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# Lyapunov functions for input-to-state stability of infinite-dimensional systems with integrable inputs

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**Abstract:** In this paper, we extend the ISS Lyapunov methodology to make it suitable for the analysis of ISS w.r.t. inputs from $L_p$-spaces. We show that the existence of a so-called $L_p$-ISS Lyapunov function implies $L_p$-ISS of a system. Also, we show that existence of a noncoercive $L_p$-ISS Lyapunov function implies $L_p$-ISS of a control system provided the flow map is continuous w.r.t. states and inputs and provided the finite-time reachability sets, corresponding to the input space $L_p$ are bounded.



*Keywords:* Nonlinear systems, infinite-dimensional systems, input-to-state stability, Lyapunov methods.

## 1. INTRODUCTION

The concept of input-to-state stability (ISS), introduced in (Sontag, 1989) for ordinary differential equations (ODEs), has unified the classical Lyapunov and input-output stability theories and became a foundation for robust stabilization of nonlinear systems, design of nonlinear observers (Arcak and Kokotović, 2001), analysis of large-scale networks (Jiang et al., 1994; Dashkovskiy et al., 2010), etc.

Recently wide-reaching generalization of the classical ISS theory to the class of infinite-dimensional systems has been proposed (Mironchenko and Prieur, 2020; Karafyllis and Krstic, 2019a). This rapidly developing research area which employs the methods of nonlinear control, functional analysis, Lyapunov theory and partial differential equations (PDEs), whose interplay resulted in a broad range of powerful techniques for ISS analysis and robust control, such as: criteria of ISS and ISS-like properties in terms of weaker stability concepts (Mironchenko and Wirth, 2018), (Jacob et al., 2018; Schmid, 2019), constructions of ISS Lyapunov functions for PDEs with in-domain and/or boundary controls (Prieur and Mazenc, 2012; Tanwani et al., 2018; Zheng and Zhu, 2018b; Edalatzadeh and Morris, 2019), efficient functional-analytic methods for the study of linear systems with unbounded input operators (e.g. linear boundary control systems) (Zheng and Zhu, 2018a; Jacob et al., 2018; Jayawardhana et al., 2008; Jacob et al., 2019b; Karafyllis and Krstic, 2016; Lhachemi and Shorten, 2019; Karafyllis and Krstic, 2019a), as well as small-gain techniques for stability analysis of networks (Karafyllis and Krstic, 2019b; Dashkovskiy and Mironchenko, 2013), etc.

For a comprehensive survey on ISS of linear and nonlinear infinite-dimensional systems and its applications to robust control, we refer to (Mironchenko and Prieur, 2020). For an overview of the linear infinite-dimensional ISS theory see also (Schwenninger, 2019).

Lyapunov functions are an indispensable tool for stability analysis of dynamical and control systems, especially nonlinear ones. The fact that the existence of an ISS Lyapunov function implies ISS, can be naturally generalized from the ODE case (Sontag, 1989; Sontag and Wang, 1995) to the case of infinite-dimensional systems (Dashkovskiy and Mironchenko, 2013, Theorem 1). However, attempts to apply the ISS Lyapunov methods to linear and nonlinear boundary control systems have faced serious obstacles. For instance, it is well-known that the classic linear heat equation with Dirichlet boundary inputs is ISS. However, no constructions of coercive ISS Lyapunov functions have been proposed, and it is not known whether such functions exist or not.

*Non-coercive Lyapunov functions*, introduced in (Mironchenko and Wirth, 2019) for stability analysis of nonlinear dynamical systems, help to tackle such obstacles and enlarge the applicability of Lyapunov methods. Noncoercive ISS Lyapunov functions have been employed in (Mironchenko and Wirth, 2018) to show that under certain requirements on the dynamics of the system already existence of a non-coercive ISS Lyapunov function implies ISS of a control system. This result has been generalized to a broad class of infinite-dimensional systems including important classes of boundary control systems in (Jacob et al., 2019a), and furthermore, construction of noncoercive ISS Lyapunov functions for a class of abstract linear systems with an $\infty$-admissible control operator has been introduced. From these results, it follows in particular, that non-coercive ISS Lyapunov functions for a heat equation with a Dirichlet boundary control do exist.

