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Small gain theorems for networks of heterogeneous systems

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Abstract: We prove a small-gain theorem for interconnections of n nonlinear heterogeneous input-to-state stable control systems of a general nature, covering partial, delay and ordinary differential equations. We use in this paper the summation formulation of the ISS property, but the method can be adapted to other formulations of the ISS concept as well. The proof is based on the recent characterizations of the input-to-state stability for infinite-dimensional systems in terms of weaker stability properties.

Keywords: infinite-dimensional systems, input-to-state stability, interconnected systems, nonlinear systems, small-gain theorems.

1. INTRODUCTION

The notion of input-to-state stability (ISS), introduced in Sontag (1989) for ordinary differential equations (ODEs), has become a backbone for much of nonlinear control theory, and is currently a well developed theory with a firm theoretical basis and such powerful tools for ISS analysis, as Lyapunov and small-gain methods. Broad applications of ISS theory include design of robust controllers and observers for nonlinear systems Arcak and Kokotović (2001), analysis of large-scale networks Jiang et al. (1994); Dashkovskiy et al. (2007, 2010), etc.

The impact of finite-dimensional ISS theory and the need of proper tools for robust stability analysis of distributed parameter systems resulted in generalizations of ISS concepts to broad classes of distributed parameter systems, including partial differential equations (PDEs) with distributed and boundary controls, nonlinear evolution equations in Banach spaces with bounded and unbounded input operators, etc. Techniques developed within the infinite-dimensional ISS theory include characterizations of ISS and ISS-like properties in terms of weaker stability concepts Mironchenko and Wirth (2018), Jacob et al. (2018b), constructions of ISS Lyapunov functions for PDEs with distributed and boundary controls Dashkovskiy and Mironchenko (2013); Mironchenko and Ito (2015); Tanwani et al. (2018); Zheng and Zhu (2018), non-coercive ISS Lyapunov functions Mironchenko and Wirth (2018); Jacob et al. (2018a), efficient methods for study of boundary control systems Zheng and Zhu (2017); Jacob et al. (2018b); Karafyllis and Krstic (2019), transfer functions Jayawardhana et al. (2008) etc.

One of the central topics of the mathematical control theory is the analysis of coupled systems. Large-scale nonlinear systems can be very complex, so that a direct stability analysis of such systems is rarely possible. Small-gain theorems allow us to overcome this obstacle by ensuring ISS of an interconnected system, provided all subsystems are ISS and the interconnection structure, described by gains, satisfies the small-gain condition.

1.1 Existing ISS small-gain results

There are two types of nonlinear small-gain theorems: theorems in terms of trajectories and in terms of Lyapunov functions. In *small-gain theorems in the trajectory formulation* one assumes that each subsystem is ISS both w.r.t. external inputs and internal inputs from other subsystems, and the so-called internal gains of subsystems characterizing the influence of subsystems on each other are known. The small-gain theorem states that the coupled system is ISS provided the gains satisfy the small-gain condition. First small-gain theorems of this type have been developed in Jiang et al. (1994) for feedback couplings of two ODE systems and in Dashkovskiy et al. (2007) for arbitrary couplings of n ODE systems.

In *Lyapunov small-gain theorems* it is assumed that all subsystems are ISS w.r.t. external and internal inputs and the ISS Lyapunov functions for subsystems are given together with the corresponding Lyapunov gains. If Lyapunov gains satisfy the small-gain condition, then the whole interconnection is ISS and moreover, an ISS Lyapunov function for the overall system can be constructed. For couplings of 2 systems such theorems have been shown in Jiang et al. (1996) and this result has been extended to couplings of n nonlinear ODE systems in Dashkovskiy et al. (2010).

As was argued in Dashkovskiy and Mironchenko (2013), ISS and integral ISS small-gain theorems in a Lyapunov formulation can be extended to interconnections of n infinite-dimensional systems without radical changes in the formulation and proof technique, although in the integral ISS case one should carefully choose the state spaces for subsystems, see Mironchenko and Ito (2015).

The case of trajectory-based infinite-dimensional small-gain theorems for couplings of $n > 2$ *systems is significantly more complicated* since the proof of such theorems in Dashkovskiy et al. (2007) is based on the fundamental result that ISS of ODE systems is equivalent to uniform

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global stability (UGS) combined with the asymptotic gain (AG) property shown in Sontag and Wang (1996). Such a characterization is not valid for infinite-dimensional systems, as argued in Mironchenko and Wirth (2018) which makes the proof of Dashkovskiy et al. (2007) not applicable to infinite-dimensional systems without substantial modifications.

