Small-gain theorems for stability of infinite networks

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***Abstract*— We derive a small-gain theorem for infinite networks of input-to-state stable systems. It shows that the whole interconnection is ISS provided that the gains, characterizing the interconnection structure satisfy a version of a small-gain condition, and some other natural conditions hold. For the special case of networks with linear gains, we show that the network is ISS provided that the gain operator is compact and non-increasing at any nonzero element of a positive cone.**

Keywords: input-to-state stability, large-scale interconnections, infinite-dimensional systems, small-gain theorems.

I. Introduction

This work is a continuation of the paper [25], where the small-gain stability criteria for couplings of $n \in \mathbb{N}$ infinite-dimensional systems have been obtained. In this paper we extend some of those results to couplings of countably many systems.

Stability and control of infinite interconnections have received significant attention during the last decades. In particular, a large body of literature is devoted to spatially invariant systems consisting of an infinite number of components, interconnected with each other by means of the same pattern [2], [7], [3], [5], etc. Recently a development of stability criteria for infinite interconnections without spatial invariance assumption has been initiated in [9], [10]. Methods, developed in [9], [10] are based on the input-to-state stability paradigm, which is the backbone for a great deal of the nonlinear control theory, including robust stabilization, nonlinear observer design and analysis of large-scale networks, see [23], [1], [31]. Powerful nonlinear small-gain theorems, developed during the last two decades for couplings of $n \in \mathbb{N}$ input-to-state stable ordinary differential equations (ODEs) [16], [15], [12], [11], time-delay systems [29], [35], [28], [36] and distributed parameter systems [8], [26], [4], [25], are especially relevant in the context of coupled systems. We refer to [25] for more details and references.

Several new challenges appear in the case of infinite interconnections. Even if each subsystem is finite-dimensional, the whole network is infinite-dimensional which makes the analysis of the ISS of the whole system much more involved, as many results and criteria valid in ODE setting become wrong for infinite-dimensional systems, see [24], [27]. In particular, a crucial step for the proof of small-gain theorems for couplings of ODE systems in [12], is the equivalence between ISS and a combination of global stability and asymptotic gain properties, shown in [32], [33]. *The characterizations from [32], [33] are not applicable to infinite-dimensional systems*, which significantly complicates the proof of small-gain theorems in case if the overall coupled system is infinite-dimensional. This obstacle has been overcome: in [27], [24] novel criteria for ISS of distributed parameter systems have been developed and in [25] they have been used for stability analysis of couplings of a finite number of infinite-dimensional systems. For a recent account of ISS theory of infinite-dimensional systems we refer to [27], [14], [21], [38], [34], [25] and references therein.

Another obstacle, arising in dealing with infinite networks is in the fact that the gain operator, which collects all the information about the internal gains, acts in the infinite-dimensional space as well, in contrast to couplings of $n \in \mathbb{N}$ systems of arbitrary nature. We are going to treat this problem in this paper.

A. Contribution

Our main results are ISS small-gain theorems (in trajectory formulation) for well-posed interconnections of an infinite number of heterogeneous control systems whose components belong to a broad class of control systems covering PDEs, time-delay systems, ODE etc.

First in Theorem 3.1 we state a nonlinear small-gain theorem, showing ISS of an infinite interconnection under assumptions that each subsystem is ISS and the gains satisfy a certain condition, which is roughly speaking a monotone invertibility of an operator $I - \Gamma_\otimes$, where $\Gamma_\otimes$ is the gain operator. For finite interconnections this condition is implied by the strong small-gain condition, see [12, Lemma 13].

Next we investigate infinite couplings with linear gains. Using Krein-Rutman theorem we show that monotone invertibility property of $I - \Gamma_\otimes$ holds, if $\Gamma_\otimes$ is compact and non-increasing at any nonzero element of a positive cone. We show by means of a counterexample that compactness cannot be omitted.

B. Relations to other results

Stability of infinite interconnections has been studied in [9], [10], where it was assumed that each subsystem of an infinite network is finite-dimensional, input-to-state stable and the ISS Lyapunov functions for all subsystems are known together with so-called Lyapunov gains, characterizing the interconnection structure of subsystems. On the basis of

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this information it was shown that if the Lyapunov gains satisfy a certain type of small-gain conditions, then the whole interconnection is input-to-state stable.

The main differences in compare to the papers [9], [10] are due to the fact that we consider couplings of general infinite-dimensional systems and not only of ODEs, and secondly we study small-gain theorems in terms of trajectories, whereas in [9], [10] the small-gain theorems in terms of Lyapunov functions have been studied. Secondly, the small-gain conditions which we impose are different from those in [9], [10], and it is an interesting problem for the future research to find the relationships between these two conditions and in this way to unify the small-gain results for infinite networks.

Recently in [4] the small-gain theorems for couplings of n input-to-output stable (IOS) evolution equations in Banach spaces have been derived. As a special case of these results, the authors obtain a small-gain theorem for networks of n ISS systems in the maximum formulation. Application of small-gain theorems for stability analysis of coupled parabolic-hyperbolic PDEs has been performed in [20]. Small-gain based boundary feedback design for global exponential stabilization of 1-D semilinear parabolic PDEs has been proposed in [19].

C. Notation

In the following $\mathbb{R}_+ := [0,\infty)$.

For a function $v:\mathbb{R}_+ \to X$, where X is a certain set, we define its restriction to the interval $[s_1,s_2]$ by

$$v_{[s_1,s_2]}(t) := \begin{cases} v(t) & \text{if } t \in [s_1,s_2],\\ 0 & \text{else,}\end{cases}$$

and by $\|v\|_{[0,t]}$ we denote the supremum norm of $v_{[0,t]}$, i.e. $\|v\|_{[0,t]} := \sup_{s\in[0,t]} \|v(s)\|_X$.

