (IOS) evolution equations in Banach spaces have been derived. As a special case of these results, the authors obtain a small-gain theorem for networks of n ISS systems in the maximum formulation. Application of small-gain theorems for stability analysis of coupled parabolic-hyperbolic PDEs has been performed in [19]. Small-gain based boundary feedback design for global exponential stabilization of 1-D semilinear parabolic PDEs has been proposed in [18].

1.2. Contribution. *Our main results are ISS small-gain theorems (in summation, semimaximum and maximum formulation) for feedback interconnections of n nonlinear heterogeneous control systems whose components belong to a broad class of control systems covering PDEs, time-delay systems, ODE etc. Furthermore, for the same class of control systems we show small-gain theorems for AG, UGS and weak ISS properties.*

For the description of interconnections of control systems we adopt an approach described in [14]. Furthermore, in this paper we use a variation of a uniform asymptotic gain (UAG) property, which we call bounded input uniform asymptotic gain (bUAG) property. Although it was used so far not so often as the standard UAG property (but see e.g. [38], [22, Proposition 1.4.3.]), it is more flexible in use and in most cases as powerful as the standard UAG property.

The ISS small-gain theorem for the summation formulation of the ISS property (Theorem 5.3) is achieved in 3 steps:

(i) We derive UGS property of the interconnection (which is the UGS small-gain Theorem 5.1) using the methods developed in [6].
(ii) We show that the interconnection satisfies bUAG property (the main technical step).
(iii) We show that UGS $\wedge$ bUAG is equivalent to ISS, which concludes the proof. Here we base ourselves on characterizations of ISS obtained in [26].

Using similar steps we prove in Section 5.4 ISS small-gain theorems in semimaximum (Theorem 5.11) and maximum (Corollary 5.13) formulations.

Motivated by the notion of strong input-to-state stability (sISS), introduced in [26] and studied in [26], [28], in [32, 33] the concept of weak input-to-state stability (wISS) has been introduced and investigated, in particular, in the context of robust stabilization of port-Hamiltonian systems, see [33]. The system is called weak ISS, if it is uniformly globally stable and has an asymptotic gain property. In Section 5.3 we derive a small-gain result for a weak ISS property.

As asymptotic gain properties for bounded inputs were very useful for the proof of small-gain theorems, it is natural to surmise that they can be useful in other contexts as well. In Section 6 we derive several new characterizations of ISS using properties of this kind.

1.3. Relation to previous research. This paper is motivated by the ISS small-gain theorems for networks of $n \in \mathbb{N}$ ODE systems, reported in [6], and recovers these results in a special case of ODE systems.

As a particular application of our general small-gain theorems one can obtain novel small-gain results for couplings of n nonlinear time-delay systems. In contrast to [30, 39, 29], we obtain not only UGS $\wedge$ AG (i.e. weak ISS) small-gain results, but also ISS small-gain theorems. And in contrast to ISS small-gain theorems from [40], our approach is not time-delay specific, does not require a transformation of retarded systems into delay-free ones and is applicable to the sum formulation of the ISS property.

In [2] small-gain theorems for couplings of n evolution equations in Banach spaces with Lipschitz continuous nonlinearities have been derived, by using a rather different proof technique, which is applicable if the small-gain property is formulated in the so-called maximization formulation. Instead, we focus in this work on the summation and semimaximum formulations of the ISS property and thus the developments in [2] are complementary to this paper.

The approach which we use in this paper is very flexible as it is valid for a broad class of infinite-dimensional systems and applies to many different formulations of the ISS property (and not only for the maximum formulation). In the case of summation and semimaximum formulations our small-gain theorems are also tight, as shown in Section 5.5. However, for the maximum formulation of the ISS property the results in [2] are stronger.

Finally, we note that a key ingredient in the proof of the small-gain theorems in this paper are the characterizations of ISS of infinite-dimensional systems in terms of weaker stability properties, derived in [26]. These characterizations have been already useful in several further contexts as the non-coercive Lyapunov function theory, see [26, 8] and the characterization of the practical ISS property [24]. We hope that refined and more flexible versions of these characterizations derived in Section 6 will find further applications within the infinite-dimensional ISS theory.

1.4. Notation. In the following $\mathbb{R}_+ := [0,\infty)$. For arbitrary $x, y \in \mathbb{R}^n$ define the relation "$\geq$" on $\mathbb{R}^n$ by: $x \geq y \;\Leftrightarrow\; x_i \geq y_i,\ \forall i = 1,\dots,n$.

By "$\not\geq$" we understand the logical negation of "$\geq$", that is $x \not\geq y$ iff $\exists i$: $x_i < y_i$.

Further define $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x \geq 0\}$.

For a function $v : \mathbb{R}_+ \to X$, where X is a certain set, we define its restriction to the interval $[s_1, s_2]$ by

$$v_{[s_1,s_2]}(t) := \begin{cases} v(t) & \text{if } t \in [s_1, s_2], \\ 0 & \text{else.} \end{cases}$$

Also we will use the following classes of comparison functions.

$$\begin{aligned}
\mathcal{K} &:= \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \ :\ \gamma \text{ is continuous and strictly increasing, } \gamma(0) = 0\} \\
\mathcal{K}_\infty &:= \{\gamma \in \mathcal{K} \ :\ \gamma \text{ is unbounded}\} \\
\mathcal{L} &:= \left\{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \ :\ \gamma \text{ is continuous and decreasing with } \lim_{t\to\infty} \gamma(t) = 0\right\} \\
\mathcal{KL} &:= \left\{\beta : \mathbb{R}^2_+ \to \mathbb{R}_+ \ :\ \beta(\cdot,t) \in \mathcal{K},\ \forall t \geq 0,\ \beta(r,\cdot) \in \mathcal{L},\ \forall r > 0\right\}
\end{aligned}$$

2. Problem formulation.

2.1. Definition of control systems. We define the concept of a (time-invariant) system in the following way:

Definition 2.1. *Consider the triple $\Sigma = (X, \mathcal{U}, \phi)$ consisting of*

(i) A normed linear space $(X, \|\cdot\|_X)$, called the state space, endowed with the norm $\|\cdot\|_X$.

(ii) A set of input values U, which is a nonempty subset of a certain normed linear space.

(iii) A space of inputs $\mathcal{U} \subset \{f : \mathbb{R}_+ \to U\}$ endowed with a norm $\|\cdot\|_{\mathcal{U}}$ which satisfies the following two axioms:

The axiom of shift invariance*: for all $u \in \mathcal{U}$ and all $\tau \geq 0$ the time shift $u(\cdot+\tau)$ belongs to $\mathcal{U}$ with $\|u\|_{\mathcal{U}} \geq \|u(\cdot+\tau)\|_{\mathcal{U}}$.*

The axiom of concatenation*: for all $u_1, u_2 \in \mathcal{U}$ and for all $t > 0$ the concatenation of u_1 and u_2 at time t*

$$u(\tau) := \begin{cases} u_1(\tau), & \textit{if } \tau \in [0,t], \\ u_2(\tau - t), & \textit{otherwise,} \end{cases} \tag{2.1}$$

belongs to $\mathcal{U}$.

(iv) *A map* $\phi : D_\phi \to X$, $D_\phi \subseteq \mathbb{R}_+ \times X \times \mathcal{U}$ *(called transition map), so that for all* $(x,u) \in X \times \mathcal{U}$ *there is a* $t > 0$ *so that* $[0,t] \times \{(x,u)\} \subset D_\phi$.

The triple Σ *is called a (forward complete) dynamical system, if the following properties hold:*

(Σ1*)* The identity property: *for every* $(x,u) \in X \times \mathcal{U}$ *it holds that* $\phi(0,x,u) = x$.

(Σ2*)* Causality: *for every* $(t,x,u) \in \mathbb{R}_+ \times X \times \mathcal{U}$, *for every* $\tilde{u} \in \mathcal{U}$, *such that* $u(s) = \tilde{u}(s)$ *for all* $s \in [0,t]$ *it holds that* $\phi(t,x,u) = \phi(t,x,\tilde{u})$.

(Σ3*)* Continuity: *for each* $(x,u) \in X \times \mathcal{U}$ *the map* $t \mapsto \phi(t,x,u)$ *is continuous.*

(Σ4*)* The cocycle property: *for all* $x \in X$, $u \in \mathcal{U}$, *for all* $t, h \geq 0$ *so that* $[0,t+h] \times \{(x,u)\} \subset D_\phi$, *we have* $\phi(h, \phi(t,x,u), u(t+\cdot)) = \phi(t+h,x,u)$.

This class of systems encompasses control systems generated by ordinary differential equations (ODEs), switched systems, time-delay systems, many classes of partial differential equations (PDEs), important classes of boundary control systems and many other systems.

Next we define several important properties of control systems:

Definition 2.2. *We say that a control system (as introduced in Definition 2.1 is* forward complete (FC)*, if* $D_\phi = \mathbb{R}_+ \times X \times \mathcal{U}$, *that is for every* $(x,u) \in X \times \mathcal{U}$ *and for all* $t \geq 0$ *the value* $\phi(t,x,u) \in X$ *is well-defined.*

Remark 2.3. *The class of forward complete control systems considered in this paper is precisely the class of control systems considered in [26], where forward completeness axiom was a part of a definition of a control system. Hence the results obtained in [26] are applicable to forward complete systems considered in this paper.*

An important property of ordinary differential equations with Lipschitz continuous right-hand sides states that if the solution stays bounded over $[0,t)$, then it can be prolonged to $[0,t+\varepsilon)$ for a certain $\varepsilon > 0$. Similar properties have evolution equations in Banach spaces with bounded control operators and Lipschitz continuous right hand sides [3, Theorem 4.3.4] and many other classes of systems [16, Chapter 1]. The next property, adopted from [16, Definition 1.4] formalizes this behavior for general control systems.

Definition 2.4. *We say that a system* Σ *satisfies the* boundedness-implies-continuation (BIC) property *if for each* $(x,u) \in X \times \mathcal{U}$, *there exists* $t_{\max} \in (0,+\infty]$, *called a* maximal existence time*, such that* $[0,t_{\max}) \times \{(x,u)\} \subset D_\phi$ *and for all* $t \geq t_{\max}$, *it holds that* $(t,x,u) \notin D_\phi$. *In addition, if* $t_{\max} < +\infty$, *then for every* $M > 0$, *there exists* $t \in [0,t_{\max})$ *with* $\|\phi(t,x,u)\|_X > M$.