In particular, considerable attention has been devoted to small-gain theorems in terms of trajectories for time-delay systems. The small-gain results for AG $\wedge$ UGS property for time-delay systems have been obtained in Polushin et al. (2006) for couplings of 2 systems and in Tiwari et al. (2009) and Polushin et al. (2013) for interconnections of n systems. As AG $\wedge$ UGS is (possibly) weaker than ISS for time-delay systems, ISS small-gain theorems have not been obtained in these works.

The first ISS small-gain theorems, applicable for time-delay systems have been achieved in Karafyllis and Jiang (2007), as a special case of a rather abstract result on stability of couplings of two systems.

The obstacle that ISS is (at least potentially) not equivalent to AG $\wedge$ UGS, was overcome in Tiwari et al. (2012) where ISS small-gain theorems for couplings of $n \geq 2$ time-delay systems have been obtained by using a Razumikhin-type argument. In this approach the delayed state in the right hand side of a time-delay system is treated as an input to the system, which makes the time-delay system a delay-free system with an additional input. However, the transformation of time-delay systems to the delay-free form is not always straightforward.

Recently in Bao et al. (2018) the small-gain theorems for couplings of n input-to-output stable (IOS) evolution equations in Banach spaces have been derived. As a special case of these results, the authors obtain a small-gain theorem for networks of n ISS systems in the maximum formulation. Application of small-gain theorems for stability analysis of coupled parabolic-hyperbolic PDEs has been performed in Karafyllis and Krstic (2018b). Small-gain based boundary feedback design for global exponential stabilization of 1-D semilinear parabolic PDEs has been proposed in Karafyllis and Krstic (2018a).

1.2 Contribution

Our main result is the ISS small-gain theorem for feedback interconnections of n nonlinear heterogeneous systems whose components belong to a broad class of control systems covering PDEs, time-delay systems, ODEs etc.

For the description of interconnections of control systems we adopt an approach described in Karafyllis and Jiang (2007).

Another key ingredient for our approach is a powerful characterization of ISS obtained in Mironchenko and Wirth (2018), which proved to be ultimately useful also in other contexts, e.g. for non-coercive Lyapunov function theory Mironchenko and Wirth (2018); Jacob et al. (2018a), study of practical ISS Mironchenko (2019a), etc. To apply it to our problem, we exploit bounded input uniform asymptotic gain (bUAG) property, which is more flexible in use than standard uniform asymptotic gain (UAG) property.

The ISS small-gain theorem (Theorem 5.2) is achieved in 3 steps:

(i) UGS property of the interconnection (see Theorem 5.1) is established using the methods developed in Dashkovskiy et al. (2007).
(ii) bUAG property of the interconnection is verified (the main technical step and the main difference to the method of Dashkovskiy et al. (2007)).
(iii) We show that UGS $\wedge$ bUAG is equivalent to ISS (based on Mironchenko and Wirth (2018)), which concludes the proof.

1.3 Relation to previous research

This paper is motivated by ISS small-gain theorems for networks of $n \in \mathbb{N}$ ODE systems, reported in Dashkovskiy et al. (2007), and recovers these results in the special case of ODE systems.

As a particular application of our general small-gain theorems one can obtain novel small-gain results for couplings of n nonlinear time-delay systems. Unlike Polushin et al. (2006); Tiwari et al. (2009); Polushin et al. (2013), we obtain not only UGS $\wedge$ AG (i.e. weak ISS) small-gain results, but also ISS small-gain theorems. In contrast to ISS small-gain theorems from Tiwari et al. (2012), our approach is not time-delay specific, does not require a transformation of retarded systems into delay-free ones and is applicable to the summation formulation of the ISS property.

In Bao et al. (2018) small-gain theorems for couplings of n evolution equations in Banach spaces with Lipschitz continuous nonlinearities have been derived, by using a rather different proof technique, which is applicable if the small-gain property is formulated in the so-called maximization formulation. Instead, we focus in this work on the summation formulation of the ISS property and thus the developments in Bao et al. (2018) are complementary to this paper.