By ℓ_∞ we denote the Banach space of sequences $\{x_i\}_{i\in\mathbb{N}}$, $x_i \in \mathbb{R}$ so that there is $M>0$ with $|x_i| \le M$ for all $i \in \mathbb{N}$. By ℓ_∞^+ we denote the set of sequences $x = \{x_i\}_{i\in\mathbb{N}} \in \ell_\infty$: $x_i \ge 0$ for all $i \in \mathbb{N}$. ℓ_∞^+ is a generating cone in ℓ_∞, which endows ℓ_∞ with a natural order $\ge$.

By "$\not\geq$" we understand the logical negation of "$\geq$". In particular, for $x,y \in \ell_\infty$ the condition $x \not\geq y$ means that $\exists i \in \mathbb{N}$ so that $x_i < y_i$.

We use the following classes of comparison functions

$$\begin{aligned}
\mathcal{K} &:= \{\gamma:\mathbb{R}_+\to\mathbb{R}_+ \mid \gamma(0)=0,\ \gamma \text{ is continuous and strictly increasing}\}\\
\mathcal{K}_\infty &:= \{\gamma\in\mathcal{K} \mid \gamma \text{ is unbounded}\}\\
\mathcal{L} &:= \{\gamma:\mathbb{R}_+\to\mathbb{R}_+ \mid \gamma \text{ is continuous and strictly decreasing with } \lim_{t\to\infty}\gamma(t)=0\}\\
\mathcal{KL} &:= \{\beta:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+ \mid \beta \text{ is continuous, } \beta(\cdot,t)\in\mathcal{K},\ \beta(r,\cdot)\in\mathcal{L},\ \forall t,r\ge 0\}
\end{aligned}$$

II. Problem formulation

A. Definition of control systems

We define the concept of a (time-invariant) system in the following way:

Definition 2.1: Consider the triple $\Sigma = (X,\mathcal{U},\phi)$ consisting of

(i) A normed vector space $(X,\|\cdot\|_X)$, called the *state space*, endowed with the norm $\|\cdot\|_X$.

(ii) A set of input values U, which is a nonempty subset of a certain normed vector space.

(iii) A *space of inputs* $\mathcal{U} \subset \{f:\mathbb{R}_+\to U\}$ endowed with a norm $\|\cdot\|_{\mathcal{U}}$ which satisfies the following two axioms:
The axiom of shift invariance: for all $u\in\mathcal{U}$ and all $\tau\ge 0$ the time shift $u(\cdot+\tau)$ belongs to $\mathcal{U}$ with $\|u\|_{\mathcal{U}} \ge \|u(\cdot+\tau)\|_{\mathcal{U}}$.
The axiom of concatenation: for all $u_1,u_2\in\mathcal{U}$ and for all $t>0$ the concatenation of u_1 and u_2 at time t

$$u(\tau) := \begin{cases} u_1(\tau), & \text{if } \tau\in[0,t],\\ u_2(\tau-t), & \text{otherwise,}\end{cases} \tag{1}$$

belongs to $\mathcal{U}$.

(iv) A map $\phi: D_\phi \to X$, $D_\phi \subseteq \mathbb{R}_+\times X\times\mathcal{U}$ (called *transition map*), so that for all $(x,u)\in X\times\mathcal{U}$ there is a $t>0$ so that $[0,t]\times\{(x,u)\}\subset D_\phi$.

The triple Σ is called a *(control) system*, if the following properties hold:

(Σ1) *The identity property:* for every $(x,u)\in X\times\mathcal{U}$ it holds that $\phi(0,x,u)=x$.

(Σ2) *Causality:* for every $(t,x,u)\in\mathbb{R}_+\times X\times\mathcal{U}$, for every $\tilde{u}\in\mathcal{U}$, such that $u(s)=\tilde{u}(s)$ for all $s\in[0,t]$ it holds that $\phi(t,x,u)=\phi(t,x,\tilde{u})$.

(Σ3) *Continuity:* for each $(x,u)\in X\times\mathcal{U}$ the map $t\mapsto\phi(t,x,u)$ is continuous.

(Σ4) *The cocycle property:* for all $x\in X$, $u\in\mathcal{U}$, for all $t,h\ge 0$ so that $[0,t+h]\times\{(x,u)\}\subset D_\phi$, we have $\phi(h,\phi(t,x,u),u(t+\cdot)) = \phi(t+h,x,u)$.

This class of systems encompasses control systems generated by ordinary differential equations (ODEs), switched systems, time-delay systems, many classes of partial differential equations (PDEs), important classes of boundary control systems and many other systems.

Next we define several important properties of control systems:

Definition 2.2: We say that a control system (as introduced in Definition 2.1) is *forward complete (FC)*, if $D_\phi = \mathbb{R}_+\times X\times\mathcal{U}$, that is for every $(x,u)\in X\times\mathcal{U}$ and for all $t\ge 0$ the value $\phi(t,x,u)\in X$ is well-defined.

The main concept in this paper is:

Definition 2.3: A system $\Sigma=(X,\mathcal{U},\phi)$ is called *(uniformly) input-to-state stable (ISS)*, if there exist $\beta\in\mathcal{KL}$ and $\gamma\in\mathcal{K}$ such that for all $(t,x,u)\in D_\phi$ it holds that

$$\|\phi(t,x,u)\|_X \le \beta(\|x\|_X,t) + \gamma(\|u\|_{\mathcal{U}}). \tag{2}$$

An important property of ordinary differential equations with Lipschitz continuous right-hand sides states that if the solution stays bounded over $[0,t)$, then it can be prolonged to $[0,t+\varepsilon)$ for a certain $\varepsilon>0$. Similar properties have evolution equations in Banach spaces with bounded control operators and Lipschitz continuous right hand sides [6, Theorem 4.3.4] and many other classes of systems [18, Chapter 1]. The next property, adopted from [18, Definition 1.4] formalizes this behavior for general control systems.