2.2. Interconnections of control systems. Let $(X_i, \|\cdot\|_{X_i})$, $i = 1,\dots,n$ be normed linear spaces endowed with the corresponding norms. Define for each $i = 1,\dots,n$ the normed linear space

$$X_{\neq i} := X_1 \times \ldots \times X_{i-1} \times X_{i+1} \times \ldots \times X_n, \qquad \|x\|_{X_{\neq i}} := \sqrt{\sum_{j=1,\ j\neq i}^{n} \|x_j\|^2_{X_j}}. \tag{2.2}$$

Let control systems $\Sigma_i := (X_i, PC(\mathbb{R}_+, X_{\neq i}) \times \mathcal{U}, \bar{\phi}_i)$ be given and assume that each Σ_i possesses a BIC property. We call $X_{\neq i}$ the *space of internal input values*, $PC(\mathbb{R}_+, X_{\neq i})$ the *space of internal inputs*. The norm on $PC(\mathbb{R}_+, X_{\neq i}) \times \mathcal{U}$ we define as

$$\|(v,u)\|_{PC(\mathbb{R}_+,X_{\neq i})\times\mathcal{U}} := \sqrt{\sum_{j\neq i} \|v_j\|^2_{PC(\mathbb{R}_+,X_j)} + \|u\|^2_{\mathcal{U}}}. \tag{2.3}$$

Define also the normed linear space

$$X = X_1 \times \ldots \times X_n, \qquad \|x\|_X := \sqrt{\sum_{i=1}^n \|x_i\|^2_{X_i}}, \tag{2.4}$$

and assume that there is a map $\phi = (\phi_1, \ldots, \phi_n) : D_\phi \to X$, defined over a certain domain $D_\phi \subseteq \mathbb{R}_+ \times X \times \mathcal{U}$ so that for each $x = (x_1, x_2, \ldots, x_n) \in X$, each $u \in \mathcal{U}$ and all $t \in \mathbb{R}_+$ so that $(t, x, u) \in D_\phi$ and for every $i = 1, \ldots, n$, it holds that

$$\phi_i(t, x_i, u) = \bar{\phi}_i\big(t, x_i, (v_i, u)\big), \tag{2.5}$$

where

$$v_i(t) = (\phi_1(t, x, u), \ldots, \phi_{i-1}(t, x, u), \phi_{i+1}(t, x, u), \ldots, \phi_n(t, x, u)).$$

Assume further that $\Sigma := (X, \mathcal{U}, \phi)$ is a control system with the state space X, input space $\mathcal{U}$ and with a BIC property. Then Σ is called *a (feedback) interconnection* of systems $\Sigma_1, \ldots, \Sigma_n$.

In other words, condition (2.5) means that if the modes $\phi_j(\cdot, x, u)$, $j \neq i$ of the system Σ will be sent to Σ_i as the internal inputs (together with an external input u), and the initial state will be chosen as x_i (the i-th mode of x), then the resulting trajectory of the system Σ_i, which is $\bar{\phi}_i(\cdot, x_i, v, u)$ will coincide with the trajectory of the i-th mode of the system Σ on the interval of existence of ϕ_i.

Note that the trajectory of each Σ_i depends continuously on time due to the continuity axiom. However, as the space of continuous functions does not satisfy the concatenation property, we enlarge it to include the piecewise continuous functions. This motivates the choice of the space $PC(\mathbb{R}_+, X_{\neq i}) \times \mathcal{U}$ as the input space for the i-th subsystem.

Remark 2.5. *This definition of feedback interconnections, which we adopted from [14, Definition 3.3], does not depend on a particular type of control systems which are coupled, and is applicable to large-scale systems, consisting of heterogeneous components as PDEs, time-delay systems, ODE systems etc. The definition also applies to different kinds of interconnections, e.g. both for in-domain and boundary interconnections of PDE systems.*

One of the key problems is to check whether the interconnection is well-defined, i.e. whether the solution exists in a certain sense. This problem can be very hard, especially in case of boundary interconnections of PDE systems. In our definition of the interconnection we assume that the interconnection is well-defined, and thus we avoid the well-posedness problem and concentrate our attention on stability analysis of coupled systems. □

Next we show how the couplings of ODEs and of evolution equations in Banach spaces can be represented in our approach. Many further classes of systems can be treated in a similar way.

2.3. Example: interconnections of ODE systems. Consider the interconnected systems of the following form

$$\begin{cases} \dot{x}_i = f_i(x_1, \ldots, x_n, u), \\ i = 1, \ldots, n, \end{cases} \tag{2.6}$$

Here we assume that the state space of the i-th subsystem is $X_i := \mathbb{R}^{p_i}$ endowed with the Euclidean norm. Define $N := p_1 + \ldots + p_n$. Then the state space of the whole system, defined by (2.4) is $X := \mathbb{R}^N$, endowed with the Euclidean norm. We assume that inputs belong to the space $\mathcal{U} := L_\infty(\mathbb{R}_+, \mathbb{R}^m)$. Assuming that f is Lipschitz continuous w.r.t. the state, for each initial condition and for each external input $u \in \mathcal{U}$ there is a unique absolutely continuous (mild) solution of (2.6) and (2.6) is a well-defined control system, which is an interconnection of the systems Σ_i, $i = 1, \dots, n$.

Defining for $x_i \in \mathbb{R}^{p_i}$, $i = 1, \dots, n$ the state $x = (x_1, \dots, x_n)^T$, $f(x, u) = (f_1(x, u), \dots, f_n(x, u))^T$, we can rewrite the coupled system in the form

$$\dot{x}(t) = f(x(t), u(t)), \quad u(t) \in U. \tag{2.7}$$

2.4. Example: interconnections of evolution equations in Banach spaces. Consider a system of the following form

$$\begin{cases} \dot{x}_i = A_i x_i + f_i(x_1, \dots, x_n, u), \\ i = 1, \dots, n, \end{cases} \tag{2.8}$$

where the state space of the i-th subsystem X_i is a Banach space and A_i with the domain of definition $D(A_i)$ is the generator of a C_0-semigroup on X_i, $i = 1, \dots, n$. In the sequel we assume that the set of input values U is a normed linear space and that the input functions belong to the space $\mathcal{U} := PC(\mathbb{R}_+, U)$ of globally bounded, piecewise continuous functions $u : \mathbb{R}_+ \to U$, which are right continuous. The norm of $u \in \mathcal{U}$ is given by $\|u\|_{\mathcal{U}} := \sup_{t \geq 0} \|u(t)\|_U$.

Define the state space X of the whole system (2.8) by (2.4). We choose further the input space to the i-th subsystem as (2.3).

For $x_i \in X_i$, $i = 1, \dots, n$ define $x = (x_1, \dots, x_n)^T$, $f(x, u) = (f_1(x, u), \dots, f_n(x, u))^T$. By A we denote the diagonal operator $A := \operatorname{diag}(A_1, \dots, A_n)$, i.e.:

$$A = \begin{pmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_n \end{pmatrix}$$

Domain of definition of A is given by $D(A) = D(A_1) \times \dots \times D(A_n)$. It is well-known that A is the generator of a C_0-semigroup on X.

With this notation the coupled system (2.8) takes the form

$$\dot{x}(t) = Ax(t) + f(x(t), u(t)), \quad u(t) \in U. \tag{2.9}$$

Assuming that f is Lipschitz continuous w.r.t. x guarantees that the mild solutions of (2.9) exists and is unique for every initial condition and for any admissible input. Here mild solutions $x : [0, \tau] \to X$ are the solutions of the integral equation

$$x(t) = T(t)x(0) + \int_0^t T(t-s) f(x(s), u(s)) ds, \tag{2.10}$$

belonging to the space of continuous functions $C([0, \tau], X)$ for some $\tau > 0$.

Under these assumptions the system (2.9) can be seen as a well-defined interconnection of the systems Σ_i, $i = 1, \dots, n$, and each Σ_i is a well-defined system in the sense of Definition 2.1. Moreover, by a variation of [3, Theorem 4.3.4] one can show that (2.10) possesses the BIC property.

3. Stability notions.

3.1. Stability notions of single control systems. The main concept in this paper is:

Definition 3.1. *A system $\Sigma = (X,\mathcal{U},\phi)$ is called (uniformly) input-to-state stable (ISS), if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for all $(t,x,u) \in D_\phi$ it holds that*

$$\|\phi(t,x,u)\|_X \leq \beta(\|x\|_X,t) + \gamma(\|u\|_\mathcal{U}). \tag{3.1}$$

Another important definition is

Definition 3.2. *A system $\Sigma = (X,\mathcal{U},\phi)$ is called uniformly globally asymptotically stable (0-UGAS) for a zero input, if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for all $(t,x,0) \in D_\phi$ it holds that*

$$\|\phi(t,x,0)\|_X \leq \beta(\|x\|_X,t). \tag{3.2}$$

Clearly, ISS implies 0-UGAS. Another important property implied by ISS is

Definition 3.3. *A system $\Sigma = (X,\mathcal{U},\phi)$ is called uniformly globally stable (UGS), if there exist $\sigma \in \mathcal{K}_\infty$, $\gamma \in \mathcal{K}_\infty \cup \{0\}$ such that for all $(t,x,u) \in D_\phi$ the following holds:*

$$\|\phi(t,x,u)\|_X \leq \sigma(\|x\|_X) + \gamma(\|u\|_\mathcal{U}). \tag{3.3}$$

A local counterpart of the UGS property is

Definition 3.4. *A system $\Sigma = (X,\mathcal{U},\phi)$ is called uniformly locally stable (ULS), if there exist $\sigma \in \mathcal{K}_\infty$, $\gamma \in \mathcal{K}_\infty \cup \{0\}$ and $r > 0$ such that for all $x \in B_r$, $u \in \overline{B_{r,\mathcal{U}}}$ and all t so that $(t,x,u) \in D_\phi$, the estimate* (3.3) *holds.*

Lemma 3.5. *Let $\Sigma = (X,\mathcal{U},\phi)$ be an UGS control system. If Σ has the BIC property, then Σ is forward complete.*

Proof. Pick any $x \in X$ and $u \in \mathcal{U}$. Then there is a maximal existence time t^* so that $(t,x,u) \in D_\phi$ for all $t \in [0,t^*)$. Assume that $t^* < \infty$. As Σ is UGS, $\limsup_{t\to t^*} \|\phi(t,x,u)\|_X < \infty$, and we obtain a contradiction to the BIC property of Σ. Hence $t^* = +\infty$ and Σ is forward complete. □

For forward complete systems we introduce the following asymptotic properties

Definition 3.6. *A forward complete system $\Sigma = (X,\mathcal{U},\phi)$ has the*

(i) asymptotic gain (AG) property, if there is a $\gamma \in \mathcal{K}_\infty \cup \{0\}$ such that for all $\varepsilon > 0$, for all $x \in X$ and for all $u \in \mathcal{U}$ there exists a $\tau = \tau(\varepsilon,x,u) < \infty$ such that

$$t \geq \tau \quad \Rightarrow \quad \|\phi(t,x,u)\|_X \leq \varepsilon + \gamma(\|u\|_\mathcal{U}). \tag{3.4}$$

(ii) strong asymptotic gain (sAG) property, if there is a $\gamma \in \mathcal{K}_\infty \cup \{0\}$ such that for all $x \in X$ and for all $\varepsilon > 0$ there exists a $\tau = \tau(\varepsilon,x) < \infty$ such that for all $u \in \mathcal{U}$

$$t \geq \tau \quad \Rightarrow \quad \|\phi(t,x,u)\|_X \leq \varepsilon + \gamma(\|u\|_\mathcal{U}). \tag{3.5}$$

(iii) bounded input uniform asymptotic gain (bUAG) property, if there exists a $\gamma \in \mathcal{K}_\infty \cup \{0\}$ such that for all $\varepsilon, r > 0$ there is a $\tau = \tau(\varepsilon,r) < \infty$ such that for all $u \in \mathcal{U}$: $\|u\|_\mathcal{U} \leq r$ and all $x \in B_r$

$$t \geq \tau \quad \Rightarrow \quad \|\phi(t,x,u)\|_X \leq \varepsilon + \gamma(\|u\|_\mathcal{U}). \tag{3.6}$$

(iii) uniform asymptotic gain (UAG) property, if there exists a $\gamma \in \mathcal{K}_\infty \cup \{0\}$ *such that for all* $\varepsilon, r > 0$ *there is a* $\tau = \tau(\varepsilon, r) < \infty$ *such that for all* $u \in \mathcal{U}$ *and all* $x \in B_r$ *the implication* (3.6) *holds.*

All types of asymptotic gain properties imply that all trajectories converge to the ball of radius $\gamma(\|u\|_\mathcal{U})$ around the origin as $t \to \infty$. The difference between AG, sAG, bUAG and UAG is in the kind of dependence of τ on the states and inputs. In UAG systems this time depends (besides ε) only on the norm of the state, in sAG systems, it depends on the state x (and may vary for different states with the same norm), but it does not depend on u. In AG systems τ depends both on x and on u.