The approach which we use in this paper is very flexible as it is valid for a broad class of infinite-dimensional systems, independent on the type of the couplings between subsystems (in-domain or boundary couplings) and can be extended also to the maximum and semimaximum formulations. Furthermore, the small-gain theorems in the summation formulation presented in this paper are tight (however, for the maximum formulation of the ISS property the results in Bao et al. (2018) are stronger). A counterpart for the weak ISS property, introduced in Schmid and Zwart (2018) can be shown as well. More on these topics can be found in the full journal paper Mironchenko (2019b). In the same paper one can find strengthenings of the characterizations of ISS property shown in Mironchenko and Wirth (2018), using the notion of bUAG and a related bounded inputs uniform limit property.

1.4 Notation

In the following $\mathbb{R}_+ := [0, \infty)$. For arbitrary $x, y \in \mathbb{R}^n$ define the relation "$\geq$" on $\mathbb{R}^n$ by: $x \geq y \Leftrightarrow x_i \geq y_i$, $\forall i = 1, \ldots, n$. Further define $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x \geq 0\}$.

By "$\not\geq$" we understand the logical negation of "$\geq$", that is $x \not\geq y$ iff $\exists i$: $x_i < y_i$.

For a function $v : \mathbb{R}_+ \to X$, where X is a certain set, we define its restriction to the interval $[s_1, s_2]$ by

$$v_{[s_1,s_2]}(t) := \begin{cases} v(t) & \text{if } t \in [s_1, s_2], \\ 0 & \text{else.} \end{cases}$$

For a normed vector space U, we denote by $PC_b(\mathbb{R}_+, U)$ the space of globally bounded piecewise continuous (right-continuous) functions from $\mathbb{R}_+$ to U with the norm $\|u\|_{PC_b(\mathbb{R}_+,U)} = \|u\|_{C(\mathbb{R}_+,U)}$.

We use the following classes of comparison functions

$$\begin{aligned} \mathcal{K} &:= \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous, } \gamma(0) = 0 \text{ and strictly increasing}\} \\ \mathcal{K}_\infty &:= \{\gamma \in \mathcal{K} \mid \gamma \text{ is unbounded}\} \\ \mathcal{L} &:= \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous and strictly decreasing with } \lim_{t\to\infty} \gamma(t) = 0\} \\ \mathcal{KL} &:= \{\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \mid \beta \text{ is continuous, } \beta(\cdot, t) \in \mathcal{K},\ \beta(r, \cdot) \in \mathcal{L},\ \forall t \geq 0, r > 0\} \end{aligned}$$

2. PROBLEM FORMULATION

2.1 Definition of control systems

First we define the concept of a (time-invariant) system:

Definition 2.1. *Consider the triple $\Sigma = (X, \mathcal{U}, \phi)$ consisting of*

(i) A normed linear space $(X, \|\cdot\|_X)$, called the state space, endowed with the norm $\|\cdot\|_X$.

(ii) A set of input values U, which is a nonempty subset of a certain normed linear space.

(iii) A space of inputs $\mathcal{U} \subset \{f : \mathbb{R}_+ \to U\}$ endowed with a norm $\|\cdot\|_\mathcal{U}$ which satisfies the following two axioms:

The axiom of shift invariance*: for all $u \in \mathcal{U}$ and all $\tau \geq 0$ the time shift $u(\cdot + \tau)$ belongs to $\mathcal{U}$ with $\|u\|_\mathcal{U} \geq \|u(\cdot + \tau)\|_\mathcal{U}$.*

The axiom of concatenation*: for all $u_1, u_2 \in \mathcal{U}$ and all $t > 0$ the concatenation of u_1 and u_2 at time t*

$$u(\tau) := \begin{cases} u_1(\tau), & \text{if } \tau \in [0, t], \\ u_2(\tau - t), & \text{otherwise,} \end{cases} \tag{1}$$

belongs to $\mathcal{U}$.

(iv) A map $\phi : D_\phi \to X$, $D_\phi \subseteq \mathbb{R}_+ \times X \times \mathcal{U}$ (called transition map), so that for all $(x, u) \in X \times \mathcal{U}$ there is a $t > 0$ so that $[0, t] \times \{(x, u)\} \subset D_\phi$.