Definition 2.4: We say that a system Σ satisfies the *boundedness-implies-continuation (BIC) property* if for each $(x,u) \in X \times \mathscr{U}$, there exists $t_{\max} \in (0,+\infty]$, called a *maximal existence time*, such that $[0,t_{\max}) \times \{(x,u)\} \subset D_\phi$ and for all $t \geq t_{\max}$, it holds that $(t,x,u) \notin D_\phi$. In addition, if $t_{\max} < +\infty$, then for every $M > 0$, there exists $t \in [0,t_{\max})$ with $\|\phi(t,x,u)\|_X > M$.

Lemma 2.5: Assume that Σ is ISS and possesses the BIC property. Then Σ is forward complete.

B. Orders and monotonicity

Consider a normed linear space X. For two sets $A,B \subset X$ we define $A+B := \{a+b : a \in A, b \in B\}$, $-A := \{-a : a \in A\}$, and $\mathbb{R}_+ \cdot A := \{r \cdot a : a \in A, r \in \mathbb{R}_+\}$. Given a topology, by $\operatorname{int} A$ we denote the *interior of* A and by $\overline{A}$ its *closure*.

Definition 2.6: We say that $K \subset X$ is a positive cone in X if $K \cap (-K) = \{0\}$, $\mathbb{R}_+ K \subset K$ and $K + K \subset K$.

The cone K introduces the partial order relation $\leq$ on X as $x \leq y \;\Leftrightarrow\; y - x \in K$.

Definition 2.7: Let X be a Banach space ordered by a cone K. Then (X,K) is called an *ordered Banach space (OBS)*, if the cone K is closed.

Definition 2.8: Let X be OBS, ordered via a cone K.

- The cone K is called *total* if $\overline{K + (-K)} = X$ and *generating* if $K + (-K) = X$.
- The norm in X is called *monotone*, if there for any $x_1, x_2 \in X$: $0 \leq x_1 \leq x_2$ it follows that $\|x_1\|_X \leq \|x_2\|_X$.

Having defined an order on a set, it is natural to consider maps on this set, which preserve the order in a certain sense.

Definition 2.9: Let X,Y be ordered sets with positive cones K and Q, respectively. The map $f : X \to Y$ is called:

(i) *monotone*, if $x \leq y$ implies $f(x) \leq f(y)$ for all $x,y \in X$.
(ii) *positive* if $f(K) \subset Q$
(iii) *strictly positive* if $f(K \setminus \{0\}) \subset Q \setminus \{0\}$

C. Infinite Interconnections

Let $(X_i, \|\cdot\|_{X_i})$, $i \in \mathbb{N}$ be normed vector spaces endowed with the corresponding norms. For all $i \in \mathbb{N}$ define the set $X_{\neq i}$, consisting of all elements of the space $X_1 \times \ldots \times X_{i-1} \times X_{i+1} \times \ldots$ with the finite norm

$$\|x\|_{X_{\neq i}} := \sup_{j \in \mathbb{N},\ j \neq i} \big\{\|x_j\|_{X_j}\big\}. \tag{3}$$

It is easy to see that $X_{\neq i}$ is a normed vector space and a linear subspace in $X_1 \times \ldots \times X_{i-1} \times X_{i+1} \times \ldots$.

Let control systems $\Sigma_i := (X_i, PC(\mathbb{R}_+, X_{\neq i}) \times \mathscr{U}, \bar{\phi}_i)$, $i \in \mathbb{N}$ be given and assume that each Σ_i possesses the BIC property. We call $X_{\neq i}$ the *space of internal input values*, $PC(\mathbb{R}_+, X_{\neq i})$ the *space of internal inputs*. The norm on $PC(\mathbb{R}_+, X_{\neq i}) \times \mathscr{U}$ we define as

$$\|(v,u)\|_{PC(\mathbb{R}_+,X_{\neq i}) \times \mathscr{U}} := \max\{\sup_{j \in \mathbb{N} \setminus \{i\}} \|v_j\|_{PC(\mathbb{R}_+,X_j)}, \|u\|_{\mathscr{U}}\}. \tag{4}$$

Define also the state space X for the whole system as a set of all elements in $X_1 \times X_2 \times X_3 \times \ldots$, which have a finite norm

$$\|x\|_X := \sup_{i \in \mathbb{N}} \|x_i\|_{X_i}. \tag{5}$$

Clearly, X is a normed vector space and a linear subspace in $X_1 \times X_2 \times X_3 \times \ldots$. Assume that there is a map $\phi = (\phi_1, \ldots, \phi_n, \ldots) : D_\phi \to X$, defined over a certain domain $D_\phi \subseteq \mathbb{R}_+ \times X \times \mathscr{U}$ so that for each $x = (x_1, x_2, \ldots) \in X$, each $u \in \mathscr{U}$ and all $t \in \mathbb{R}_+$ so that $(t,x,u) \in D_\phi$ and for every $i \in \mathbb{N}$, it holds that

$$\phi_i(t,x_i,u) = \bar{\phi}_i\big(t,x_i,(v_i,u)\big), \tag{6}$$

where

$$v_i(t) = (\phi_1(t,x,u), \ldots, \phi_{i-1}(t,x,u), \phi_{i+1}(t,x,u), \phi_{i+2}(t,x,u), \ldots).$$

Assume further that $\Sigma := (X, \mathscr{U}, \phi)$ is a control system with the state space X, input space $\mathscr{U}$ and with a BIC property. Then Σ is called *a (feedback) interconnection* of systems Σ_i, $i \in \mathbb{N}$.

In other words, condition (6) means that if the modes $\phi_j(\cdot,x,u)$, $j \neq i$ of the system Σ are sent to Σ_i as the internal inputs (together with an external input u), and the initial state is chosen as x_i (the i-th mode of x), then the resulting trajectory of the system Σ_i, which is $\bar{\phi}_i(\cdot,x_i,v,u)$ coincides with the trajectory of the i-th mode of the system Σ on the interval of existence of ϕ_i.