The following lemma shows how bUAG property can be 'upgraded' to the UAG and ISS properties.

LEMMA 3.7. *Let* $\Sigma = (X, \mathcal{U}, \phi)$ *be a control system with a BIC property. If* Σ *is UGS and bUAG, then* Σ *is forward compete, UAG and ISS.*

Proof. As Σ satisfies BIC property and is UGS, then it is forward complete (in particular, the property bUAG assumed for Σ makes sense).

Pick arbitrary $\varepsilon > 0$, $r > 0$ and let τ and γ be as in the formulation of the bUAG property. Let $x \in B_r$ and let $u \in \mathcal{U}$ arbitrary. If $\|u\|_\mathcal{U} \le r$, then (3.6) is the desired estimate.

Let $\|u\|_\mathcal{U} > r$. Hence it holds that $\|u\|_\mathcal{U} > \|x\|_X$. Due to uniform global stability of Σ, it holds for all t, x, u that

$$\|\phi(t, x, u)\|_X \le \sigma(\|x\|_X) + \gamma(\|u\|_\mathcal{U}),$$

where we assume that γ is same as in the definition of a bUAG property (otherwise pick the maximum of both). For $\|u\|_\mathcal{U} > \|x\|_X$ we obtain that

$$\|\phi(t, x, u)\|_X \le \sigma(\|u\|_\mathcal{U}) + \gamma(\|u\|_\mathcal{U}),$$

and thus for all $x \in X$, $u \in \mathcal{U}$ it holds that

$$t \ge \tau \quad \Rightarrow \quad \|\phi(t, x, u)\|_X \le \varepsilon + \gamma(\|u\|_\mathcal{U}) + \sigma(\|u\|_\mathcal{U}),$$

which shows UAG property with the asymptotic gain $\gamma + \sigma$.

As Σ is forward complete, UAG and UGS, the ISS property of Σ follows by [26, Theorem 5]. □

4. Coupled systems and gain operators. In this subsection we consider n systems $\Sigma_i := (X_i, PC(\mathbb{R}_+, X_{\neq i}) \times \mathcal{U}, \bar{\phi}_i)$, $i = 1, \dots, n$, where all X_i, $i = 1, \dots, n$ and $\mathcal{U}$ are normed linear spaces. Furthermore, we assume that all Σ_i, $i = 1, \dots, n$ are forward complete.

Stability properties introduced in Section 3.1 are defined in terms of the norms of the whole input, and this is not suitable for consideration of coupled systems, as we are interested not only in the collective influence of all inputs over a subsystem, but in the influence of particular subsystems over a given subsystem.

Therefore we reformulate the ISS property for a subsystem in the following form:

LEMMA 4.1. *A forward complete system* Σ_i is ISS (in summation formulation) *if there exist* $\gamma_{ij}, \gamma_i \in \mathcal{K} \cup \{0\}$, $j = 1, \dots, n$ *and* $\beta_i \in \mathcal{KL}$, *such that for all initial values* $x_i \in X_i$, *all internal inputs* $w_{\neq i} := (w_1, \dots, w_{i-1}, w_{i+1}, \dots, w_n) \in PC(\mathbb{R}_+, X_{\neq i})$, *all external inputs* $u \in \mathcal{U}$ *and all* $t \in \mathbb{R}_+$ *the following estimate holds:*

$$\|\bar{\phi}_i(t, x_i, (w_{\neq i}, u))\|_{X_i} \le \beta_i\left(\|x_i\|_{X_i}, t\right) + \sum_{j \neq i} \gamma_{ij}\left(\|w_j\|_{[0,t]}\right) + \gamma_i\left(\|u\|_\mathcal{U}\right). \tag{4.1}$$

Proof. As Σ_i is ISS, there is a $\gamma \in \mathcal{K}_\infty$ so that for all $t, x_i, w_{\neq i}, u$ it holds that

$$\begin{aligned}\|\bar{\phi}_i(t, x_i, (w_{\neq i}, u))\|_{X_i} &\le \beta_i\left(\|x_i\|_{X_i}, t\right) + \gamma(\|(w_{\neq i}, u)\|_{C(\mathbb{R}_+, X_{\neq i})\times\mathcal{U}})\\ &= \beta_i\big(\|x_i\|_{X_i}, t\big) + \gamma\Big(\Big(\sum_{j\neq i}\|w_j\|^2_{PC(\mathbb{R}_+,X_j)} + \|u\|^2_{\mathcal{U}}\Big)^{1/2}\Big)\\ &\le \beta_i\big(\|x_i\|_{X_i}, t\big) + \gamma\Big(\sum_{j\neq i}\|w_j\|_{PC(\mathbb{R}_+,X_j)} + \|u\|_{\mathcal{U}}\Big)\\ &\le \beta_i\big(\|x_i\|_{X_i}, t\big) + \sum_{j\neq i}\gamma\big(n\|w_j\|_{PC(\mathbb{R}_+,X_j)}\big) + \gamma(n\|u\|_{\mathcal{U}}),\end{aligned}$$

where in the last estimate we have exploited the estimate $\gamma(s_1 + \ldots + s_n) \le \gamma(ns_1) + \ldots + \gamma(ns_n)$, which holds for all $\gamma \in \mathcal{K}$ and all $s_1, \ldots, s_n \ge 0$.

Defining $\gamma_{ij}(r) := \gamma(nr)$ and $\gamma_i(r) := \gamma(nr)$, we obtain due to causality of Σ_i that

$$\|\bar{\phi}_i(t, x_i, (w_{\neq i}, u))\|_{X_i} \le \beta_i\big(\|x_i\|_{X_i}, t\big) + \sum_{j\neq i}\gamma_{ij}\big(\|w_j\|_{[0,t]}\big) + \gamma_i(\|u\|_{\mathcal{U}}).$$

Conversely, let the property in the statement of the lemma holds. Define $\gamma(r) := \gamma_i(r) + \sum_{j\neq i}\gamma_{ij}(r)$, $r \in \mathbb{R}_+$. Then we have for all $t, x_i, w_{\neq i}, u$ that

$$\begin{aligned}\|\bar{\phi}_i(t, x_i, (w_{\neq i}, u))\|_{X_i} &\le \beta_i\left(\|x_i\|_{X_i}, t\right) + \sum_{j\neq i}\gamma_{ij}\left(\|w_j\|_{[0,t]}\right) + \gamma_i\left(\|u\|_{\mathcal{U}}\right)\\ &\le \beta_i\left(\|x_i\|_{X_i}, t\right) + \sum_{j\neq i}\gamma\left(\|w_j\|_{[0,t]}\right) + \gamma\left(\|u\|_{\mathcal{U}}\right)\\ &\le \beta_i\left(\|x_i\|_{X_i}, t\right) + \gamma\left(\max_{j\neq i}\{\|w_j\|_{[0,t]}, \|u\|_{\mathcal{U}}\}\right)\\ &\le \beta_i\left(\|x_i\|_{X_i}, t\right) + \gamma(\|(w_{\neq i}, u)\|_{C(\mathbb{R}_+, X_{\neq i})\times\mathcal{U}}).\end{aligned}$$

This shows the claim. □

The functions γ_{ij} and γ_i in the statement of Lemma 4.1 are called *(nonlinear) gains*. For notational simplicity we allow the case $\gamma_{ij} \equiv 0$ and require $\gamma_{ii} \equiv 0$ for all i.

Analogously, one can restate the definitions of UGS, AG and bUAG properties, which we leave without the proof:

LEMMA 4.2. *Σ_i is UGS (in summation formulation) if and only if there exist γ_{ij}, $\gamma_i \in \mathcal{K}\cup\{0\}$ and $\sigma_i \in \mathcal{KL}$, such that for all initial values $x_i \in X_i$, all internal inputs $w_{\neq i} := (w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_n) \in PC(\mathbb{R}_+, X_{\neq i})$, all external inputs $u \in \mathcal{U}$ and all $t \in \mathbb{R}_+$ the following inequality holds*

$$\|\bar{\phi}_i(t, x_i, (w_{\neq i}, u))\|_{X_i} \le \sigma_i\left(\|x_i\|_{X_i}\right) + \sum_{j\neq i}\gamma_{ij}\big(\|w_j\|_{[0,t]}\big) + \gamma_i\left(\|u\|_{\mathcal{U}}\right). \tag{4.2}$$

LEMMA 4.3. *Σ_i is AG if and only if there exist γ_{ij}, $\gamma_i \in \mathcal{K}\cup\{0\}$ and $\beta_i \in \mathcal{KL}$, such that for all $x_i \in X_i$,$u \in \mathcal{U}$, $w_{\neq i} := (w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_n) \in PC(\mathbb{R}_+, X_{\neq i})$, $\varepsilon > 0$ there is a time $\tau_i := \tau_i(x_i, u, w_{\neq i}, \varepsilon) < \infty$ so that it holds that*

$$t \ge \tau_i \quad\Rightarrow\quad \|\bar{\phi}_i(t, x_i, (w_{\neq i}, u))\|_{X_i} \le \varepsilon + \sum_{j\neq i}\gamma_{ij}\left(\|w_j\|_\infty\right) + \gamma_i(\|u\|_{\mathcal{U}}). \tag{4.3}$$

LEMMA 4.4. *Σ_i is bUAG if and only if there exist γ_{ij}, $\gamma_i \in \mathcal{K} \cup \{0\}$ and $\beta_i \in \mathcal{KL}$, such that for all $\varepsilon > 0$, for all $r > 0$ there is $\tau_i = \tau_i(\varepsilon, r) < \infty$ such that for all $u \in \mathcal{U}$: $\|u\|_{\mathcal{U}} \le r$ and all $x_i \in B_r(0, X_i)$, for all $w_{\neq i} := (w_1, \dots, w_{i-1}, w_{i+1}, \dots, w_n) \in PC(\mathbb{R}_+, X_{\neq i})$: $\|w_j\|_\infty \le r$ the implication (4.3) holds.*

In above definitions w_j, $j = 1, \dots, n$ are general inputs, which are not necessarily related to the states of other subsystems (i.e. we have considered all Σ_i as disconnected systems). Now assume that the interconnection of forward complete systems $\Sigma_1, \dots, \Sigma_n$ as introduced in Section 2.2, which we call $\Sigma := (X, \mathcal{U}, \phi)$, is a well-defined control system with a BIC property.

Pick arbitrary $x \in X$ and $u \in \mathcal{U}$ and define for $i = 1, \dots, n$ the following quantities:

$$\phi_i := \phi_i(\cdot, x, u), \quad \phi_{\neq i} = (\phi_1, \dots, \phi_{i-1}, \phi_{i+1}, \dots, \phi_n). \tag{4.4}$$

We would like to rewrite the definitions of ISS and UGS, specialized for the inputs $w_j := \phi_j$, in a shorter vectorized form, using the shorthand notation from [6], introduced next.