ISS Lyapunov functions methodology developed in (Dashkovskiy and Mironchenko, 2013; Mironchenko and Wirth, 2018; Jacob et al., 2019a) and briefly explained above is well-suited for stability analysis with respect to the spaces of inputs endowed with some sort of supre-

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mum norm (as $L_\infty$ space, or space of piecewise-continuous functions). At the same time, as argued in Jacob et al. (2019a), the usual definition of an ISS Lyapunov function is too restrictive for ISS analysis of control systems with inputs from $L_q$-spaces with $q \in [1, +\infty)$, widely used in the infinite-dimensional systems theory (Jacob et al., 2018).

In this paper, we propose the ISS Lyapunov methodology for the analysis of ISS w.r.t. inputs from $L_p$-spaces. *We show in Theorem 1 that existence of a so-called $L_p$-ISS Lyapunov function implies $L_p$-ISS of a system, provided the flow of the system depends continuously on external inputs. In Theorem 3 we show that the existence of a noncoercive $L_p$-ISS Lyapunov function implies $L_p$-ISS of a control system, provided that the flow map is continuous w.r.t. states and inputs from $L_p$-space and provided the finite-time reachability sets, corresponding to this input space are bounded.*

**Notation:** The nonnegative reals are denoted by $\mathbb{R}_+ := [0, \infty)$. The open ball of radius $r$ around 0 in a normed vector space $X$ is denoted by $B_r := B_{r,X} := \{x \in X : \|x\|_X < r\}$. Similarly, $B_{r,\mathcal{U}} := \{u \in \mathcal{U} : \|u\|_\mathcal{U} < r\}$.

For any normed linear space $X$, for any $S \subset X$ we denote the closure of $S$ by $\overline{S}$.

Let $U$ be a Banach space, $I$ be a closed subset of $\mathbb{R}$ and $p \in [1, +\infty)$. We define the following spaces (see (Jacob and Zwart, 2012, Definition A.1.14) for details)

$$\begin{aligned} M(\mathbb{R}_+, U) &:= \{f : \mathbb{R}_+ \to U : f \text{ is strongly measurable}\}, \\ L_p(\mathbb{R}_+, U) &:= \{f \in M(\mathbb{R}_+, U) : \\ &\qquad \|f\|_{L_p(\mathbb{R}_+,U)} := \Big(\int_0^\infty \|f(s)\|_U^p ds\Big)^{1/p} < \infty\}, \\ L_\infty(\mathbb{R}_+, U) &:= \{f \in M(\mathbb{R}_+, U) : \\ &\qquad \|f\|_{L_\infty(\mathbb{R}_+,U)} := \operatorname*{ess\,sup}_{s\in\mathbb{R}_+} \|f(s)\|_U < \infty\}. \end{aligned}$$

Identifying the functions, which differ on a set with a Lebesgue measure zero, the spaces $L_p(\mathbb{R}_+, U)$, $p \in [1, +\infty]$ are Banach spaces.

We use the following classes of comparison functions:

$$\begin{aligned} \mathcal{K} &:= \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous, strictly increasing and } \gamma(0) = 0\}, \\ \mathcal{K}_\infty &:= \{\gamma \in \mathcal{K} \mid \gamma \text{ is unbounded}\}, \\ \mathcal{L} &:= \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous and strictly decreasing with } \lim_{t\to\infty} \gamma(t) = 0\}, \\ \mathcal{KL} &:= \{\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \mid \beta \text{ is continuous, } \beta(\cdot, t) \in \mathcal{K},\ \beta(r, \cdot) \in \mathcal{L},\ \forall t \geq 0,\ \forall r > 0\}. \end{aligned}$$

## 2. GENERAL FRAMEWORK

We start with a general definition of a control system.