Remark 2.10: This definition of feedback interconnections, which we adopted from [17, Definition 3.3], does not depend on a particular type of control systems which are coupled, and is applicable to large-scale systems, consisting of heterogeneous components as PDEs, time-delay systems, ODE systems etc. The definition also applies to different kinds of interconnections, e.g. both for in-domain and boundary interconnections of PDE systems.

D. Stability properties for infinite couplings

In this subsection we consider a countable family of forward complete systems $\Sigma_i := (X_i, PC(\mathbb{R}_+, X_{\neq i}) \times \mathscr{U}, \bar{\phi}_i)$, $i \in \mathbb{N}$, where all X_i, $i \in \mathbb{N}$ and $\mathscr{U}$ are normed linear spaces.

Stability properties introduced previously are defined in terms of the norms of the whole input, and this is not suitable for consideration of coupled systems, as we are interested not only in the collective influence of all inputs over a subsystem, but in the influence of particular subsystems over a given subsystem.

Definition 2.11: We say that a forward complete system Σ_i is *input-to-state stable (ISS) in a semimaximum formulation*, if there exist $\gamma_{ij}, \gamma_i \in \mathscr{K} \cup \{0\}$, $j \in \mathbb{N}$ and $\beta_i \in \mathscr{KL}$, such that for all initial values $x_i \in X_i$, all internal inputs $w_{\neq i} := (w_1, \ldots, w_{i-1}, w_{i+1}, w_{i+2}, \ldots) \in PC(\mathbb{R}_+, X_{\neq i})$, all external inputs $u \in \mathscr{U}$ and all $t \in \mathbb{R}_+$ the following estimate holds:

$$\begin{aligned}&\|\bar{\phi}_i\big(t,x_i,(w_{\neq i},u)\big)\|_{X_i} \\ &\quad \leq \beta_i\left(\|x_i\|_{X_i}, t\right) + \sup_{j \in \mathbb{N} \setminus \{i\}} \{\gamma_{ij}\big(\|w_j\|_{[0,t]}\big)\} + \gamma_i\big(\|u\|_{\mathscr{U}}\big).\end{aligned} \tag{7}$$

If all γ_{ij}, $j \in \mathbb{N}$ are linear functions, we say that Σ_i *is ISS (in semimaximum formulation) with linear gains.*

Above gain structure was called in [25] the semimaximum formulation of the ISS property. There are also other formulations of the ISS property, e.g. summation and maximum formulations. These formulations are equivalent for finite interconnections, but are not equivalent for infinite couplings. The functions γ_{ij} and γ_i in the statement of Definition 2.11 are called *(nonlinear) gains*. For notational simplicity we allow the case $\gamma_{ij} \equiv 0$ and require $\gamma_{ii} \equiv 0$ for all i.

We are going to collect all the internal gains in the matrix $\Gamma^{ISS} := (\gamma_{ij})_{i,j\in\mathbb{N}}$, which we call the *gain matrix*.

Now for a given gain matrix Γ^{ISS} define for any $s = (s_1, s_2, \dots)^T \in \ell_\infty^+$ the operator $\Gamma_\otimes^{ISS} : \ell_\infty^+ \to \ell_\infty^+$ by

$$\Gamma_\otimes^{ISS}(s) := \Big(\sup_{j\in\mathbb{N}}\{\gamma_{1j}(s_j)\}, \sup_{j\in\mathbb{N}}\{\gamma_{2j}(s_j)\}, \dots \Big)^T. \tag{8}$$

In what follows we assume that there exist $\beta \in \mathcal{KL}$ and $\gamma, \bar{\gamma} \in \mathcal{K}$ so that for all $r \in \mathbb{R}_+,\ t \geq 0,\ i,j \in \mathbb{N}$ it holds that

$$\beta_i(r,t) \leq \beta(r,t), \quad \gamma_i(r) \leq \gamma(r), \quad \gamma_{ij}(r) \leq \bar{\gamma}(r). \tag{9}$$

The last condition guarantees that $\Gamma_\otimes^{ISS}$ is well-defined.

Note that by the properties of γ_{ij} for $s_1, s_2 \in \ell_\infty^+$ we have the implication

$$s_1 \geq s_2 \quad \Rightarrow \quad \Gamma_\otimes^{ISS}(s_1) \geq \Gamma_\otimes^{ISS}(s_2), \tag{10}$$

so that $\Gamma_\otimes^{ISS}$ defines a monotone (w.r.t. the order $\geq$ in ℓ_∞) map.

In order to guarantee stability of the interconnection Σ, the properties of the operator $\Gamma_\otimes^{ISS}$ are central.

III. Small-gain theorems for infinite interconnections

In this section we state a general small-gain theorem for infinite interconnections.

Theorem 3.1 (ISS Small-gain theorem): Let $\Sigma_i := (X_i, PC(\mathbb{R}_+, X_{\neq i}) \times \mathcal{U}, \bar{\phi}_i)$, $i \in \mathbb{N}$ be control systems, where all X_i, $i \in \mathbb{N}$ and $\mathcal{U}$ are normed linear spaces. Suppose that Σ_i, $i \in \mathbb{N}$ are forward complete, satisfy the ISS estimates as in Definition 2.11, and that the interconnection $\Sigma = (X, \mathcal{U}, \phi)$ is well-defined and possesses the BIC property.

Furthermore, let the following conditions hold:

(i) There are $\beta \in \mathcal{KL}$ and $\gamma, \bar{\gamma} \in \mathcal{K}$ so that (9) holds.

(ii) there exists a $\xi \in \mathcal{K}_\infty$ such that for all $w, v \in \ell_\infty^+$ the inequality
$$(\mathrm{Id} - \Gamma_\otimes^{ISS})(w) \leq v \tag{11}$$
implies $\|w\|_{\ell_\infty} \leq \xi(\|v\|_{\ell_\infty})$.

Then Σ is ISS.