For vector functions $w = (w_1, \dots, w_n)^T : \mathbb{R}_+ \to X_1 \times \ldots \times X_n$ such that $w_i \in C(\mathbb{R}_+, X_i), i = 1, \dots, n$ and times $0 \le t_1 \le t_2$ we define

$$\| w_{[t_1,t_2]} \| := \begin{pmatrix} \|w_{1,[t_1,t_2]}\|_\infty \\ \vdots \\ \|w_{n,[t_1,t_2]}\|_\infty \end{pmatrix} \in \mathbb{R}^n_+.$$

Furthermore, we introduce for all t, u and all $x = (x_1, \dots, x_n) \in X$ the following notation:

$$\left| \bar{\phi}(t, x, u) \right| := \begin{pmatrix} \|\bar{\phi}_1(t, x_1, (\phi_{\neq 1}, u))\|_{X_1} \\ \vdots \\ \|\bar{\phi}_n(t, x_n, (\phi_{\neq n}, u))\|_{X_n} \end{pmatrix} \in \mathbb{R}^n_+, \quad \gamma(\|u\|_{\mathcal{U}}) := \begin{pmatrix} \gamma_1(\|u\|_{\mathcal{U}}) \\ \vdots \\ \gamma_n(\|u\|_{\mathcal{U}}) \end{pmatrix} \in \mathbb{R}^n_+ \tag{4.5}$$

$$\text{and} \quad \beta(s, t) := \begin{pmatrix} \beta_1(s_1, t) \\ \vdots \\ \beta_n(s_n, t) \end{pmatrix}. \tag{4.6}$$

We are going to collect all the internal gains in the matrix $\Gamma := (\gamma_{ij})_{i,j=1,\dots,n}$, which we call the *gain matrix*. If the gains are taken from the ISS restatement (4.1), then we call the corresponding gain matrix Γ^{ISS}. Analogously, the gain matrices Γ^{UGS}, Γ^{AG}, Γ^{UAG} are defined.

Now for a given gain matrix Γ define the operator $\Gamma_\oplus : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ by

$$\Gamma_\oplus s := \left(\sum_{j=1}^n \gamma_{1j}(s_j), \dots, \sum_{j=1}^n \gamma_{nj}(s_j) \right)^T, \quad s = (s_1, \dots, s_n)^T \in \mathbb{R}^n_+. \tag{4.7}$$

Again, in order to emphasize that the gains are from the ISS restatement (4.1), the corresponding gain operators of sum and max type will be denoted by $\Gamma^{ISS}_\oplus$ respectively.

Note that by the properties of γ_{ij} for $s_1, s_2 \in \mathbb{R}^n_+$ we have the implication

$$s_1 \geq s_2 \quad \Rightarrow \quad \Gamma_\oplus(s_1) \geq \Gamma_\oplus(s_2), \tag{4.8}$$

so that $\Gamma_\oplus$ defines a monotone (w.r.t. the order $\geq$ in $\mathbb{R}^n$) map.

The ISS conditions (4.1) with this notation imply that for $t \geq 0$ it holds that

$$\left|\!\left| \bar{\phi}(t,x,u) \right|\!\right| \leq \beta(\left|\!\left| \bar{\phi}(0,x,u) \right|\!\right|, t) + \Gamma_\oplus^{ISS}(\| \phi_{[0,t]} \|) + \gamma(\|u\|_{\mathcal{U}}). \tag{4.9}$$

Analogously, the UGS conditions (4.2) imply that for $t \geq 0$ it holds that

$$\left|\!\left| \bar{\phi}(t,x,u) \right|\!\right| \leq \sigma(\left|\!\left| \bar{\phi}(0,x,u) \right|\!\right|) + \Gamma_\oplus^{UGS}(\| \phi_{[0,t]} \|) + \gamma(\|u\|_{\mathcal{U}}). \tag{4.10}$$

In order to guarantee stability of the interconnection Σ, the properties of the operators $\Gamma_\oplus^{UGS}$ and $\Gamma_\oplus^{ISS}$ will be crucial.

We introduce the following notation. For $\alpha_i \in \mathcal{K}_\infty, i = 1, \dots, n$ define $D : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ by

$$D(s_1, \dots, s_n)^T := \begin{pmatrix} (\mathrm{Id} + \alpha_1)(s_1) \\ \vdots \\ (\mathrm{Id} + \alpha_n)(s_n) \end{pmatrix}. \tag{4.11}$$

A fundamental role will be played by the following operator conditions:

Definition 4.5. *We say that a nonlinear operator $A : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ satisfies*

- *the* small-gain condition*, if*
$$A(s) \not\geq s, \qquad \forall s \in \mathbb{R}^n_+ \setminus \{0\}. \tag{4.12}$$
- *the* strong small-gain condition*, if there exists a mapping D as in* (4.11)*, such that*
$$(A \circ D)(s) \not\geq s, \qquad \forall s \in \mathbb{R}^n_+ \setminus \{0\}. \tag{4.13}$$

For the proof of small-gain theorems we use a technical result, see [6, Lemma 13, p. 102]. We state it in a somewhat more general form than it was done in [6], although the proof remains basically the same.

Lemma 4.6. *Let $A : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be a monotone w.r.t. the order in $\mathbb{R}^n$ and continuous operator with $A(0) = 0$, satisfying the strong small gain condition* (4.13)*. Then there exists a $\xi \in \mathcal{K}_\infty$ such that for all $w, v \in \mathbb{R}^n_+$ inequality*

$$(Id - A)(w) \leq v \tag{4.14}$$

implies $|w| \leq \xi(|v|)$.

Proof. In the proof of [6, Lemma 13, p. 102] it was shown that for all $w, v \in \mathbb{R}^n_+$ satisfying (4.14), it holds that

$$w \leq R(|v|_{\max}, \dots, |v|_{\max})^T, \tag{4.15}$$

where $R := \Big(D \circ (D - \mathrm{Id})^{-1} \circ (\mathrm{Id} + A)\Big)^n$. As $A(0) = 0$, we have $R(0) = 0$ as well. Furthermore, R is continuous and monotone as a composition of continuous and monotone maps. Moreover, as $D \circ (D - \mathrm{Id})^{-1} \in {\mathcal{K}_\infty}^n$, it holds that if $a_i > b_i$ for all $i = 1, \dots, n$, then also $R(a)_i > R(b)_i$ for all $i = 1, \dots, n$.

Define $\xi(r) := |R(r, \dots, r)|$. Then $\xi(0) = 0$, ξ is continuous, strictly increasing and unbounded (since $D \circ (D - \mathrm{Id})^{-1} \in {\mathcal{K}_\infty}^n$). With this ξ we obtain $|w| \leq \xi(|v|)$. □

5. Small-gain theorems for control systems. In this section we show small-gain theorems for UGS, ISS, AG and weak ISS properties.

5.1. UGS small-gain theorem. We start with a small-gain theorem which guarantees that a coupling of UGS systems is a UGS system again provided the strong small-gain condition (4.13) holds. This will in particular show that the coupled system is forward complete. This result and its proof are an infinite-dimensional version of [6, Theorem 8].

THEOREM 5.1 (UGS Small-gain theorem). *Let* $\Sigma_i := (X_i, PC(\mathbb{R}_+, X_{\neq i}) \times \mathcal{U}, \bar{\phi}_i)$, $i = 1, \dots, n$ *be control systems, where all* X_i, $i = 1, \dots, n$ *and* $\mathcal{U}$ *are normed linear spaces. Assume that* Σ_i, $i = 1, \dots, n$ *are forward complete systems, satisfying the UGS estimates as in Lemma 4.2, and that the interconnection* $\Sigma = (X, \mathcal{U}, \phi)$ *is well-defined and possesses the BIC property.*

If $\Gamma_\oplus^{UGS}$ *satisfies the strong small gain condition* (4.13)*, then* Σ *is forward complete and UGS.*

Proof. Pick any $u \in \mathcal{U}$ and any initial condition $x \in X$. As we assume that the interconnection $\Sigma = (X, \mathcal{U}, \phi)$ is well-defined, by definition of a control system, the solution of Σ exists on a certain interval $[0, t]$, where $t \in (0, +\infty]$. Define ϕ_i and $\phi_{\neq i}$ as in (4.4).

According to the definition of the interconnection, for all $i = 1, \dots, n$ it holds that $\phi_i(s, x, u) = \bar{\phi}_i(s, x_i, (\phi_{\neq i}, u))$, $s \in [0, t]$ and hence we have

$$\begin{pmatrix} \sup_{s\in[0,t]} \|\bar{\phi}_1(s, x_1, (\phi_{\neq 1}, u))\|_{X_1} \\ \vdots \\ \sup_{s\in[0,t]} \|\bar{\phi}_n(s, x_n, (\phi_{\neq n}, u))\|_{X_n} \end{pmatrix} = \begin{pmatrix} \sup_{s\in[0,t]} \|\phi_1(s, x, u)\|_{X_1} \\ \vdots \\ \sup_{s\in[0,t]} \|\phi_n(s, x, u)\|_{X_n} \end{pmatrix} = \begin{pmatrix} \|\phi_{1,[0,t]}\|_\infty \\ \vdots \\ \|\phi_{n,[0,t]}\|_\infty \end{pmatrix} =: \Vert \phi_{[0,t]} \Vert . \tag{5.1}$$

By assumptions, on $[0, t]$ the estimate (4.10) is valid. Taking in this estimate the supremum over $[0, t]$, and making use of (5.1), we see that

$$\Vert \phi_{[0,t]} \Vert \leq \sigma(\Vert \bar{\phi}(0, x, u) \Vert) + \Gamma_\oplus^{UGS}(\Vert \phi_{[0,t]} \Vert) + \gamma(\|u\|_{\mathcal{U}}), \tag{5.2}$$

and thus

$$(I - \Gamma_\oplus^{UGS}) \Vert \phi_{[0,t]} \Vert \leq \sigma(\Vert \bar{\phi}(0, x, u) \Vert) + \gamma(\|u\|_{\mathcal{U}}), \tag{5.3}$$

As $\Gamma_\oplus^{UGS}$ is a monotone operator satisfying the strong small-gain condition, by Lemma 4.6 there is a $\xi \in \mathcal{K}_\infty$ so that

$$\begin{aligned} |\Vert \phi_{[0,\tau]} \Vert| &\leq \xi\big(\sigma(|\Vert \bar{\phi}(0, x, u) \Vert|) + \gamma(\|u\|_{\mathcal{U}})\big) \\ &\leq \xi\big(2\sigma(|\Vert \bar{\phi}(0, x, u) \Vert|)\big) + \xi\big(2\gamma(\|u\|_{\mathcal{U}})\big). \end{aligned} \tag{5.4}$$

Finally, $|\Vert \bar{\phi}(0, x, u) \Vert| = |(\|x_1\|_{X_1}, \dots, \|x_n\|_{X_n})| = \|x\|_X$ and

$$\|\phi(t, x, u)\|_X = \sqrt{\sum_{i=1}^n \|\phi_i(t, x, u)\|_{X_i}^2} \leq \sqrt{\sum_{i=1}^n \|\phi_{i,[0,t]}\|_\infty^2} = |\Vert \phi_{[0,\tau]} \Vert|. \tag{5.5}$$

Combining this estimate with (5.4), we see that a UGS estimate

$$\|\phi(t,x,u)\|_X \le \xi\big(2\sigma(\|x\|_X)\big) + \xi\big(2\gamma(\|u\|_{\mathcal{U}})\big) \tag{5.6}$$

is valid on the certain maximal interval of existence $[0,t^*)$ of $\phi(\cdot,x,u)$. As we assume that Σ possesses the BIC property, then Σ is forward complete, see Lemma 3.5. □

5.2. ISS small-gain theorem. We start with a technical lemma:

LEMMA 5.2. *Let $g:\mathbb{R}_+\to\mathbb{R}^p_+$, $p\in\mathbb{N}$ be a globally bounded function and let $f:\mathbb{R}_+\to\mathbb{R}_+$ be an unbounded monotone function. Then*

$$\lim_{t\to\infty}\sup_{s\ge f(t)} g(s) = \lim_{t\to\infty}\sup_{s\ge t} g(s). \tag{5.7}$$

Proof. Define $a := \lim_{t\to\infty}\sup_{s\ge f(t)} g(s)$ and $b := \lim_{t\to\infty}\sup_{s\ge t} g(s)$. As g is globally bounded, both a and b are well-defined and finite. By definition it follows that for all $\varepsilon>0$ there is a time $T>0$ so that

$$t\ge T \quad\Rightarrow\quad \sup_{s\ge f(t)} g(s) < a+\varepsilon.$$

As f is monotone, this is equivalent to the fact that

$$t\ge f(T) \quad\Rightarrow\quad \sup_{s\ge t} g(s) < a+\varepsilon.$$

As ε can be chosen arbitrarily small, this shows that $b\le a$.