The triple Σ is called a (control) system, if the following properties hold:

(Σ1) The identity property: *for every $(x, u) \in X \times \mathcal{U}$ it holds that $\phi(0, x, u) = x$.*

(Σ2) Causality: *for every $(t, x, u) \in \mathbb{R}_+ \times X \times \mathcal{U}$, for every $\tilde{u} \in \mathcal{U}$, such that $u(s) = \tilde{u}(s)$ for all $s \in [0, t]$ it holds that $\phi(t, x, u) = \phi(t, x, \tilde{u})$.*

(Σ3) Continuity: *for each $(x, u) \in X \times \mathcal{U}$ the map $t \mapsto \phi(t, x, u)$ is continuous.*

(Σ4) The cocycle property: *for all $x \in X$, $u \in \mathcal{U}$, for all $t, h \geq 0$ so that $[0, t + h] \times \{(x, u)\} \subset D_\phi$, we have $\phi(h, \phi(t, x, u), u(t + \cdot)) = \phi(t + h, x, u)$.*

This class of systems encompasses control systems generated by ordinary differential equations (ODEs), switched systems, time-delay systems, many classes of partial differential equations (PDEs), important classes of boundary control systems and many other systems.

We define several important properties of control systems:

Definition 2.2. *We say that a system is* forward complete (FC)*, if $D_\phi = \mathbb{R}_+ \times X \times \mathcal{U}$, that is for every $(x, u) \in X \times \mathcal{U}$ and for all $t \geq 0$ the value $\phi(t, x, u) \in X$ is well-defined.*

An important property of ordinary differential equations with Lipschitz continuous right-hand sides states that if the solution stays bounded over $[0, t)$, then it can be prolonged to $[0, t+\varepsilon)$ for a certain $\varepsilon > 0$. Similar properties have evolution equations in Banach spaces with bounded control operators and Lipschitz continuous right hand sides (Cazenave and Haraux, 1998, Theorem 4.3.4) and many other classes of systems (Karafyllis and Jiang, 2011, Chapter 1). The next property, adopted from (Karafyllis and Jiang, 2011, Definition 1.4) formalizes this behavior for general control systems.

Definition 2.3. *We say that a system Σ satisfies the* boundedness-implies-continuation (BIC) property *if for each $(x, u) \in X \times \mathcal{U}$, there exists $t_{\max} \in (0, +\infty]$, called a* maximal existence time*, such that $[0, t_{\max}) \times \{(x, u)\} \subset D_\phi$ and for all $t \geq t_{\max}$, it holds that $(t, x, u) \notin D_\phi$. In addition, if $t_{\max} < +\infty$, then for every $M > 0$, there exists $t \in [0, t_{\max})$ with $\|\phi(t, x, u)\|_X > M$.*

2.2 Interconnections of control systems

Let $(X_i, \|\cdot\|_{X_i})$, $i = 1, \ldots, n$ be normed linear spaces. Define for each $i = 1, \ldots, n$ the normed linear space

$$X_{\neq i} := X_1 \times \ldots \times X_{i-1} \times X_{i+1} \times \ldots \times X_n, \tag{2}$$

endowed with the norm $\|x\|_{X_{\neq i}} := \Big(\sum_{j=1,\ j\neq i}^n \|x_j\|^2_{X_j}\Big)^{1/2}$.

Let control systems $\Sigma_i := (X_i, PC_b(\mathbb{R}_+, X_{\neq i}) \times \mathcal{U}, \bar{\phi}_i)$ be given and assume that each Σ_i possesses a BIC property. We call $X_{\neq i}$ the *space of internal input values* and $PC_b(\mathbb{R}_+, X_{\neq i})$ the *space of internal inputs*. The norm on $PC_b(\mathbb{R}_+, X_{\neq i}) \times \mathcal{U}$ we introduce as

$$\|(v, u)\|_{PC_b(\mathbb{R}_+,X_{\neq i})\times\mathcal{U}} := \Big(\sum_{j\neq i} \|v_j\|^2_{PC_b(\mathbb{R}_+,X_j)} + \|u\|^2_\mathcal{U}\Big)^{\frac{1}{2}}. \tag{3}$$

Define also the normed linear space

$$X = X_1 \times \ldots \times X_n, \qquad \|x\|_X := \Big(\sum_{i=1}^n \|x_i\|^2_{X_i}\Big)^{1/2}, \tag{4}$$

and assume that there is a map $\phi = (\phi_1, \ldots, \phi_n) : D_\phi \to X$, defined over a certain domain $D_\phi \subseteq \mathbb{R}_+ \times X \times \mathcal{U}$ so that for each $x = (x_1, x_2, \ldots, x_n) \in X$, each $u \in \mathcal{U}$ and all $t \in \mathbb{R}_+$ so that $(t, x, u) \in D_\phi$ and for every $i = 1, \ldots, n$, it holds that

$$\phi_i(t, x_i, u) = \bar{\phi}_i\big(t, x_i, (v_i, u)\big), \text{ with} \tag{5}$$

$$v_i(t) = (\phi_1(t,x,u), \ldots, \phi_{i-1}(t,x,u), \phi_{i+1}(t,x,u), \ldots, \phi_n(t,x,u)).$$

Assume further that $\Sigma := (X, \mathcal{U}, \phi)$ is a control system with the state space X, input space $\mathcal{U}$ and with a BIC property. Then Σ is called *a (feedback) interconnection* of systems $\Sigma_1, \ldots, \Sigma_n$.