Proof: The proof follows closely the corresponding proof for finite networks, see [25] and consists of three steps. First one shows so-called uniform global stability (UGS) of the coupled system, then one shows the uniform asymptotic gain (UAG) property of the interconnection, and finally one uses the equivalence between UGS $\wedge$ UAG and ISS, which is special case of much stronger characterizations of ISS, shown in [27]. We omit the details due to space reasons. ■

Remark 3.2: The condition (ii) in the formulation of Theorem 3.1 seems technical and the naming of the above result as a small-gain theorem may look strange as no small-gain condition is assumed. However, for finite networks the strong small-gain condition implies (ii), as shown in [12, Lemma 13] and in Theorem 4.7 we will see that for infinite networks with linear internal gains the condition (ii) will be implied by the small-gain condition provided the operator $\Gamma_\otimes^{ISS}$ is compact.

Example 3.3 (Finite interconnections): Having n coupled systems Σ_i, $i = 1, \dots, n$, we can formally introduce the systems Σ_i, $i > n$, which are disconnected from other systems, have no inputs and are uniformly exponentially stable (e.g. one could pick $\dot{x}_i = -x_i$, $i > n$). Then ISS of the interconnection of all $\{\Sigma_i\}_{i\in\mathbb{N}}$ is equivalent to ISS of the finite coupling $\{\Sigma_i\}_{i=1}^n$. Thus, we can treat finite interconnections as a special case of our results on infinite couplings. The infinite matrix Γ^{ISS} is in this case a block-diagonal matrix with a nonzero $n \times n$ component in the upper left corner, and with zero entries otherwise.

In the case of finite interconnections it is known that if $\Gamma_\otimes^{ISS}$ satisfies the small-gain condition, then the assumption (ii) is fulfilled. The condition (i) always holds with $\sigma := \max\{\sigma_1, \dots, \sigma_n\}$.

Thus, Theorem 3.1 is a natural extension of the small-gain theorems for the finite interconnections of ISS systems, see [25, Theorem 5.1] and [12, Theorem 8].

For infinite interconnections the condition (i) cannot be dropped in the formulation of Theorem 3.1, and the condition (ii) cannot be in general substituted to the strong small-gain condition, as we analyze next.

Example 3.4 (Importance of the assumption (i)): Consider the system

$$\dot{x}_i = -\frac{1}{i} x_i, \quad i \in \mathbb{N}, \tag{12}$$

as an interconnection of $|\mathbb{N}|$ x_i-systems with $x_i(t) \in \mathbb{R}$, for each i and with the state space ℓ_∞ of the whole system. As there are no couplings between subsystems, the gain matrix is $\Gamma^{ISS} := 0$ and the assumption (ii) of Theorem 3.1 trivially holds. However, as $x_i(t) = e^{-\frac{1}{i}t} x_i(0)$, we have that $\beta_i(r,t) := e^{-\frac{1}{i}t} r$, $i \in \mathbb{N}$, and clearly, there is no $\beta \in \mathcal{KL}$ so that $\beta_i(r,t) \leq \beta(r,t)$ for all $r, t \in \mathbb{R}_+$.

It is easy to check that for $x_0 := (1, 1, \dots)$ it holds that $\|\phi(t, x_0)\|_{\ell_\infty} = 1$ for all times $t \geq 0$, and thus (12) is not UGAS.

IV. Interconnections with linear gains

A. Stability of discrete-time systems

Let X be a normed vector space and let $A \in L(X)$. Consider a linear *discrete-time system*

$$x(k+1) = Ax(k), \quad k \in \mathbb{N}. \tag{13}$$

The solution of this system at time $k \in \mathbb{N}$ subject to an initial condition $x \in X$ we denote by $\phi(k, x) \in X$.

Definition 4.1: Discrete-time system (13) is called *uniformly globally exponentially stable (UGES)*, if there are $a \in (0,1)$ and $M > 0$ so that for each $k \geq 0$ and all $x \in X$ it holds that

$$\|\phi(k,x)\|_X \leq M a^k \|x\|_X. \tag{14}$$

We will use the famous theorem due to Krein and Rutman [22].

Theorem 4.2 (Krein-Rutman): Let (X,K) be an OBS with a total positive cone K. Suppose that $A \in L(X)$ is compact, positive and has a positive spectral radius $\rho(A)$. Then $\rho(A)$ is an eigenvalue of A and of the dual operator A^* with eigenvectors in K and in K^*, respectively.

Next we state a useful criterion for UGES of a discrete-time system (13). The result seems to be quite classic, but we give a short argument for the sake of completeness.

Proposition 4.3: Let (X,K) be a normed vector space with an order given by a positive cone K. Further, let $A \in L(X)$ be monotone with respect to the order in (X,K).

Consider the following statements

(i) $\rho(A) < 1$,

(ii) $A^k \to 0$, for $k \to \infty$,

(iii) (13) is UGES

(iv) there exists a strictly positive operator $P \in L(X)$ so that

$$A(I+P)x \not\geq x, \quad \forall x \in K \setminus \{0\}. \tag{15}$$

(v) The following condition holds

$$Ax \not\geq x, \quad \forall x \in K \setminus \{0\}. \tag{16}$$

Then the following holds:

(A) (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$(iii) $\Rightarrow$ (iv) $\Rightarrow$ (v).

(B) Furthermore, if (X,K) is an OBS, K is a total cone and $A \in L(X)$ is a compact positive operator with a positive spectral radius $\rho(A)$, then all above statements (i)-(v) are equivalent.

Proof: **(A).** The equivalence between items (i), (ii), (iii) is well-known, see e.g. [30, Theorem 2.1, p. 516].

Let (i) holds. Then there is $\varepsilon > 0$ so that $\rho((1+\varepsilon)A) < 1$ and by the equivalence between (i) and (iii), the system (13) with $\tilde{A} := (1+\varepsilon)A$ instead of A is UGES. Furthermore, $\tilde{A} = A(I+\varepsilon I)$ is monotone and εI is a strictly positive operator.