**Definition 1.** *Consider the triple $\Sigma = (X, \mathcal{U}, \phi)$ consisting of*

*(i)* *A normed vector space $(X, \|\cdot\|_X)$, called the* state space*, endowed with the norm $\|\cdot\|_X$.*

*(ii)* *A normed vector* space of inputs $\mathcal{U} \subset \{u : \mathbb{R}_+ \to U\}$ *endowed with a norm $\|\cdot\|_\mathcal{U}$, where $U$ is a normed vector* space of input values. *We assume that the following two axioms hold:*

The axiom of shift invariance*: for all $u \in \mathcal{U}$ and all $\tau \geq 0$ the time shift $u(\cdot + \tau)$ belongs to $\mathcal{U}$ with $\|u\|_\mathcal{U} \geq \|u(\cdot + \tau)\|_\mathcal{U}$.*

The axiom of concatenation*: for all $u_1, u_2 \in \mathcal{U}$ and for all $t > 0$ the concatenation of $u_1$ and $u_2$ at time $t$, defined by*

$$u_1 \underset{t}{\Diamond} u_2(\tau) := \begin{cases} u_1(\tau), & \text{if } \tau \in [0, t], \\ u_2(\tau - t), & \text{otherwise}, \end{cases} \tag{1}$$

*belongs to $\mathcal{U}$.*

*(iii)* *A map $\phi : D_\phi \to X$, $D_\phi \subseteq \mathbb{R}_+ \times X \times \mathcal{U}$ (called* transition map*), such that for all $(x, u) \in X \times \mathcal{U}$ it holds that $D_\phi \cap (\mathbb{R}_+ \times \{(x, u)\}) = [0, t_m) \times \{(x, u)\} \subset D_\phi$, for a certain $t_m = t_m(x, u) \in (0, +\infty]$.*

*The corresponding interval $[0, t_m)$ is called the* maximal domain of definition *of $t \mapsto \phi(t, x, u)$.*

*The triple $\Sigma$ is called a* (control) system*, if the following properties hold:*

*(Σ1)* The identity property: *for every $(x, u) \in X \times \mathcal{U}$ it holds that $\phi(0, x, u) = x$.*

*(Σ2)* Causality: *for every $(t, x, u) \in D_\phi$, for every $\tilde{u} \in \mathcal{U}$, such that $u(s) = \tilde{u}(s)$ for all $s \in [0, t]$ it holds that $[0, t] \times \{(x, \tilde{u})\} \subset D_\phi$ and $\phi(t, x, u) = \phi(t, x, \tilde{u})$.*

*(Σ3)* Continuity: *for each $(x, u) \in X \times \mathcal{U}$ the map $t \mapsto \phi(t, x, u)$ is continuous on its maximal domain of definition.*

*(Σ4)* The cocycle property: *for all $x \in X$, $u \in \mathcal{U}$, for all $t, h \geq 0$ so that $[0, t + h] \times \{(x, u)\} \subset D_\phi$, we have $\phi(h, \phi(t, x, u), u(t + \cdot)) = \phi(t + h, x, u)$.*

**Definition 2.** *We say that a control system $\Sigma = (X, \mathcal{U}, \phi)$ is* forward complete*, if $D_\phi = \mathbb{R}_+ \times X \times \mathcal{U}$, that is for every $(x, u) \in X \times \mathcal{U}$ and for all $t \geq 0$ the value $\phi(t, x, u) \in X$ is well-defined.*

Verification of forward completeness for nonlinear systems is often a complex task. A weaker property that helps on this way and which is satisfied for broad classes of control systems is a possibility to prolong bounded solutions to a larger interval.

**Definition 3.** *(Karafyllis and Jiang, 2011, Definition 1.4) We say that a system $\Sigma$ satisfies the* boundedness-implies-continuation (BIC) property *if for each $(x, u) \in X \times \mathcal{U}$ such that the maximal existence time $t_m = t_m(x, u)$ is finite, and for all $M > 0$, there exists $t \in [0, t_{\max})$ with $\|\phi(t, x, u)\|_X > M$.*

Finally, we introduce another property, which will be used frequently in this work

**Definition 4.** *Let $\Sigma := (X, \mathcal{U}, \phi)$ be a control system. We say that $\phi$* depends continuously on inputs*, if for all $x \in X$, $u \in \mathcal{U}$, $T \in (0, t_m(x, u))$ and all $\varepsilon > 0$ there is $\delta > 0$, such that for all $\tilde{u} \in \mathcal{U}$: $\|u - \tilde{u}\|_\mathcal{U} < \delta$ it holds that $t_m(x, \tilde{u}) \geq T$ and*

$$\|\phi(t, x, u) - \phi(t, x, \tilde{u})\|_X < \varepsilon, \quad \forall t \in [0, T].$$