In other words, condition (5) means that if the modes $\phi_j(\cdot, x, u)$, $j \neq i$ of the system Σ will be sent to Σ_i as the internal inputs (together with an external input u), and the initial state will be chosen as x_i (the i-th mode of x), then the resulting trajectory of the system Σ_i, which is $\bar{\phi}_i(\cdot, x_i, v, u)$ will coincide with the trajectory of the i-th mode of the system Σ on the interval of existence of ϕ_i.

Note that the trajectory of each Σ_i depends continuously on time due to the continuity axiom. However, as the space of continuous functions does not satisfy the concatenation property, we enlarge it to include the piecewise continuous functions. This motivates the choice of the space $PC_b(\mathbb{R}_+, X_{\neq i}) \times \mathcal{U}$ as the input space for the i-th subsystem.

Remark 2.4. *This definition of feedback interconnections, which we adopted from (Karafyllis and Jiang, 2007, Definition 3.3), does not depend on a particular type of control systems which are coupled, and is applicable to large-scale systems, consisting of heterogeneous components as PDEs, time-delay systems, ODE systems etc. The definition also applies to different kinds of interconnections, e.g. both for in-domain and boundary interconnections of PDE systems.*

Next we show how the couplings of evolution equations in Banach spaces can be represented in our approach. Many further classes of systems can be treated in a similar way.

2.3 Example: interconnections of evolution equations in Banach spaces

Consider a system of the following form

$$\begin{cases} \dot{x}_i = A_i x_i + f_i(x_1, \dots, x_n, u), \\ i = 1, \dots, n, \end{cases} \tag{6}$$

where the state space of the i-th subsystem X_i is a Banach space and A_i with the domain of definition $D(A_i)$ is the generator of a C_0-semigroup on X_i, $i = 1, \dots, n$. In the sequel we assume that the set of input values U is a normed linear space and that the input functions belong to the space $\mathcal{U} := PC_b(\mathbb{R}_+, U)$.

Define the state space X of the whole system (6) by (4). We choose further the input space to the i-th subsystem as (3).

For $x_i \in X_i$, $i = 1, \dots, n$ define $x = (x_1, \dots, x_n)^T$, $f(x,u) = (f_1(x,u), \dots, f_n(x,u))^T$. By A we denote the diagonal operator $A := \mathrm{diag}(A_1, \dots, A_n)$ with the domain of definition $D(A) = D(A_1) \times \dots \times D(A_n)$. It is well-known that A is the generator of a C_0-semigroup on X.

With this notation the coupled system (6) takes the form

$$\dot{x}(t) = Ax(t) + f(x(t), u(t)), \quad u(t) \in U. \tag{7}$$

Assuming that f is Lipschitz continuous w.r.t. x guarantees that the mild solutions of (7) exists and is unique for every initial condition and for any admissible input. Here mild solutions $x : [0, \tau] \to X$ are the solutions of the integral equation

$$x(t) = T(t)x(0) + \int_0^t T(t-s) f(x(s), u(s)) ds, \tag{8}$$

belonging to the space of continuous functions $C([0,\tau], X)$ for some $\tau > 0$.

Under these assumptions the system (7) can be seen as a well-defined interconnection of the systems Σ_i, $i = 1, \dots, n$, and each Σ_i is a well-defined system in the sense of Definition 2.1. Moreover, by a variation of (Cazenave and Haraux, 1998, Theorem 4.3.4) one can show that (8) possesses the BIC property.