Now, let (iv) does not hold. Then there is a certain $x \in X$ so that $\tilde{A}x \geq x$. But then in view of monotonicity of $\tilde{A}$ with respect to the order generated by K, we have that

$$\tilde{A}^n x \geq \tilde{A}^{n-1} x \geq \ldots \geq x,$$

for all $n \in \mathbb{N}$, and thus the solution of (13) governed by the operator $\tilde{A}$ does not converge to 0 for an initial condition x, which contradicts to UGES of this system.

(iv) $\Rightarrow$ (v). Assume that (v) does not hold, and thus there is $y \in K \backslash \{0\}$ so that $Ay \geq y$. Then for any positive operator $P \in L(X)$ it holds that $(I+P)y = y + Py \geq y$ (we use here that the order on X is linear) and as A is an increasing operator it holds that $A(I+P)y \geq Ay \geq y$. This contradicts to (iv).

(B). Now assume that (X,K) is an OBS, K is a total cone and $A \in L(X)$ is a compact operator with a positive spectral radius $\rho(A)$. We are going to show that the implication (v) $\Rightarrow$ (i) holds. Assume that this is not the case, and (v) holds but $\rho(A) \geq 1$. By Krein-Rutman Theorem 4.2 it holds that $\rho(A)$ is an eigenvalue of A and the corresponding eigenvector $x \in K$. Thus, $Ax = \rho(A)x \geq x$, which contradicts to (v). ■

Example 4.4: In this example we show that without additional assumptions in the part (B) in Proposition 4.3, the property (v) does not imply (iii). Consider a system over the state space $X := \ell_\infty$, with an order given by the cone $X^+ := \ell_\infty^+$:

$$x(k+1) = cRx(k), \quad k \in \mathbb{N}, \tag{17}$$

where $c \geq 1$ and the operator R is the right shift on X, that is for $x = (x_1, x_2, x_3, \ldots)$ it holds that $Rx := (0, x_1, x_2, x_3, \ldots)$.

First let $c := 1$. It is easy to see that $\|Rx\|_X = \|x\|_X$, i.e. R is an isometry, and thus $\|\phi(k,x)\|_X = \|x\|_X$ for all $k \in \mathbb{N}$, and hence (17) is not asymptotically stable. Furthermore, R is a positive operator over $(\ell_\infty, \ell_\infty^+)$.

Consider an arbitrary strictly positive diagonal operator $D : X \to X$, defined for $x := (x_1, x_2, \ldots) \in X$ by $Dx := (a_1 x_1, a_2 x_2, \ldots)$, where $\{a_i\}_{i \in \mathbb{N}}$ is a sequence of positive numbers with $a_i \leq M$ for certain $M > 0$ and all $i \in \mathbb{N}$.

Let $x = (x_1, x_2, \ldots) \in X^+ \backslash \{0\}$ and let i be the index of the first nonzero component of x (which is well-defined and finite as $x \in X^+$ and $x \neq 0$). Then the components of $R(I+D)x$ with indices $j = 1, \ldots, i$ are equal to 0, which shows that $R(I+D)x \not\geq x$.

In this example only one condition of Proposition 4.3, part (B) is not satisfied: R is not a compact operator, as the image of the unit ball is clearly not relatively compact. All other assumptions are fulfilled.

Finally, note (see [13, Example B.7]) that $\sigma(R) = \overline{B(0,1)}$, where $B(0,1)$ is an open ball of radius 1 with the center at 0 in the complex plane and at the same time the point spectrum of R is empty.

Analyzing as above (13) with $c > 1$, we obtain by a completely similar argument an exponentially unstable system with $A := cR$, which satisfies condition (iv) in Proposition 4.3.

Stability of (13) has the following important consequence:

Proposition 4.5: Assume that X is a Banach space with a closed cone and let $A \in L(X)$ be a positive operator with $\rho(A) < 1$. Then the operator $I - A$ is invertible and positive.

In particular, if $(I-A)x \leq y$ for certain $x, y \in X$, then it holds also $x \leq (I-A)^{-1} y$.

Proof: As X is a Banach space and $\rho(A) < 1$, the operator A is invertible and the inverse is given by the Neumann series: $(I-A)^{-1} = \sum_{k=0}^{\infty} A^k$, see e.g. [37, p. 69].

Pick any $x \in K$. As A is a positive operator, A^k is positive as well for any $k \in \mathbb{N}$ and as X is a linear space, it holds that $z_n := \sum_{k=0}^{n} A^k x \in K$. Furthermore, as the cone K is closed we have $(I-A)^{-1} x = \lim_{n \to \infty} z_n \in K$. ■

B. Small-gain theorems: case of linear gains

Assume as in Section III, that Σ_i, $i \in \mathbb{N}$ are forward complete systems, satisfying the ISS estimates as in Definition 2.11, and that the interconnection $\Sigma = (X, \mathcal{U}, \phi)$ is well-defined and possesses the BIC property. Furthermore, we assume that all internal gains γ_{ij} are linear functions. This enables us to use the results from Section IV-A to obtain a more usable small-gain criterion for ISS of the coupled system.

Denote by $\mathbb{R}_+^{\mathbb{N} \times \mathbb{N}}$ the set of infinite matrices $M = (m_{ij})_{i,j \in \mathbb{N}}$, $m_{ij} \in \mathbb{R}_+$. The next lemma gives a simple criterion of well-posedness of the linear operator $\Gamma_\otimes$.

Lemma 4.6: $\Gamma_\otimes$ is a well-defined positive linear bounded operator if and only if $\Gamma \in \mathbb{R}_+^{\mathbb{N}\times\mathbb{N}}$ with $\sup_{i,j\in\mathbb{N}} \gamma_{ij} =: C < \infty$. In this case $\|\Gamma_\otimes\|_{L(X)} = C$.