Conversely, we have that for all $\varepsilon>0$ there is a time $T>0$ so that

$$t\ge T \quad\Rightarrow\quad \sup_{s\ge t} g(s) < b+\varepsilon. \tag{5.8}$$

As f is unbounded, there is a time $\tau=\tau(T)$ so that $f(\tau)>T$.

Thus, (5.8) shows that $\sup_{s\ge f(\tau)} g(s) < b+\varepsilon$ and as f is monotone it follows that

$$t\ge \tau \quad\Rightarrow\quad \sup_{s\ge f(t)} g(s) < b+\varepsilon.$$

This implies that $a\le b$. Overall, $a=b$. □

Now we are able to prove the main result of this paper.

THEOREM 5.3 (ISS Small-gain theorem). *Let $\Sigma_i := (X_i, PC(\mathbb{R}_+, X_{\neq i})\times\mathcal{U}, \bar\phi_i)$, $i=1,\dots,n$ be control systems, where all X_i, $i=1,\dots,n$ and $\mathcal{U}$ are normed linear spaces. Assume that Σ_i, $i=1,\dots,n$ are forward complete systems, satisfying the ISS estimates as in Lemma 4.1, and that the interconnection $\Sigma=(X,\mathcal{U},\phi)$ is well-defined and possesses the BIC property.*

If $\Gamma^{ISS}_\oplus$ satisfies the strong small gain condition (4.13)*, then Σ is ISS.*

Proof. We show UGS and bUAG properties of the interconnection $\Sigma=(X,\mathcal{U},\phi)$, and infer ISS of Σ by Lemma 3.7.

UGS. From the assumptions of the theorem it follows that all Σ_i are UGS with the gain matrix Γ^{ISS}. As $\Gamma^{ISS}_\oplus$ satisfies the strong small gain condition, Theorem 5.1 shows that the coupled system Σ is forward complete and UGS.

bUAG. We use the notation (4.4) for ϕ_i and $\phi_{\neq i}$. As Σ is a well-defined and forward complete interconnection, we have that $\phi_i(t,x,u) = \bar\phi_i(t,x_i,(\phi_{\neq i},u))$ for all $i=1,\dots,n$ and for all $t\ge 0$.

Pick any $r>0$, any $u\in\mathcal{U}$: $\|u\|_{\mathcal{U}}\leq r$ and any $x\in B_r$. As Σ is UGS, the estimate (3.3) is valid for some $\sigma_{UGS},\gamma_{UGS}\in\mathcal{K}_\infty$ and it holds that

$$\|\phi(t,x,u)\|_X \leq \mu(r) := \sigma_{UGS}(r)+\gamma_{UGS}(r),\quad t\geq 0. \tag{5.9}$$

On the other hand, for all $i=1,\dots,n$ in view of the cocycle property it holds for any $t,\tau\geq 0$ that

$$\begin{aligned}\phi_i(t+\tau,x,u) &= \bar{\phi}_i(t+\tau,x_i,(\phi_{\neq i},u))\\ &= \bar{\phi}_i\Big(\tau,\bar{\phi}_i\big(t,x_i,(\phi_{\neq i},u)\big),\big(\phi_{\neq i}(\cdot+t),u(\cdot+t)\big)\Big).\end{aligned} \tag{5.10}$$

Note that due to the axiom of shift invariance, it holds that $u(t+\cdot)\in\mathcal{U}$.

In view of (5.9) we have for all $i=1,\dots,n$ that

$$\|\bar{\phi}_i(t,x_i,\phi_{\neq i},u)\|_{X_i} = \|\phi_i(t,x,u)\|_{X_i} \leq \|\phi(t,x,u)\|_X \leq \mu(r) \tag{5.11}$$

and

$$\|\phi_{\neq i}\|_\infty \leq \mu(r). \tag{5.12}$$

Furthermore, as $\sigma_{UGS}(r)\geq r$ for all $r\in\mathbb{R}_+$ (this follows from (3.3) by setting $u:=0$ and $t:=0$), we have also

$$\|u\|_{\mathcal{U}} \leq \mu(r). \tag{5.13}$$

By the bUAG property of Σ_i (which is implied by ISS of Σ_i), there is a time $\tau_i=\tau_i(\varepsilon,\mu(r))$ so that

$$\begin{aligned}&\|x\|_X\leq r\ \wedge\ \|u\|_{\mathcal{U}}\leq r\ \wedge\ \tau\geq\tau_i\\ &\qquad\Rightarrow\quad \|\phi_i(t+\tau,x,u)\|_{X_i} \leq \varepsilon+\sum_{j\neq i}\gamma_{ij}\left(\left\|\phi_{j,[t,+\infty)}\right\|_\infty\right)+\gamma_i(\|u_{[t,+\infty)}\|_{\mathcal{U}})\\ &\qquad\qquad\qquad\qquad\qquad\quad \leq \varepsilon+\sum_{j\neq i}\gamma_{ij}\left(\left\|\phi_{j,[t,+\infty)}\right\|_\infty\right)+\gamma_i(\|u\|_{\mathcal{U}}).\end{aligned} \tag{5.14}$$

Note that dependence of τ_i on r and ε only (and not on x, u and t) follows from the estimates (5.11), (5.12) and (5.13).

Define the convergence time which is uniform for all subsystems (it is finite, as we have finitely many subsystems):

$$\tau^*(\varepsilon,r) := \max_{i=1,\dots,n}\tau_i(\varepsilon,\mu(r)). \tag{5.15}$$

Pick any $k\in\mathbb{N}$. Taking supremum of (5.14) over $x\in B_r$ and over all $u\in\mathcal{U}$: $\|u\|_{\mathcal{U}}\in[2^{-k}r,2^{1-k}r]$, we obtain for all $i=1,\dots,n$ and all $\tau\geq\tau^*$ that

$$\begin{aligned}&\sup_{\|u\|_{\mathcal{U}}\in[2^{-k}r,2^{1-k}r]}\ \sup_{x\in B_r}\|\phi_i(t+\tau,x,u)\|_{X_i}\\ &\qquad\leq \varepsilon+\sum_{j\neq i}\gamma_{ij}\left(\sup_{\|u\|_{\mathcal{U}}\in[2^{-k}r,2^{1-k}r]}\ \sup_{x\in B_r}\left\|\phi_{j,[t,+\infty)}\right\|_\infty\right)+\gamma_i(2^{1-k}r)\end{aligned} \tag{5.16}$$

and thus

$$\sup_{s\geq t+\tau^*}\ \sup_{\|u\|_{\mathcal{U}}\in[2^{-k}r,2^{1-k}r]}\ \sup_{x\in B_r}\|\phi_i(s,x,u)\|_{X_i}$$
$$\leq \varepsilon+\sum_{j\neq i}\gamma_{ij}\left(\sup_{\|u\|_{\mathcal{U}}\in[2^{-k}r,2^{1-k}r]}\ \sup_{x\in B_r}\ \sup_{s\geq t}\|\phi_j(s,x,u)\|_{X_j}\right)+\gamma_i(2^{1-k}r)$$
$$= \varepsilon+\sum_{j\neq i}\gamma_{ij}\left(\sup_{s\geq t}\ \sup_{\|u\|_{\mathcal{U}}\in[2^{-k}r,2^{1-k}r]}\ \sup_{x\in B_r}\|\phi_j(s,x,u)\|_{X_j}\right)+\gamma_i(2^{1-k}r). \tag{5.17}$$

Define

$$y_i(r,k):=\lim_{t\to+\infty}\sup_{s\geq t}\ \sup_{\|u\|_{\mathcal{U}}\in[2^{-k}r,2^{1-k}r]}\ \sup_{x\in B_r}\|\phi_i(s,x,u)\|_{X_i}$$
$$=\limsup_{t\to+\infty}\ \sup_{\|u\|_{\mathcal{U}}\in[2^{-k}r,2^{1-k}r]}\ \sup_{x\in B_r}\|\phi_i(t,x,u)\|_{X_i}.$$

By Lemma 5.2 it holds that

$$y_i(r,k)=\lim_{t\to+\infty}\sup_{s\geq t+\tau^*}\ \sup_{\|u\|_{\mathcal{U}}\in[2^{-k}r,2^{1-k}r]}\ \sup_{x\in B_r}\|\phi_i(s,x,u)\|_{X_i}$$

and thus taking a limit $t\to\infty$ in (5.17), we have that

$$y_i(r,k)\leq \varepsilon+\sum_{j\neq i}\gamma_{ij}\left(y_j(r,k)\right)+\gamma_i(2^{1-k}r). \tag{5.18}$$

As (5.18) is valid for arbitrarily small $\varepsilon>0$, we obtain by computing the limit $\varepsilon\to 0$ that

$$y_i(r,k)\leq \sum_{j\neq i}\gamma_{ij}\left(y_j(r,k)\right)+\gamma_i(2^{1-k}r),\quad i=1,\dots,n. \tag{5.19}$$

With the notation (4.5) and by defining $y(r,k):=(y_1(r,k),\dots,y_n(r,k))^T\in\mathbb{R}^n_+$ we can rewrite (5.19) in a vector form:

$$y(r,k)\leq \Gamma^{ISS}_{\oplus}\left(y(r,k)\right)+\gamma(2^{1-k}r). \tag{5.20}$$

Since $\Gamma^{ISS}_{\oplus}$ is a monotone operator satisfying the strong small-gain condition, we obtain by applying Lemma 4.6 to the inequality

$$(Id-\Gamma^{ISS}_{\oplus})y(r,k)\leq\gamma(2^{1-k}r)$$

that there is $\xi\in\mathcal{K}_\infty$ so that

$$|y(r,k)|\leq\xi(|\gamma(2^{1-k}r)|).$$

Using a technical computation

$$\begin{aligned}\limsup_{t\to+\infty} \sup_{\|u\|_{\mathcal{U}}\in[2^{-k}r,2^{1-k}r]} \sup_{x\in B_r} \|\phi(t,x,u)\|_X &\\ = \limsup_{t\to+\infty} \sup_{\|u\|_{\mathcal{U}}\in[2^{-k}r,2^{1-k}r]} \sup_{x\in B_r} \sqrt{\sum_{i=1}^n \|\phi_i(t,x,u)\|^2_{X_i}} &\\ \le \sqrt{\sum_{i=1}^n \limsup_{t\to+\infty} \sup_{\|u\|_{\mathcal{U}}\in[2^{-k}r,2^{1-k}r]} \sup_{x\in B_r} \|\phi_i(t,x,u)\|^2_{X_i}} &\\ = \sqrt{\sum_{i=1}^n y_i^2(r,k)} = |y(r,k)|,&\end{aligned}$$

we conclude, that for any $r>0$ and any $k\in\mathbb{N}$ it holds that

$$\limsup_{t\to+\infty} \sup_{\|u\|_{\mathcal{U}}\in[2^{-k}r,2^{1-k}r]} \sup_{x\in B_r} \|\phi(s,x,u)\|_X \le \xi(|\gamma(2^{1-k}r)|).$$

which means that for any $\varepsilon>0$, any $r>0$ and any $k\in\mathbb{N}$ there is a time $\tilde\tau=\tilde\tau(\varepsilon,r,k)$ so that

$$\begin{aligned}&\|x\|_X\le r \ \wedge\ \|u\|_{\mathcal{U}}\in[2^{-k}r,2^{1-k}r] \ \wedge\ t\ge\tilde\tau(\varepsilon,r,k)\\ &\qquad\qquad \Rightarrow \quad \|\phi(s,x,u)\|_X \le \varepsilon+\xi(|\gamma(2^{1-k}r)|).\end{aligned} \tag{5.21}$$

Define $k_0=k_0(\varepsilon,r)\in\mathbb{N}$ as the minimal k so that $\xi(|\gamma(2^{1-k}r)|)\le\varepsilon$ (clearly, such k_0 always exists and is finite) and let

$$\hat\tau(\varepsilon,r):=\max\{\tilde\tau(\varepsilon,r,k):\ k=1,\dots,k_0(\varepsilon,r)\}.$$

As k_0 is finite, $\hat\tau(\varepsilon,r)$ is finite as well.