3. STABILITY NOTIONS

The main concept in this paper is:

Definition 3.1. *A system $\Sigma = (X, \mathcal{U}, \phi)$, where $\phi : D_\phi \to X$ is called* (uniformly) input-to-state stable (ISS)*, if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty \cup \{0\}$ such that for all $(t,x,u) \in D_\phi$ it holds that*

$$\|\phi(t,x,u)\|_X \le \beta(\|x\|_X, t) + \gamma(\|u\|_{\mathcal{U}}). \tag{9}$$

An important property implied by ISS is

Definition 3.2. *A system $\Sigma = (X, \mathcal{U}, \phi)$, where $\phi : D_\phi \to X$ is called* uniformly globally stable (UGS)*, if there exist $\sigma \in \mathcal{K}_\infty$, $\gamma \in \mathcal{K}_\infty \cup \{0\}$ such that for all $(t,x,u) \in D_\phi$ the following holds:*

$$\|\phi(t,x,u)\|_X \le \sigma(\|x\|_X) + \gamma(\|u\|_{\mathcal{U}}). \tag{10}$$

Lemma 3.3. *Let $\Sigma = (X, \mathcal{U}, \phi)$ be an UGS control system. If Σ has the BIC property, then Σ is forward complete.*

Proof. The proof is an easy exercise and is omitted. □

For forward complete systems we introduce the following asymptotic properties

Definition 3.4. *A forward complete system $\Sigma = (X, \mathcal{U}, \phi)$*

(i) satisfies bounded input uniform asymptotic gain (bUAG) property*, if there exists a $\gamma \in \mathcal{K}_\infty \cup \{0\}$ such that for all $\varepsilon, r > 0$ there is a $\tau = \tau(\varepsilon, r) < \infty$ s.t. for all $u \in \mathcal{U}$: $\|u\|_{\mathcal{U}} \le r$ and all $x \in X$: $\|x\|_X \le r$*

$$t \ge \tau \quad \Rightarrow \quad \|\phi(t,x,u)\|_X \le \varepsilon + \gamma(\|u\|_{\mathcal{U}}). \tag{11}$$

(ii) satisfies uniform asymptotic gain (UAG) property*, if there exists a $\gamma \in \mathcal{K}_\infty \cup \{0\}$ such that for all $\varepsilon, r > 0$ there is a $\tau = \tau(\varepsilon, r) < \infty$ such that for all $u \in \mathcal{U}$ and all $x \in X$: $\|x\|_X \le r$ the implication (11) holds.*

Both bUAG and UAG properties show that all trajectories converge uniformly to the ball of radius $\gamma(\|u\|_{\mathcal{U}})$ around the origin as $t \to \infty$. However, in bUAG property the uniformity is only over the set of inputs with a given magnitude, and in UAG property over all inputs.

The following lemma shows how bUAG property can be 'upgraded' to the UAG and ISS properties.

Lemma 3.5. *Let $\Sigma = (X, \mathcal{U}, \phi)$ be a control system with a BIC property. If Σ is UGS and bUAG, then Σ is forward compete, UAG and ISS.*

Proof. As Σ satisfies BIC property and is UGS, Σ is forward complete by Lemma 3.3 (in particular, the property bUAG assumed for Σ makes sense).

Pick arbitrary $\varepsilon > 0$, $r > 0$ and let τ and γ be as in the formulation of the bUAG property. Let $x \in B_r$ and let $u \in \mathcal{U}$ arbitrary. If $\|u\|_{\mathcal{U}} \le r$, then (11) is the desired estimate.

Let $\|u\|_{\mathcal{U}} > r$. Hence it holds that $\|u\|_{\mathcal{U}} > \|x\|_X$. Due to uniform global stability of Σ, it holds for all t, x, u that

$$\|\phi(t,x,u)\|_X \le \sigma(\|x\|_X) + \gamma(\|u\|_{\mathcal{U}}),$$

where we assume that γ is same as in the definition of a bUAG property (otherwise pick the maximum of both). For $\|u\|_{\mathcal{U}} > \|x\|_X$ we obtain that

$$\|\phi(t,x,u)\|_X \le \sigma(\|u\|_{\mathcal{U}}) + \gamma(\|u\|_{\mathcal{U}}),$$

and thus for all $x \in X$, $u \in \mathcal{U}$ it holds that

$$t \ge \tau \quad \Rightarrow \quad \|\phi(t,x,u)\|_X \le \varepsilon + \gamma(\|u\|_{\mathcal{U}}) + \sigma(\|u\|_{\mathcal{U}}),$$

which shows UAG property with the asymptotic gain $\gamma + \sigma$.