Proof: Clearly, $\Gamma_\otimes$ is a linear operator. For each $j \in \mathbb{N}$ it holds that $\Gamma_\otimes e_j = (\gamma_{1j}, \ldots, \gamma_{nj}, \ldots)^T$. Clearly,

$$\|\Gamma_\otimes\|_{L(X)} = \sup_{\|x\|_{\ell_\infty}=1} \|\Gamma_\otimes x\|_{\ell_\infty} \geq \sup_{j\in\mathbb{N}} \|\Gamma_\otimes e_j\|_{\ell_\infty} = \sup_{j\in\mathbb{N}} \sup_{i\in\mathbb{N}} |\gamma_{ij}| = C$$

On the other hand,

$$\|\Gamma_\otimes\|_{L(X)} = \sup_{\|x\|_{\ell_\infty}=1} \sup_{i\in\mathbb{N}} \sup_{j\in\mathbb{N}} |\gamma_{ij} x_j| \leq C \sup_{\|x\|_{\ell_\infty}=1} \sup_{j\in\mathbb{N}} |x_j| = C.$$

For a well-defined operator $\Gamma_\otimes$ the positivity is clearly equivalent to the fact that $\Gamma \in \mathbb{R}_+^{\mathbb{N}\times\mathbb{N}}$. ■

Net we present a criterion for ISS of countable networks with linear internal gains.

Theorem 4.7 (ISS small-gain theorem: linear gains): Let $\Sigma_i := (X_i, PC(\mathbb{R}_+, X_{\neq i}) \times \mathcal{U}, \bar{\phi}_i)$, $i \in \mathbb{N}$ be control systems, where all X_i, $i \in \mathbb{N}$ and $\mathcal{U}$ are normed linear spaces. Assume that Σ_i, $i \in \mathbb{N}$ are forward complete systems, satisfying the ISS estimates as in Definition 2.11 with linear internal gains γ_{ij}, $i, j \in \mathbb{N}$, and that the interconnection $\Sigma = (X, \mathcal{U}, \phi)$ is well-defined and possesses the BIC property. Furthermore, let the following conditions hold:

(i) There is a $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ so that (9) holds.
(ii) $\Gamma_\otimes^{ISS}$ is a compact operator.
(iii) $\Gamma_\otimes^{ISS}$ satisfies the small-gain condition

$$A(s) \not\geq s, \qquad \forall s \in \ell_\infty^+ \setminus \{0\}.$$

Then Σ is ISS.

Proof: To show the claim we need merely to show that the assumption (ii) of Theorem 3.1 holds. The operator $\Gamma_\otimes^{ISS} \in L(X)$ acts on ℓ_∞, which we endow with a cone ℓ_∞^+. It is well-known that ℓ_∞ is a Banach space, and ℓ_∞^+ is a closed and generating cone in ℓ_∞. Hence, ℓ_∞ is an ordered Banach space with a total cone. As $\Gamma_\otimes^{ISS}$ is a compact positive operator, Proposition 4.3 implies that the assumption (iii) is equivalent to the fact that $\rho(\Gamma_\otimes^{ISS}) < 1$. In turn, this implies by Proposition 4.5 that for all $w, v \in \ell_\infty^+$ so that $(\mathrm{Id} - \Gamma_\otimes^{ISS})(w) \leq v$ it follows that $w \leq (\mathrm{Id} - \Gamma_\otimes^{ISS})^{-1} v$, and as the norm in ℓ_∞ is monotone w.r.t. the order, this implies that

$$\|w\|_{\ell_\infty} \leq \|(\mathrm{Id} - \Gamma_\otimes^{ISS})^{-1}\| \|v\|_{\ell_\infty},$$

which verifies the assumption (ii) of Theorem 3.1 with $\xi(r) = \|(\mathrm{Id} - \Gamma_\otimes^{ISS})^{-1}\| r$, $r \in \mathbb{R}_+$. Now, application of Theorem 3.1 shows the claim. ■

V. Conclusion and outlook

We have shown ISS small-gain theorems for general interconnections with nonlinear gains (Theorem 3.1) as well as with linear gains (Theorem 4.7). These theorems ensure input-to-state stability of countably infinite networks consisting of ISS components. For the future research it is interest to obtain more checkable conditions for ISS of infinite networks in the case of nonlinear internal gains as well as to compare the small-gain conditions exploited in this paper with the so-called robust strong small-gain condition introduced in [9] for establishing a Lyapunov small-gain theorem for countably infinite couplings.

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References

[1] M. Arcak and P. Kokotović. Nonlinear observers: a circle criterion design and robustness analysis. *Automatica*, 37(12):1923–1930, 2001.

[2] B. Bamieh, F. Paganini, and M. A. Dahleh. Distributed control of spatially invariant systems. *IEEE Transactions on Automatic Control*, 47(7):1091–1107, 2002.

[3] B. Bamieh and P. G. Voulgaris. A convex characterization of distributed control problems in spatially invariant systems with communication constraints. *Systems Control Lett.*, 54(6):575–583, 2005.

[4] A. Bao, T. Liu, Z.-P. Jiang, and L. Zhang. A nonlinear small-gain theorem for large-scale infinite-dimensional systems. *Journal of Systems Science & Complexity*, 31(1):188–199, 2018.

[5] B. Besselink and K. H. Johansson. String stability and a delay-based spacing policy for vehicle platoons subject to disturbances. *IEEE Trans. Automat. Control*, 62(9):4376–4391, 2017.

[6] T. Cazenave and A. Haraux. *An Introduction To Semilinear Evolution Equations*. Oxford University Press, New York, 1998.

[7] R. Curtain, O. V. Iftime, and H. Zwart. System theoretic properties of a class of spatially invariant systems. *Automatica*, 45(7):1619–1627, 2009.

[8] S. Dashkovskiy and A. Mironchenko. Input-to-state stability of infinite-dimensional control systems. *Mathematics of Control, Signals, and Systems*, 25(1):1–35, 2013.