Pick any $u\in\mathcal{U}$ so that $\|u\|_{\mathcal{U}}\le r$. Then there is $k\in\mathbb{N}$ so that $\|u\|_{\mathcal{U}}\in(2^{-k}r,2^{1-k}r]$. If $k\le k_0$ (i.e. if inputs are large enough), then for $t\ge\hat\tau(\varepsilon,r)$ it holds that

$$\|\phi(s,x,u)\|_X \le \varepsilon+\xi(|\gamma(2^{1-k}r)|) \le \varepsilon+\xi(|\gamma(2\|u\|_{\mathcal{U}})|). \tag{5.22}$$

It remains to consider the case when $k>k_0$, i.e. when inputs are small. The estimate (5.21) gives convergence time, which depends on k and it is not clear whether the supremum of $\tilde\tau(\varepsilon,r,k)$ over all $k\ge k_0$ exists. In order to overcome this obstacle and to find the uniform time, we mimic above argument once again, namely: for any $q\in[0,r]$ one can take supremum of (5.14) over $x\in B_r$ and over all $u\in\mathcal{U}$: $\|u\|_{\mathcal{U}}\le q$, to obtain for all $i=1,\dots,n$ and all $\tau\ge\tau^*$ that

$$\begin{aligned}&\sup_{\|u\|_{\mathcal{U}}\le q}\sup_{x\in B_r}\|\phi_i(t+\tau,x,u)\|_{X_i}\\ &\qquad\le \varepsilon+\sum_{j\ne i}\gamma_{ij}\left(\sup_{\|u\|_{\mathcal{U}}\le q}\sup_{x\in B_r}\left\|\phi_{j,[t,+\infty)}\right\|_\infty\right)+\gamma_i(q),\end{aligned} \tag{5.23}$$

where τ^* has been defined in (5.15). Defining

$$z_i(r,q):=\lim_{t\to+\infty}\sup_{s\ge t}\sup_{\|u\|_{\mathcal{U}}\le q}\sup_{x\in B_r}\|\phi_i(s,x,u)\|_{X_i}$$

and doing analogous steps as above we obtain that for any $r > 0$ and any $q \le r$ it holds that

$$\limsup_{t\to+\infty} \sup_{\|u\|_{\mathcal{U}}\le q} \sup_{x\in B_r} \|\phi(s,x,u)\|_X \le \xi(|\gamma(q)|).$$

This means that for any $\varepsilon > 0$, any $r > 0$ and any $k \in \mathbb{N}$ there is a time $\bar\tau = \bar\tau(\varepsilon, r, q)$ so that

$$\|x\|_X \le r \ \wedge\ \|u\|_{\mathcal{U}} \le q \ \wedge\ t \ge \bar\tau(\varepsilon,r,q) \quad\Rightarrow\quad \|\phi(s,x,u)\|_X \le \varepsilon + \xi(|\gamma(q)|). \tag{5.24}$$

In particular, for $q_0 := 2^{1-k_0(\varepsilon,r)}r$ we have that

$$\begin{aligned}\|x\|_X \le r \ \wedge\ \|u\|_{\mathcal{U}} \le{}& q_0 \ \wedge\ t \ge \bar\tau(\varepsilon,r,q_0)\\ &\Rightarrow\quad \|\phi(s,x,u)\|_X \le \varepsilon + \xi(|\gamma(2^{1-k_0(\varepsilon,r)}r)|) \le \varepsilon+\varepsilon.\end{aligned} \tag{5.25}$$

Define

$$\tau(\varepsilon,r) := \max\{\hat\tau(\varepsilon,r), \bar\tau(\varepsilon,r,q)\}.$$

Combining (5.22) and (5.25), we obtain that

$$\|x\|_X \le r \ \wedge \|u\|_{\mathcal{U}} \le r \ \wedge t \ge \tau(\varepsilon,r) \quad\Rightarrow\quad \|\phi(s,x,u)\|_X \le \varepsilon + \max\{\varepsilon, \xi(|\gamma(2\|u\|_{\mathcal{U}})|)\}.$$

and finally

$$\|x\|_X \le r \ \wedge \|u\|_{\mathcal{U}} \le r \ \wedge t \ge \tau(\varepsilon,r) \quad\Rightarrow\quad \|\phi(s,x,u)\|_X \le 2\varepsilon + \xi(|\gamma(2\|u\|_{\mathcal{U}})|).$$

As $r \mapsto \xi(|\gamma(2r)|)$ is a $\mathcal{K}_\infty$-function, this implication shows that Σ is bUAG.
ISS. Since Σ is UGS $\wedge$ bUAG, Lemma 3.7 implies ISS of Σ. □

REMARK 5.4 (Discussion of the proof of Theorem 5.3). *A key step in the proof of Theorem 5.3 is the shifting of the time horizon, see* (5.14)*, achieved by means of the cocycle property* (5.17)*. It is important that we want to achieve the dependence of the convergence time τ_i on r and ε only, which follows from* (5.11), (5.12) *and* (5.13)*, which are valid in turn since we consider the inputs with a norm uniformly bounded by r. Having an arbitrary u would result that the norm of the state $\phi_i(t, x_i, \phi_{\neq i}, u)$ in* (5.10)*, and hence the time τ_i would depend on the norm of u, which makes it hard to obtain UAG property of the interconnection. Using bUAG property instead helps us to avoid these complications and this is one of the reasons for introducing this property.*

Then, in order to tackle the distinction in the time intervals over which the supremum in the left and right hand sides of (5.17) *is taken, we pass to the limit $t \to \infty$. However, before we can compute such limit we have to make the expressions in both sides of* (5.14) *independent on x, u. Taking supremum over $x \in B_r$ causes no problems, but taking a supremum over $u \in B_{r,\mathcal{U}}$ leads to overly rough estimates, and thus we slice the ball $u \in B_{r,\mathcal{U}}$ into several 'rings', which helps in the end to obtain the desired bUAG property of the whole interconnection.*

REMARK 5.5 (Applications of the ISS small-gain theorem). *Theorem 5.3 is very general and applicable for networks of heterogeneous infinite-dimensional systems, consisting of components belonging to different system classes, with boundary and in-domain couplings, as long as these systems are well-defined and possess the BIC property. In particular, our ISS small-gain theorem is applicable to the interconnections of ISS ODE systems, interconnections of evolution equation in Banach spaces* *and couplings of n time-delay systems. In all these cases the interconnections possess the BIC property, see [15] for details.*

Specialized to couplings of ODE systems, Theorem 5.3 boils down to the classic small-gain theorem derived in [6]. However, already for couplings of n time-delay systems Theorem 5.3 is new. In contrast to [30, 39, 29], we obtain not only UGS ∧ AG (i.e. weak ISS) small-gain results, but also ISS small-gain theorems obtain (uniform) ISS small-gain results. Compared to ISS small-gain theorems obtained in [40], our approach provides an alternative way for verification of ISS of a network of delay systems, which does not require a transformation of retarded systems into delay-free ones and is applicable to the sum formulation of the ISS property.

5.3. AG and weak ISS small-gain theorem. Our next result is the small-gain theorem for the asymptotic gain property.

THEOREM 5.6 (AG Small-gain theorem). *Let $\Sigma_i := (X_i, PC(\mathbb{R}_+, X_{\neq i}) \times \mathcal{U}, \bar{\phi}_i)$, $i = 1, \dots, n$ be control systems, where all X_i, $i = 1, \dots, n$ and $\mathcal{U}$ are normed linear spaces. Assume that Σ_i, $i = 1, \dots, n$ are forward complete systems, satisfying the AG estimates as in Lemma 4.3, and that the interconnection Σ is well-defined and forward complete.*

If $\Gamma_{\oplus}^{AG}$ satisfies the strong small gain condition (4.13)*, then Σ is AG.*

In this case the complexities, described in Remark 5.4 do not appear, and the proof goes along the lines of the proof of Theorem 5.3, with significant simplifications. Hence in the following rather sketchy argument we merely indicate the main differences to the detailed proof of Theorem 5.3. It is also possible to show Theorem 5.6 along the lines of the proof of the corresponding finite-dimensional counterpart [6, Theorem 9].

Proof. [Sketch] Define $\phi_i := \phi_i(\cdot, x, u)$ and $\phi_{\neq i} := (\phi_1, \dots, \phi_{i-1}, \phi_{i+1}, \dots, \phi_n)$, $i = 1, \dots, n$.

Pick any $x \in X$ and any $u \in \mathcal{U}$. The cocycle property (5.10) and AG property of Σ_i imply existence of a time $\tau_i = \tau_i(\varepsilon, x, u, t)$ which we can assume to be an increasing function of t, such that

$$\begin{aligned}\sup_{\tau \geq \tau_i + t} \|\phi_i(\tau, x, u)\|_{X_i} &= \sup_{\tau \geq \tau_i} \|\phi_i(t + \tau, x, u)\|_{X_i} \\ &\leq \varepsilon + \sum_{j \neq i} \gamma_{ij}\left(\left\|\phi_{j,[t,+\infty)}\right\|_\infty\right) + \gamma_i(\|u\|_{\mathcal{U}}).\end{aligned} \tag{5.26}$$

Note that the time τ_i depends on the tuple $(\varepsilon, \phi_i(t, x_i, \phi_{\neq i}, u), \phi_{\neq i}(\cdot + t), u(\cdot + t))$ in (5.10), but all these parameters depend on (ε, x, u, t) only.

Defining

$$\tau^*(\varepsilon, x, u, t) := \max_{i=1,\dots,n} \tau_i(\varepsilon, x, u, t), \quad y_i(x, u) := \lim_{t \to +\infty} \sup_{s \geq t} \|\phi_i(s, x, u)\|_{X_i},$$

and

$$y(r, k) := (y_1(r, k), \dots, y_n(r, k))^T \in \mathbb{R}^n_+,$$

and doing the same steps as in the proof of ISS small-gain theorem we obtain that there is $\xi \in \mathcal{K}_\infty$ so that

$$\limsup_{t \to +\infty} \|\phi(s, x, u)\|_X = |y(r, k)| \leq \xi(|\gamma(\|u\|_{\mathcal{U}})|), \tag{5.27}$$

which is precisely the asymptotic gain property of Σ. □

REMARK 5.7. *A notable difference in the assumptions of the AG small-gain theorem, in compare to UGS and ISS small-gain theorems is that the forward completeness of the interconnection is required. The assumption of forward-completeness cannot be relaxed to merely BIC property, as demonstrated in [6, Section 4.2] already for couplings of ODE systems.*

DEFINITION 5.8. *A control system Σ possessing the BIC property is called weakly ISS, provided Σ is UGS and possesses AG property. It is then forward complete, see Lemma 3.5.*

For ODEs weak ISS is equivalent to ISS, but it is much weaker than ISS even for linear infinite-dimensional systems.