In the infinite-dimensional systems theory one of common choices for the input space $\mathcal{U}$ are the spaces $L_p(\mathbb{R}_+, U)$, $p \in [1, +\infty)$ of $p$-th power Bochner-integrable $U$-valued functions defined on $\mathbb{R}_+$. In this work, we study the input-to-state stability of infinite-dimensional systems with such input spaces. Before we define this concept, we note that every control system $\Sigma := (X, L_p(\mathbb{R}_+, U), \phi)$ can be naturally extended to a larger control system $(X, L_{p,loc}(\mathbb{R}_+, U), \phi)$, which we again denote by $\Sigma$.

Indeed, pick any $x \in X$ and any $u \in L_{p,loc}(\mathbb{R}_+, U)$. As $u \underset{t}{\Diamond} 0 \in L_p(\mathbb{R}_+, U)$ for any $t \geq 0$, we extend $\phi$ to a larger domain, by defining for each $t$ so that $(t, x, u \underset{t}{\diamondsuit} 0) \in D_\phi$

$$\phi(t, x, u) := \phi(t, x, u \underset{t}{\diamondsuit} 0).$$

Strictly speaking, the system $\Sigma_e$ is not a control system in the sense of Definition 1, as $L_{p,loc}(\mathbb{R}_+, U)$ is not a normed linear space, but we understand it in the sense of this "causal" extension.

As $L_{q,loc}(\mathbb{R}_+, U) \subset L_{p,loc}(\mathbb{R}_+, U)$ for all $q > p$, we can study stability of an extended $\Sigma$ with respect to all $L_q$-spaces for $q \geq p$. Here we proceed to one of the main definitions in this paper:

**Definition 5.** *Let $q \geq p \geq 1$ be given. System $\Sigma = (X, L_{p,loc}(\mathbb{R}_+, U), \phi)$ is called* $L_q$-input-to-state stable (ISS)*, if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for all $x \in X$, $u \in L_q(\mathbb{R}_+, U)$ and $t \geq 0$ it holds that*

$$\|\phi(t, x, u)\|_X \leq \beta(\|x\|_X, t) + \gamma(\|u\|_{L_q(\mathbb{R}_+,U)}). \tag{2}$$

For ISS analysis we define several further concepts.

**Definition 6.** *Consider a forward complete control system $\Sigma = (X, \mathcal{U}, \phi)$.*

*(1) We call $0 \in X$ an* equilibrium point (of the undisturbed system) *if $\phi(t, 0, 0) = 0$ for all $t \geq 0$.*

*(2) We say that $\Sigma$ has the* continuity at the equilibrium point (CEP) property*, if $0$ is an equilibrium and for every $\varepsilon > 0$ and for any $h > 0$ there exists a $\delta = \delta(\varepsilon, h) > 0$, so that*

$$t \in [0, h],\ \|x\|_X \leq \delta,\ \|u\|_{\mathcal{U}} \leq \delta \ \Rightarrow\ \|\phi(t, x, u)\|_X \leq \varepsilon. \tag{3}$$

*(3) We say that $\Sigma$ has* bounded reachability sets (BRS)*, if for any $C > 0$ and any $\tau > 0$ it holds that*

$$\sup\left\{\|\phi(t, x, u)\|_X :\ x \in B_C,\ u \in B_{C,\mathcal{U}},\ t \in [0, \tau]\right\} < \infty.$$

*(4) System $\Sigma$ is called* uniformly locally stable (ULS)*, if there exist $\sigma \in \mathcal{K}_\infty$, $\gamma \in \mathcal{K}_\infty$ and $r > 0$ such that for all $x \in \overline{B_r}$ and all $u \in \overline{B_{r,\mathcal{U}}}$:*

$$\|\phi(t, x, u)\|_X \leq \sigma(\|x\|_X) + \gamma(\|u\|_{\mathcal{U}}) \quad \forall t \geq 0. \tag{4}$$