[9] S. Dashkovskiy, A. Mironchenko, J. Schmid, and F. Wirth. Stability of infinitely many interconnected systems. In *Submitted to the 11th IFAC Symposium on Nonlinear Control Systems (NOLCOS 2019)*, 2019.

[10] S. Dashkovskiy and S. Pavlichkov. Decentralized Stabilization of Infinite Networks of Systems with Nonlinear Dynamics and Uncontrollable Linearization. In *Proc. of the 20th IFAC World Congress*, pages 1728–1734, 2017.

[11] S. Dashkovskiy, B. Rüffer, and F. Wirth. Small gain theorems for large scale systems and construction of ISS Lyapunov functions. *SIAM Journal on Control and Optimization*, 48(6):4089–4118, 2010.

[12] S. Dashkovskiy, B. S. Rüffer, and F. R. Wirth. An ISS small gain theorem for general networks. *Mathematics of Control, Signals, and Systems*, 19(2):93–122, 2007.

[13] D. Hundertmark, M. Meyries, L. Machinek, and R. Schnaubelt. Operator semigroups and dispersive equations. In *16th Internet Seminar on Evolution Equations*, 2013.

[14] B. Jacob, R. Nabiullin, J. R. Partington, and F. L. Schwenninger. Infinite-Dimensional Input-to-State Stability and Orlicz Spaces. *SIAM Journal on Control and Optimization*, 56(2):868–889, 2018.

[15] Z.-P. Jiang, I. M. Y. Mareels, and Y. Wang. A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems. *Automatica*, 32(8):1211–1215, 1996.

[16] Z.-P. Jiang, A. R. Teel, and L. Praly. Small-gain theorem for ISS systems and applications. *Mathematics of Control, Signals, and Systems*, 7(2):95–120, 1994.

[17] I. Karafyllis and Z.-P. Jiang. A Small-Gain Theorem for a Wide Class of Feedback Systems with Control Applications. *SIAM Journal on Control and Optimization*, 46(4):1483–1517, 2007.

[18] I. Karafyllis and Z.-P. Jiang. *Stability and stabilization of nonlinear systems*. Springer-Verlag, London, 2011.

[19] I. Karafyllis and M. Krstic. Small-Gain-Based Boundary Feedback Design for Global Exponential Stabilization of 1-D Semilinear Parabolic PDEs. *arXiv preprint arXiv:1809.04039*, 2018.

[20] I. Karafyllis and M. Krstic. Small-gain stability analysis of certain hyperbolic–parabolic PDE loops. *Systems & Control Letters*, 118:52–61, 2018.

[21] I. Karafyllis and M. Krstic. *Input-to-State Stability for PDEs*. Springer, 2019.

[22] M. G. Krein and M. A. Rutman. Linear operators leaving invariant a cone in a Banach space. *Uspekhi Matematicheskikh Nauk*, 3(1):3–95, 1948.

[23] M. Krstic, I. Kanellakopoulos, and P. V. Kokotovic. *Nonlinear and adaptive control design*. Wiley, 1995.

[24] A. Mironchenko. Local input-to-state stability: Characterizations and counterexamples. *Systems & Control Letters*, 87:23–28, 2016.

[25] A. Mironchenko. Small gain theorems for general networks of heterogeneous infinite-dimensional systems. *Submitted to SIAM Journal on Control and Optimization*, 2019.

[26] A. Mironchenko and H. Ito. Construction of Lyapunov Functions for Interconnected Parabolic Systems: An iISS Approach. *SIAM Journal on Control and Optimization*, 53(6):3364–3382, 2015.

[27] A. Mironchenko and F. Wirth. Characterizations of input-to-state stability for infinite-dimensional systems. *IEEE Transactions on Automatic Control*, 63(6):1602–1617, 2018.

[28] I. G. Polushin, S. N. Dashkovskiy, A. Takhmar, and R. V. Patel. A small gain framework for networked cooperative force-reflecting teleoperation. *Automatica J. IFAC*, 49(2):338–348, 2013.

[29] I. G. Polushin, A. Tayebi, and H. J. Marquez. Control schemes for stable teleoperation with communication delay based on IOS small gain theorem. *Automatica*, 42(6):905–915, 2006.

[30] K. M. Przyluski. Stability of linear infinite-dimensional systems revisited. *International Journal of Control*, 48(2):513–523, 1988.

[31] E. D. Sontag. Input to State Stability: Basic Concepts and Results. In *Nonlinear and Optimal Control Theory*, chapter 3, pages 163–220. Springer, Heidelberg, 2008.

[32] E. D. Sontag and Y. Wang. On characterizations of the input-to-state stability property. *Systems & Control Letters*, 24(5):351–359, 1995.

[33] E. D. Sontag and Y. Wang. New characterizations of input-to-state stability. *IEEE Transactions on Automatic Control*, 41(9):1283–1294, 1996.

[34] A. Tanwani, C. Prieur, and S. Tarbouriech. Stabilization of linear hyperbolic systems of balance laws with measurement errors. In *Control subject to computational and communication constraints*, pages 357–374. Springer, 2018.

[35] S. Tiwari, Y. Wang, and Z.-P. Jiang. A nonlinear small-gain theorem for large-scale time delay systems. In *Proceedings of the 48th IEEE Conference on Decision and Control, 2009 held jointly with the 2009 28th Chinese Control Conference*, pages 7204–7209. IEEE, 2009.

[36] S. Tiwari, Y. Wang, and Z. P. Jiang. Nonlinear small-gain theorems for large-scale time-delay systems. *Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis*, 19(1):27–63, 2012.

[37] K. Yosida. Functional analysis. *Berlin-Heidelberg*, 1980.

[38] J. Zheng and G. Zhu. A De Giorgi Iteration-based Approach for the Establishment of ISS Properties for Burgers' Equation with Boundary and In-domain Disturbances. *Accepted to IEEE Transactions on Automatic Control*, 2018.