As a combination of UGS and AG small-gain theorems we obtain

THEOREM 5.9 (Weak ISS Small-gain theorem). *Let $\Sigma_i := (X_i, PC(\mathbb{R}_+, X_{\neq i}) \times \mathcal{U}, \bar{\phi}_i)$, $i = 1, \dots, n$ be control systems, where all X_i, $i = 1, \dots, n$ and $\mathcal{U}$ are normed linear spaces. Assume that Σ_i, $i = 1, \dots, n$ are forward complete weakly ISS systems with the same gain matrices Γ^{wISS} for both AG and UGS property (formulated as in Lemma 4.2, Lemma 4.3, and that the interconnection Σ is well-defined and possesses the BIC property.*

If $\Gamma^{wISS}_{\oplus}$ satisfies the strong small gain condition (4.13)*, then Σ is weakly ISS as well.*

Proof. As each subsystem of Σ is UGS, Σ possesses the BIC property, and $\Gamma^{wISS}_{\oplus}$ is a gain operator for UGS property, Theorem 5.1 shows that Σ is forward complete and UGS. As weakly ISS systems possess an AG property, Σ is forward complete and $\Gamma^{wISS}_{\oplus}$ is a gain operator for AG property satisfying the small-gain condition, Theorem 5.6 implies that Σ has AG property. Thus, Σ is weakly ISS. □

5.4. Semimaximum and maximum formulations of ISS. In Lemma 4.1 we have reformulated the ISS property in a way that the total influence of subsystems is the sum of the internal gains. In some cases this formulation is the most convenient, but in the other cases other restatements can be more useful.

Another important restatement, which mixes summation and maximization, is given in the next lemma:

LEMMA 5.10. *A forward complete system Σ_i is* ISS (in semimaximum formulation) *if there exist $\gamma_{ij}, \gamma_i \in \mathcal{K} \cup \{0\}$, $j = 1, \dots, n$ and $\beta_i \in \mathcal{KL}$, such that for all initial values $x_i \in X_i$, all internal inputs $w_{\neq i} := (w_1, \dots, w_{i-1}, w_{i+1}, \dots, w_n) \in PC(\mathbb{R}_+, X_{\neq i})$, all external inputs $u \in \mathcal{U}$ and all $t \in \mathbb{R}_+$ the following holds:*

$$\|\bar{\phi}_i\big(t, x_i, (w_{\neq i}, u)\big)\|_{X_i} \le \beta_i\left(\|x_i\|_{X_i}, t\right) + \max_{j \neq i}\{\gamma_{ij}\left(\|w_j\|_{[0,t]}\right)\} + \gamma_i\left(\|u\|_{\mathcal{U}}\right). \tag{5.28}$$

Proof. The proof is analogous to the proof of Lemma 4.1 and is omitted. □

We can collect all the internal gains γ_{ij} from the semimaximum reformulation (5.28) of ISS again into the matrix Γ and introduce instead of the operator $\Gamma_{\oplus}$ the operator $\Gamma_{\otimes} : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ acting for $s = (s_1, \dots, s_n)^T \in \mathbb{R}^n_+$ as

$$\Gamma_{\otimes}(s_1, \dots, s_n)^T := \left(\max_{j=1}^{n} \gamma_{1j}(s_j), \dots, \max_{j=1}^{n} \gamma_{nj}(s_j)\right)^T. \tag{5.29}$$

Similarly to $\Gamma_{\oplus}$, the operator $\Gamma_{\otimes}$ is a monotone (w.r.t. the order $\ge$ in $\mathbb{R}^n$) map, i.e.

$$s_1 \ge s_2 \quad \Rightarrow \quad \Gamma_{\otimes}(s_1) \ge \Gamma_{\otimes}(s_2). \tag{5.30}$$

A counterpart of the ISS small-gain Theorem 5.3 for the semimaximum formulation of the ISS property is given by the next result:

Theorem 5.11 (ISS Small-gain theorem in semimaximum formulation). *Let $\Sigma_i := (X_i, PC(\mathbb{R}_+, X_{\neq i}) \times \mathcal{U}, \bar{\phi}_i)$, $i = 1, \dots, n$ be control systems, where all X_i, $i = 1, \dots, n$ and $\mathcal{U}$ are normed linear spaces. Assume that Σ_i, $i = 1, \dots, n$ are forward complete systems, satisfying the ISS estimates as in Lemma 5.10, and that the interconnection $\Sigma = (X, \mathcal{U}, \phi)$ is well-defined and possesses the BIC property.*

If $\Gamma_{\otimes}^{ISS}$ satisfies the strong small gain condition (4.13), *then Σ is ISS.*

Proof. Pick any $u \in \mathcal{U}$ and any $x \in X$. As the interconnection Σ is well-defined, there is a certain $t_1 > 0$ so that $(t, x, u) \in D_\phi$ for all $t \in [0, t_1]$. Using the notation and the arguments introduced in the UGS small-gain theorem in the summation form (Theorem 5.1), we obtain the following estimate for all $t \in [0, t_1]$:

$$\Vert \phi_{[0,t]} \Vert \leq \sigma(\vert \bar{\phi}(0, x, u) \vert) + \Gamma_{\otimes}^{ISS}(\Vert \phi_{[0,t]} \Vert) + \gamma(\|u\|_{\mathcal{U}}),$$

and as $\Gamma_{\otimes}^{ISS}$ is a monotone operator satisfying the strong small gain condition (4.13), we can again apply Lemma 4.6 to show UGS property of the interconnection as it was done in Theorem 5.1.

The rest of the proof goes along the lines of the proof of Theorem 5.3 and is omitted. □

Finally, consider a reformulation of ISS in terms of maximums only:

Lemma 5.12. *A forward complete system Σ_i* is ISS (in maximum formulation) *if there exist $\gamma_{ij}, \gamma_i \in \mathcal{K} \cup \{0\}$, $j = 1, \dots, n$ and $\beta_i \in \mathcal{KL}$, such that for all initial values $x_i \in X_i$, all internal inputs $w_{\neq i} := (w_1, \dots, w_{i-1}, w_{i+1}, \dots, w_n) \in PC(\mathbb{R}_+, X_{\neq i})$, all external inputs $u \in \mathcal{U}$ and all $t \in \mathbb{R}_+$ the following holds:*

$$\|\bar{\phi}_i\big(t, x_i, (w_{\neq i}, u)\big)\|_{X_i} \leq \max_{j \neq i}\{\beta_i\left(\|x_i\|_{X_i}, t\right), \gamma_{ij}\left(\|w_j\|_{[0,t]}\right), \gamma_i\left(\|u\|_{\mathcal{U}}\right)\}. \tag{5.31}$$

As a corollary of Theorem 5.11, we obtain the following small-gain theorem for the maximum reformulation of ISS

Corollary 5.13 (ISS Small-gain theorem in maximum formulation). *Let $\Sigma_i := (X_i, PC(\mathbb{R}_+, X_{\neq i}) \times \mathcal{U}, \bar{\phi}_i)$, $i = 1, \dots, n$ be control systems, where all X_i, $i = 1, \dots, n$ and $\mathcal{U}$ are normed linear spaces. Assume that Σ_i, $i = 1, \dots, n$ are forward complete systems, satisfying the ISS estimates as in Lemma 5.10, and that the interconnection $\Sigma = (X, \mathcal{U}, \phi)$ is well-defined and possesses the BIC property.*

If $\Gamma_{\otimes}^{ISS}$ satisfies the strong small gain condition (4.13), *then Σ is ISS.*

Proof. Clearly, if the subsystems of Σ satisfy the estimates (5.31), then they satisfy the estimates (5.28). Hence the claim follows from Theorem 5.11. □

Remark 5.14. *There are many other ways for characterizing the total influence of the inputs over a given subsystem, which can be formalized using so-called monotone aggregation functions, see [5]. Our approach, based on Lemma 4.6, is not specifically designed for a particular formulation of an ISS property and can be used for various formulations of ISS.*

5.5. Tightness of ISS small-gain theorems. Small-gain theorems for UGS, ISS, AG and weak ISS properties have been derived in this paper under an assumption that the strong small-gain condition (4.13) holds. A natural question is whether the same result holds under weaker conditions.

Consider the planar system considered in [6, Example 18]:

$$\dot{x}_1 = -x_1 + x_2(1 - e^{-x_2}) + u(t), \tag{5.32a}$$

$$\dot{x}_2 = -x_2 + x_1(1 - e^{-x_1}) + u(t). \tag{5.32b}$$

In [6, Example 18] it was shown that:

- Both subsystems of (5.32) are ISS in a summation formulation with the gains $\gamma_{12}(r) = \gamma_{21}(r) := r(1 - e^{-r})$, $r \in \mathbb{R}_+$
- The corresponding operator $\Gamma_\oplus$, satisfies the small-gain condition (4.12), which boils down to checking that $\gamma_{12} \circ \gamma_{21}(r) < r$ for all $r > 0$
- $\Gamma_\oplus$ does not satisfy the strong small-gain condition
- (5.32) is not ISS

Hence in the statement of the ISS small-gain theorem in the summation formulation the requirement of the strong small-gain condition for $\Gamma_\oplus$ cannot be weakened to the requirement of a small-gain condition for $\Gamma_\oplus$.

As for the interconnection of 2 systems the summation and semimaximum formulations of the ISS property coincide and $\Gamma_\oplus = \Gamma_\otimes$, the same example shows the tightness of Theorem 5.11 in the above sense. Note that although the system (5.32) is not ISS, it is 0-UGAS, which can be shown using the Lyapunov function $V(x_1, x_2) = x_1^2 + x_2^2$.

In the next proposition we show that for any gain matrix for which the corresponding gain operator $\Gamma_\oplus$ does not satisfy the small-gain condition (4.12), one can construct a system with this gain matrix, so that each subsystem of this system is ISS in the summation formulation, but the interconnection is not 0-UGAS.

Proposition 5.15. *Let a gain matrix $\Gamma := (\gamma_{ij})_{i,j=1,\dots,n} \subset (\mathcal{K} \cup \{0\})^{n\times n}$, with $\gamma_{ii} = 0$ for all $i = 1, \dots, n$ be given. If the corresponding gain operator $\Gamma_\oplus$ does not satisfy the small-gain condition* (4.12)*, then there exists $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ so that each subsystem of the ODE system* (2.6) *is ISS, for all $i = 1, \dots, n$ the estimates* (4.9) *hold for all $x \in \mathbb{R}^n$, $u \in \mathcal{U}$, $t \geq 0$, but the whole interconnection* (2.7) *has a non-trivial equilibrium and thus is not 0-UGAS.*

Proof. Let $\Gamma_\oplus$ does not satisfy the small-gain condition (4.12). Then there is $s \in \mathbb{R}^n_+ \backslash \{0\}$ so that $\Gamma_\oplus(s) \geq s$. Take $\varepsilon > 0$ so that

$$(1 - \varepsilon)\Gamma_\oplus(s) = s \tag{5.33}$$

and enlarge the domain of definition of functions γ_{ij} to $\mathbb{R}$, defining $\gamma_{ij}(-r) = -\gamma_{ij}(r)$ $\forall r > 0$, $i, j = 1, \dots n$. With such definitions we can consider the operator $\Gamma_\oplus$ as an operator acting on $\mathbb{R}^n$, with the same defining formula (4.7).

Consider the following ODE system on $\mathbb{R}^n$:

$$\dot{x} = -x + (1 - \varepsilon)\Gamma_\oplus(x). \tag{5.34}$$

The point $x = s$ is a non-trivial equilibrium of this system, and hence (5.34) is not 0-UGAS.