*(5) We say that $\Sigma$ has the* uniform limit property (ULIM)*, if there exists $\gamma \in \mathcal{K}_\infty$ so that for every $\varepsilon > 0$ and for every $r > 0$ there exists a $\tau = \tau(\varepsilon, r)$ such that for all $x$ with $\|x\|_X \leq r$ and all $u \in \mathcal{U}$ there is a $t \leq \tau$ such that*

$$\|\phi(t, x, u)\|_X \leq \varepsilon + \gamma(\|u\|_{\mathcal{U}}). \tag{5}$$

*(6) We call $\Sigma$* $L_q$-integral-to-integral ISS *if there are $\alpha \in \mathcal{K}$ and $\psi \in \mathcal{K}_\infty$ and $c > 0$ so that for all $x \in X$, $u \in C(\mathbb{R}_+, U)$ and all $t \geq 0$ it holds that*

$$\int_0^t \alpha(\|\phi(s, x, u)\|_X) ds \leq \psi(\|x\|_X) + c \int_0^t \|u(s)\|_U^q ds. \tag{6}$$

*By $L_q$-ULS, $L_q$-BRS etc. we understand the corresponding property with respect to inputs from the space $L_q(\mathbb{R}_+, U)$.*

## 3. ISS LYAPUNOV FUNCTIONS FOR $L_P$-ISS PROPERTY

As argued in (Jacob et al., 2019a), existing concepts of ISS Lyapunov functions, used for ISS analysis of control systems with continuous, piecewise-continuous or $L_\infty$ inputs (see (Mironchenko and Wirth, 2018, Definition 12) and (Dashkovskiy and Mironchenko, 2013, Definition 7)), are not applicable for ISS analysis of control systems with integrable inputs. *In this paper we introduce a novel concept of an $L_p$-ISS Lyapunov function which is fine tuned specifically for such control systems, and derive criteria for $L_p$-ISS of general control systems in terms of coercive and non-coercive ISS Lyapunov functions.*

**Definition 7.** *Let $+\infty > p \geq d \geq 1$ be given. Consider a control system $\Sigma := (X, \mathcal{U}, \phi)$, with $\mathcal{U} := L_{d,loc}(\mathbb{R}_+, U)$, where $X$ and $U$ are normed linear spaces. A continuous function $V : X \to \mathbb{R}_+$ is called a* non-coercive $L_p$-ISS Lyapunov function for $\Sigma$*, if there exist $\psi_2 \in \mathcal{K}_\infty$, $\alpha \in \mathcal{K}_\infty$ and $c > 0$, such that:*

$$0 < V(x) \leq \psi_2(\|x\|_X), \quad \forall x \in X, \tag{7}$$

*and Lie derivative of $V$ along the trajectories of $\Sigma$ satisfies*

$$\dot{V}_u(x) \leq -\alpha(V(x)) + c\|u(0)\|_U^p \tag{8}$$

*for all $x \in X$ and $u \in C(\mathbb{R}_+, U)$, where the Lie derivative of $V$ corresponding to the input $u$ is defined by*

$$\dot{V}_u(x) = \limsup_{t \to +0} \frac{1}{t}\big(V(\phi(t, x, u)) - V(x)\big). \tag{9}$$

*If additionally there is $\psi_1 \in \mathcal{K}_\infty$ so that*

$$\psi_1(\|x\|_X) \leq V(x) \leq \psi_2(\|x\|_X), \quad \forall x \in X, \tag{10}$$

*then $V$ is called a* (coercive) $L_p$-ISS Lyapunov function for $\Sigma$.

**Remark 1.** We require the property (8) only for continuous inputs, as for general $u \in L_p(\mathbb{R}_+, U)$ the expression $u(0)$ is not defined. We exclude in Definition 7 the case $p = +\infty$, as $C(\mathbb{R}_+, U)$ is not dense in $L_\infty(\mathbb{R}_+, U)$. ∘

For any continuous function $y : \mathbb{R} \to \mathbb{R}$, let $D^+y$ denote the right upper Dini derivative of $y$, i.e.

$$D^+y(t) := \limsup_{h \to +0} \frac{y(t + h) - y(t)}{h}.$$

The following result is due to (Mironchenko and Ito, 2016, Corollary 1), which is a slight generalization of (Angeli et al., 2000, Corollary IV.3) and which can be understood as a nonlinear extension of the Grönwall's inequality.