As Σ is forward complete, UAG and UGS, the ISS property of Σ follows by (Mironchenko and Wirth, 2018, Theorem 5). □

4. COUPLED SYSTEMS AND GAIN OPERATORS

Consider n forward complete systems

$$\Sigma_i := (X_i, PC_b(\mathbb{R}_+, X_{\neq i}) \times \mathcal{U}, \bar{\phi}_i), \quad i = 1, \dots, n,$$

where all X_i, $i = 1, \ldots, n$ and $\mathcal{U}$ are normed linear spaces.

Stability properties introduced in Section 3 are defined in terms of the norms of the whole input, and this is not suitable for consideration of coupled systems, as we are interested not only in the collective influence of all inputs over a subsystem, but in the influence of particular subsystems over a given subsystem.

Therefore we reformulate the ISS property for a subsystem in the following form:

Lemma 4.1. *A forward complete system Σ_i is* ISS (in summation formulation) *if there exist $\gamma_{ij}, \gamma_i \in \mathcal{K} \cup \{0\}$, $j = 1, \ldots, n$ and $\beta_i \in \mathcal{KL}$, such that for all initial values $x_i \in X_i$, all internal inputs $w_{\neq i} := (w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_n) \in PC_b(\mathbb{R}_+, X_{\neq i})$, all external inputs $u \in \mathcal{U}$ and all $t \in \mathbb{R}_+$ the following estimate holds:*

$$\|\bar{\phi}_i(t, x_i, (w_{\neq i}, u))\|_{X_i} \leq \beta_i \left(\|x_i\|_{X_i}, t\right) + \sum_{j \neq i} \gamma_{ij} \left(\|w_j\|_{[0,t]}\right) + \gamma_i \left(\|u\|_{\mathcal{U}}\right). \quad (12)$$

Proof. The argument is omitted due to page limits. Please consult Mironchenko (2019b) for the proof. □

The functions γ_{ij} and γ_i in the statement of Lemma 4.1 are called *(nonlinear) gains*. For notational simplicity we allow the case $\gamma_{ij} \equiv 0$ and require $\gamma_{ii} \equiv 0$ for all i.

Analogously, one can restate the definition of UGS:

Lemma 4.2. *Σ_i is* UGS (in summation formulation) *if and only if there exist γ_{ij}, $\gamma_i \in \mathcal{K} \cup \{0\}$ and $\sigma_i \in \mathcal{KL}$, such that for all initial values $x_i \in X_i$, all internal inputs $w_{\neq i} := (w_1, \ldots, w_{i-1}, w_{i+1}, \ldots, w_n) \in PC_b(\mathbb{R}_+, X_{\neq i})$, all $u \in \mathcal{U}$ and all $t \in \mathbb{R}_+$ the following inequality holds*

$$\|\bar{\phi}_i(t, x_i, (w_{\neq i}, u))\|_{X_i} \leq \sigma_i \left(\|x_i\|_{X_i}\right) + \sum_{j \neq i} \gamma_{ij}\left(\|w_j\|_{[0,t]}\right) + \gamma_i \left(\|u\|_{\mathcal{U}}\right). \quad (13)$$

We are going to collect all internal gains in the matrix $\Gamma := (\gamma_{ij})_{i,j=1,\ldots,n}$, which we call the *gain matrix*. If the gains are taken from the ISS restatement (12), then we call the corresponding gain matrix Γ^{ISS}.

Now for a given gain matrix Γ define the *gain operator* $\Gamma_\oplus : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ for all $s = (s_1, \ldots, s_n)^T \in \mathbb{R}^n_+$ by

$$\Gamma_\oplus s := \Big(\sum_{j=1}^n \gamma_{1j}(s_j), \ldots, \sum_{j=1}^n \gamma_{nj}(s_j)\Big)^T. \quad (14)$$

Again, in order to emphasize that the gains are from the ISS restatement (12), the corresponding gain operator will be denoted by $\Gamma^{ISS}_\oplus$.

As $\gamma_{ij} \in \mathcal{K}_\infty$ for all $s_1, s_2 \in \mathbb{R}^n_+$, we have the implication

$$s_1 \geq s_2 \quad \Rightarrow \quad \Gamma_\oplus(s_1) \geq \Gamma_\oplus(s_2), \quad (15)$$

so that $\Gamma_\oplus$ defines a monotone (w.r.t. the partial order $\geq$ in $\mathbb{R}^n$) map.

To guarantee stability of the interconnection Σ, the properties of the operators $\Gamma^{UGS}_\oplus$ and $\Gamma^{ISS}_\oplus$ will be crucial.