However, all subsystems of (5.34) satisfy the following estimates

$$\begin{aligned} |x_i(t)| &\leq |x_i(0)|\, e^{-t} + e^{-t} \int_0^t e^s \sum_{j=1}^n (1 - \varepsilon)\gamma_{ij}(|x_j(s)|) ds \\ &\leq |x_i(0)|\, e^{-t} + \sum_{j=1}^n \gamma_{ij}\left(\|x_j\|_\infty\right). \end{aligned}$$

and hence are ISS. This shows the claim. □

REMARK 5.16. *It is possible to show a counterpart of Proposition 5.15 for the semimaximum formulation of ISS, see [22, Theorem 1.5.9] and [22, Theorem 1.5.10] for the corresponding Lyapunov version.*

5.5.1. On maximum formulation. ISS small-gain theorem in the maximum formulation (Corollary 5.13) has been obtained as an easy consequence of our semimaximum ISS small-gain theorem. In the first step of the proof of Corollary 5.13 we make the given maximum estimates worse, by upperestimating the maximum by the 'semimaximum'. This step is needed for our general approach, because otherwise it is unclear how to use Lemma 4.6 in its present form, which is our key technique. However, this loss of a part of information about the subsystems prevents us from obtaining a tight ISS small-gain theorem in the maximum formulation.

Indeed, in [2] it was shown that at least in the special case of the evolution equations in Banach spaces with the Lipschitz continuous right-hand sides (2.8) the stronger result is valid: it is sufficient to require that $\Gamma_{\otimes}$ satisfies the small-gain condition (4.12), and the validity of the strong small-gain condition (4.13) is not required. The technique, which has been used in [2] to obtain the tight result was specifically designed for the maximum formulation of the ISS property and is rather different from our approach. This technique cannot be applied (at least without substantial modifications) for verification of ISS of coupled system for which ISS property of subsystems is formulated in a summation or semimaximum form. Hence the results in [2] are complementary to the results in this paper.

Nevertheless, since the class of the systems which we consider in this work is much wider than those in [2], Corollary 5.13 is still of interest and not fully covered by the results in [2], although it is highly probably that the technique, used in [2] can be adapted to the general systems considered in this paper (again, for the maximum formulation of the ISS property).

6. Characterizations of ISS by means of ULIM with 'bounded inputs'. As bUAG property has been useful in the proof of an ISS small-gain theorem, it is reasonable to expect, that it can be useful also in other contexts. Hence in this section we characterize ISS in terms of bUAG and bULIM properties, which gives characterizations, which are a bit stronger and more flexible than those, proved in [26].

Weaker counterparts of asymptotic gain properties are so-called limit properties.

DEFINITION 6.1. *We say that a forward complete system $\Sigma = (X, \mathcal{U}, \phi)$ has the*

(iii) bounded input uniform limit property (bULIM), if there exists $\gamma \in \mathcal{K} \cup \{0\}$ so that for every $\varepsilon > 0$ and for every $r > 0$ there exists a $\tau = \tau(\varepsilon, r)$ such that for all x with $\|x\|_X \le r$ and all $u \in \mathcal{U}$ there is a $t \le \tau$ such that

$$\|\phi(t, x, u)\|_X \le \varepsilon + \gamma(\|u\|_{\mathcal{U}}). \tag{6.1}$$

(iv) uniform limit property (ULIM), if there exists $\gamma \in \mathcal{K} \cup \{0\}$ so that for every $\varepsilon > 0$ and for every $r > 0$ there exists a $\tau = \tau(\varepsilon, r)$ such that for all x with $\|x\|_X \le r$ and all $u \in \mathcal{U}$ there is a $t \le \tau$ such that (6.1) holds.

It is easy to see that bUAG implies bULIM and UAG implies ULIM property.

Even nonuniformly globally asymptotically stable forward complete systems do not always have uniform bounds for their reachability sets on finite intervals (see [26, 27]). Systems exhibiting such bounds deserve a special name.

Definition 6.2. *We say that a forward complete system* $\Sigma = (X, \mathcal{U}, \phi)$ *has bounded reachability sets (BRS), if for any* $C > 0$ *and any* $\tau > 0$ *it holds that*

$$\sup\big\{\|\phi(t,x,u)\|_X \ : \ \|x\|_X \le C,\ \|u\|_{\mathcal{U}} \le C,\ t \in [0,\tau]\big\} < \infty.$$

Clearly, UAG property implies bUAG property. It is not completely clear, whether the converse holds without further assumptions, as e.g. BRS property. However, for a system satisfying bUAG property it is not possible in general to verify UAG (and even sAG) property without an increase of the gain, as argued in the following example

Example 6.3. *Consider a system*

$$\dot{x} = -\frac{1}{1+|u(t)|}x, \tag{6.2}$$

where $u \in \mathcal{U} := L_\infty(\mathbb{R}_+, \mathbb{R})$ *is a globally essentially bounded input and* $x(t) \in \mathbb{R}$ *is the state.*

It is easy to see that this system is forward complete and has a BRS property as $|\phi(t,x,u)| \le |x|$ *for all* $t \ge 0$, $x \in \mathbb{R}$, $u \in \mathcal{U}$. *Let us show that* (6.2) *has a bUAG property with a zero gain.*

Pick any $u \in \mathcal{U}$, $x \in \mathbb{R}$ *and* $t \ge 0$. *The corresponding solution of* (6.2) *can be estimated as*

$$|\phi(t,x,u)| = e^{-\int_0^t \frac{1}{1+|u(s)|}ds}|x| \le e^{-\int_0^t \frac{1}{1+\|u\|_\infty}ds}|x| = e^{-\frac{1}{1+\|u\|_\infty}t}|x|.$$

Now pick any $r > 0$ *and any* $\varepsilon > 0$. *Then for any* $u \in B_{r,\mathcal{U}}$ *it holds that*

$$|\phi(t,x,u)| \le e^{-\frac{1}{1+r}t}r.$$

Now, let $\tau > 0$ *be so that* $e^{-\frac{1}{1+r}\tau}r = \varepsilon$ *(clearly, such* τ *exists).*

Overall, for all $t \ge \tau$, *all* $x \in B_r$ *and all* $u \in B_{r,\mathcal{U}}$ *it holds that* $|\phi(t,x,u)| \le \varepsilon$. *This shows bUAG property of* (6.3) *with a zero gain. At the same time, it is known that* (6.3) *does not have a strong AG property with a zero gain, which is weaker than UAG with a zero gain, see [23, Remark 4].*

Similarly to Lemma 3.7, the following result can be shown:

Lemma 6.4. *Let* $\Sigma = (X, \mathcal{U}, \phi)$ *be a control system. If* Σ *is UGS and bULIM, then* Σ *is ULIM.*

Although the notion of bUAG is used not as widely as the standard UAG (but see e.g. [38], [22, Proposition 1.4.3.]), but it is in a certain sense even more natural than UAG. Indeed, in characterizations of ISS in terms of UAG and ULIM properties shown in [26] the UAG (and uniform limit) properties have been applied almost exclusively for uniformly bounded inputs, and as we see in the following theorem, validity of such characterizations is retained if we use bUAG and bULIM properties instead of UAG and ULIM properties.

Without loss of generality we restrict our analysis to fixed points of the form $(0,0) \in X \times \mathcal{U}$, so that we tacitly assume that the zero input is an element of $\mathcal{U}$.

Definition 6.5. *Consider a system* $\Sigma = (X, \mathcal{U}, \phi)$. *We call* $0 \in X$ *an equilibrium point (of the undisturbed system) if* $\phi(t,0,0) = 0$ *for all* $t \ge 0$.

For characterizations of ISS we need one more notion

Definition 6.6. *Consider a system* $\Sigma = (X, \mathcal{U}, \phi)$ *with equilibrium point* $0 \in X$. *We say that* ϕ *is continuous at the equilibrium if for every* $\varepsilon > 0$ *and for any* $h > 0$ *there exists a* $\delta = \delta(\varepsilon, h) > 0$, *so that*

$$t \in [0,h] \ \wedge\ \|x\|_X \le \delta \ \wedge\ \|u\|_{\mathcal{U}} \le \delta \ \Rightarrow\ \|\phi(t,x,u)\|_X \le \varepsilon. \tag{6.3}$$

In this case we will also say that Σ has the CEP property.

Now we show the following characterizations of ISS, which refine the main result in [26]

THEOREM 6.7. *Let $\Sigma = (X, \mathcal{U}, \phi)$ be a forward complete control system. The following statements are equivalent:*

(i) Σ is ISS.

(ii) Σ is UAG and UGS.

(iii) Σ is bUAG and UGS.

(iv) Σ is bULIM and UGS.

(v) Σ is bULIM, ULS and BRS.

(vi) Σ is bUAG, CEP and BRS.

Proof. The proof goes mostly along the lines of the characterizations of ISS, obtained in [26, Theorem 5].

(vi) $\Rightarrow$ (v). Doing some minimal changes in [26, Lemma 6] one can show that bUAG $\wedge$ CEP implies ULS property.

(v) $\Rightarrow$ (iv). Doing some minimal changes in [26, Proposition 10] one can show that bULIM $\wedge$ BRS implies UGB property. By [26, Lemma 4] UGB $\wedge$ ULS is equivalent to UGS property.

(iv) $\Rightarrow$ (iii). Follows from the proof of [26, Lemma 7].

(iii) $\Rightarrow$ (ii). Follows from Lemma 3.7.

(ii) $\Rightarrow$ (vi). Clear.

(ii) $\Leftrightarrow$ (i). Follows by [26, Lemma 8]. □

REMARK 6.8 (Characterizations and small-gain theorems for strong ISS property). *In [26] the notions of strong ISS, strong limit property (sLIM) and strong asymptotic gain (sAG) have been defined, which are weaker than ISS, ULIM and UAG properties respectively. It is possible to define the 'bounded inputs' versions of sLIM and sAG properties, which may be called bsLIM and bsAG, and show that*

$$\text{sISS} \quad \Leftrightarrow \quad \text{bsAG} \wedge \text{UGS} \quad \Leftrightarrow \quad \text{bsLIM} \wedge \text{UGS}.$$

Since the proof of this result is completely analogous to the proofs of Theorem 6.7 and Lemma 3.7, we drop definitions and precise formulations of the results.

A harder and more interesting problem is to derive the small-gain theorem for sISS property. The approach, used for the proof of ISS and weak ISS small-gain theorems cannot be straightforwardly adapted for the sISS case. Indeed, applying the cocycle property (5.10)*, which is the starting point in our proof technique, leads to the fact that the term $\bar{\phi}_i\big(t, x_i, (\phi_{\neq i}, u)\big)$ depends on an input u. Hence the time τ_i, which is obtained in the next step of the proof of small-gain theorem for ISS property, depends on u as well (since it depends on $\bar{\phi}_i\big(t, x_i, (\phi_{\neq i}, u)\big)$). But this is not what we want in order to achieve sAG (or bsAG) property of the interconnection.*

7. Conclusion. We have proved a small-gain theorem for interconnections of n nonlinear heterogeneous input-to-state stable control systems of a general nature. We expect that these results will be particularly helpful for stability analysis of couplings of time-delay systems, in-domain and boundary couplings of PDE systems as well as for ODE-PDE and delay-PDE cascades. As the derived small-gain results are valid for summation, semimaximum and maximum formulations (and potentially for further formulations) of ISS, they allow for great flexibility in the analysis of networks of distributed parameter systems. Small-gain theorems for asymptotic gain, uniform global stability and weak input-to-state stability property have been proved as well and the tightness of the obtained results has been discussed.

We introduced the notions of bounded input uniform asymptotic gain property and bounded input uniform limit property, which are easier to verify than the classical uniform asymptotic gain and uniform limit properties, but which still lead to powerful characterizations of input-to-state stability, which are an ultimately useful for ISS theory.

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