**Proposition 1.** *Let $\tilde{t} \in (0, \infty]$ and let $y : [0, \tilde{t}) \to \mathbb{R}_+$ be a continuous function satisfying for almost all $t \in (0, \tilde{t})$ the differential inequality*

$$D^+y(t) \leq -\alpha(y(t)) + v(t), \tag{11}$$

*for some $\alpha \in \mathcal{P}$ and some measurable locally essentially bounded function $v : [0, \tilde{t}) \to \mathbb{R}_+$.*

*Then there is a $\beta \in \mathcal{KL}$ so that for all $t \in [0, \tilde{t})$ it holds that*

$$y(t) \leq \beta(y(0), t) + 2\int_0^t v(s) ds. \tag{12}$$

We are going to show that the existence of a coercive $L_p$-ISS Lyapunov function implies $L_q$-ISS for all $q \in [p, +\infty)$.

We need the following instrumental lemma.

**Lemma 1.** *Let $p \in [1, +\infty)$. Consider a control system $\Sigma := (X, \mathcal{U}, \phi)$, with $\mathcal{U} := L_{p,loc}(\mathbb{R}_+, U)$, where $U$ is a normed linear space. Assume that $\phi$ depends continuously on inputs with respect to the $L_p$-norm.*

*Then for all $q \in [p, +\infty]$ the flow $\phi$ depends continuously on inputs with respect to $L_q$-norm, more precisely: for all $x \in X$, $u \in L_{q,loc}(\mathbb{R}_+, U)$, $T \in (0, t_m(x, u))$ and all $\varepsilon > 0$ there is $\delta > 0$, such that for all $\tilde{u} \in L_{q,loc}(\mathbb{R}_+, U)$ with $\|u - \tilde{u}\|_{L_q([0,T],U)} < \delta$ it holds that $t_m(x, \tilde{u}) \geq T$ and*

$$\|\phi(t, x, u) - \phi(t, x, \tilde{u})\|_X < \varepsilon \quad \forall t \in [0, T]. \tag{13}$$

**Proof.** We skip the proof due to the page limitations. □

Our first main result is:

**Theorem 1.** (Direct coercive Lyapunov theorem) *Let $p \in [1, +\infty)$. Consider a control system $\Sigma := (X, \mathcal{U}, \phi)$, with $\mathcal{U} := L_{p,loc}(\mathbb{R}_+, U)$, where $U$ is a normed linear space. Assume that $\Sigma$ has BIC property and that $\phi$ is continuous w.r.t. inputs (in $\mathcal{U}$-norm).*

*Let $V$ be an $L_p$-ISS Lyapunov function for $\Sigma$ with $\alpha \in \mathcal{K}_\infty$ as in Definition 7. Define for each $z \in [0, +\infty)$ the map $V_z : X \to \mathbb{R}_+$ by*

$$V_z(x) = \eta_z(V(x)), \quad x \in X, \tag{14}$$

*where $\eta_z(r) := \int_0^r \big(\alpha(s)\big)^z ds$, $r \geq 0$.*

*Then the following holds:*

*(i) $\Sigma$ is $L_q$-ISS for all $q \in [p, +\infty)$.*

*(ii) For each $z \geq 0$ the map $V_z$ is an $L_{(z+1)p}$-ISS Lyapunov function for $\Sigma$.*

**Proof.** We divide the proof into several parts.

**$\Sigma$ is $L_p$-ISS.** Pick any initial condition $x \in X$ and any input $u \in C(\mathbb{R}_+, U) \subset L_{p,loc}(\mathbb{R}_+, U)$. As $\Sigma$ is a control system, there is $t_m = t_m(x, u) \in (0, +\infty]$ such that $D_\phi \cap \big(\mathbb{R}_+ \times \{(x, u)\}\big) = [0, t_m) \times \{(x, u)\}$.