For given $\alpha_i \in \mathcal{K}_\infty, i = 1, \ldots, n$ define $D : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ for any $s = (s_1, \ldots, s_n)^T \in \mathbb{R}^n_+$ by

$$D(s) := ((\mathrm{Id} + \alpha_1)(s_1), \ldots, (\mathrm{Id} + \alpha_1)(s_1))^T. \quad (16)$$

A fundamental role will be played by the following operator conditions:

Definition 4.3. *We say that a nonlinear operator $A : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ satisfies the* strong small-gain condition*, if there exists a mapping D as in* (16)*, such that*

$$(A \circ D)(s) \not\geq s, \qquad \forall s \in \mathbb{R}^n_+ \setminus \{0\}. \quad (17)$$

5. SMALL-GAIN THEOREMS FOR CONTROL SYSTEMS

In this section we show small-gain theorems for UGS and ISS properties. We start with a small-gain theorem which guarantees that a coupling of UGS systems is a UGS system again provided the strong small-gain condition (17) holds. This will in particular show that the coupled system is forward complete.

Theorem 5.1 (UGS Small-gain theorem)**.** *Let $\Sigma_i := (X_i, PC_b(\mathbb{R}_+, X_{\neq i}) \times \mathcal{U}, \bar{\phi}_i)$, $i = 1, \ldots, n$ be control systems, where all X_i, $i = 1, \ldots, n$ and $\mathcal{U}$ are normed linear spaces. Assume that Σ_i, $i = 1, \ldots, n$ are forward complete systems, satisfying the UGS estimates as in Lemma 4.2, and that the interconnection $\Sigma = (X, \mathcal{U}, \phi)$ is well-defined and possesses the BIC property.*

If $\Gamma^{UGS}_\oplus$ satisfies the strong small gain condition (17)*, then Σ is forward complete and UGS.*

Proof. The argument is close to the proof of (Dashkovskiy et al., 2007, Theorem 8) and is omitted. Please consult Mironchenko (2019b) for the proof. □

The main result of this paper is

Theorem 5.2 (ISS Small-gain theorem)**.** *Let $\Sigma_i := (X_i, PC_b(\mathbb{R}_+, X_{\neq i}) \times \mathcal{U}, \bar{\phi}_i)$, $i = 1, \ldots, n$ be control systems, where all X_i, $i = 1, \ldots, n$ and $\mathcal{U}$ are normed linear spaces. Assume that Σ_i, $i = 1, \ldots, n$ are forward complete systems, satisfying the ISS estimates as in Lemma 4.1, and that the interconnection $\Sigma = (X, \mathcal{U}, \phi)$ is well-defined and possesses the BIC property.*

If $\Gamma^{ISS}_\oplus$ satisfies the strong small gain condition (17)*, then Σ is ISS.*

Proof. From the assumptions of the theorem it follows that all Σ_i are UGS with the gain matrix Γ^{ISS}. As $\Gamma^{ISS}_\oplus$ satisfies the strong small gain condition, Theorem 5.1 shows that the coupled system Σ is forward complete and UGS.

The proof of the bUAG property of the interconnection is omitted due to the space restrictions, see the journal version of this paper Mironchenko (2019b) for details.

As Σ is UGS $\wedge$ bUAG, Lemma 3.5 shows ISS of Σ. □

Theorem 5.2 is very general and applicable for networks of heterogeneous infinite-dimensional systems, consisting of components belonging to different system classes, with boundary and in-domain couplings, as long as these systems are well-defined and possess the BIC property. In particular, our ISS small-gain theorem is applicable to the interconnections of ISS ODE systems, interconnections of evolution equation in Banach spaces and couplings of n time-delay systems. In all these cases the interconnections possess the BIC property, see Karafyllis and Jiang (2011).

Specialized to couplings of ODE systems, Theorem 5.2 boils down to the classic small-gain theorem shown in Dashkovskiy et al. (2007). For couplings of n time-delay systems Theorem 5.2 is new. Unlike Polushin et al. (2006); Tiwari et al. (2009); Polushin et al. (2013), we obtain not only small-gain results for the weak ISS property (i.e. for a combination of uniform global stability and asymptotic gain properties), but also the ISS small-gain theorems. Compared to ISS small-gain theorems obtained in Tiwari et al. (2012), our approach provides an alternative way for verification of ISS of a network of delay systems, which does not require a transformation of retarded systems into delay-free ones and is applicable to the summation formulation of the ISS property.

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