Define $y(t) := V(\phi(t, x, u))$, $t \in [0, t_m)$. We have:

$$\begin{aligned} D^+ y(t) &= \frac{d}{dt} V(\phi(t, x, u)) \\ &= \lim_{\tau \to 0} \frac{1}{\tau}\Big(V(\phi(t + \tau, x, u)) - V(\phi(t, x, u))\Big) \\ &= \limsup_{\tau \to +0} \frac{1}{\tau}\Big(V\big(\phi(\tau, \phi(t, x, u), u(t + \cdot))\big) - V(\phi(t, x, u))\Big) \\ &= \dot{V}_{u(t+\cdot)}\big(\phi(t, x, u)\big). \end{aligned} \tag{15}$$

Using (8) we have for all $t \in [0, t_m)$ that

$$\begin{aligned} D^+ y(t) &\leq -\alpha\big(V(\phi(t, x, u))\big) + c\|u(t + \cdot)(0)\|_U^p \\ &= -\alpha\big(y(t)\big) + c\|u(t)\|_U^p. \end{aligned}$$

In view of Proposition 1, there is a $\tilde{\beta} \in \mathcal{KL}$ so that

$$y(t) \leq \tilde{\beta}(y(0), t) + 2\int_0^t c\|u(s)\|_U^p ds.$$

By (10) we have that

$$\psi_1(\|\phi(t, x, u)\|_X) \leq \tilde{\beta}\big(\psi_2(\|x\|_X), t\big) + 2c\int_0^t \|u(s)\|_U^p ds.$$

As $\psi_1^{-1} \in \mathcal{K}_\infty$, it holds that $\psi_1^{-1}(a + b) \leq \psi_1^{-1}(2a) + \psi_1^{-1}(2b)$ for all $a, b \geq 0$, and thus

$$\begin{aligned} \|\phi(t, x, u)\|_X &\leq \psi_1^{-1}\Big(2\tilde{\beta}\big(\psi_2(\|x\|_X), t\big)\Big) \\ &\qquad + \psi_1^{-1}\Big(4c\int_0^t \|u(s)\|_U^p ds\Big) \\ &= \beta(\|x\|_X, t) + \gamma(\|u\|_{L_p(\mathbb{R}_+, U)}), \end{aligned} \tag{16}$$

where $\beta(r, t) := \psi_1^{-1}\Big(2\tilde{\beta}\big(\psi_2(r), t\big)\Big)$ and $\gamma(r) := \psi_1^{-1}\big(4cr^p\big)$.

In particular, the solution $\phi(\cdot, x, u)$ stays bounded on $[0, t_m)$. If $t_m < +\infty$, then by BIC property it can be prolonged to a larger interval, which contradicts to the maximality of $t_m$. Hence, $t_m = +\infty$, and the solution $\phi(\cdot, x, u)$ exists on $\mathbb{R}_+$ for all $x \in X$ and all $u \in C(\mathbb{R}_+, U)$.

Now pick any $x \in X$ and any input $u \in L_p(\mathbb{R}_+, U)$. As $\Sigma$ is a control system, the corresponding solution $\phi(\cdot, x, u)$ exists on a certain maximal interval $[0, t_m)$.

It is well-known that $C([0, t_m), U)$ is dense in $\mathcal{U} = L_p([0, t_m), U)$. As the right hand side of (12) is continuous w.r.t. $u$ in the $L_p$-norm, and since we assume that $\phi$ is continuous w.r.t. $u$ as well, the estimate (12) is valid for all inputs in $\mathcal{U}$ on their interval of existence. Again, by BIC property, these solutions exist globally. Overall, we have proved that $\Sigma$ is $L_p$-ISS.

**Let us show (ii).** As $\alpha \in \mathcal{K}_\infty$, then also $\eta_z \in \mathcal{K}_\infty$ for all $z \geq 0$. Furthermore, $\eta_z$ is differentiable on $(0, +\infty)$.

Clearly, $V_z$ satisfies (10) for suitable $\psi_1, \psi_2 \in \mathcal{K}_\infty$. Let us show the dissipation inequality (8). For any $x \in X$ and any $u \in C(\mathbb{R}_+, U)$ we have that

$$\begin{aligned} \dot{V}_{z,u}(x) &= \big(\alpha(V(x))\big)^z \dot{V}(x) \\ &\leq -\big(\alpha(V(x))\big)^{z+1} + \varepsilon\big(\alpha(V(x))\big)^z \cdot \frac{c}{\varepsilon}\|u(0)\|_U^p. \end{aligned} \tag{17}$$

Recall that for all $r, q > 1$: $\frac{1}{r} + \frac{1}{q} = 1$ and all $a, b \geq 0$ the Young's inequality is valid:

$$ab \leq \frac{a^r}{r} + \frac{b^q}{q}.$$

Applying it for the last term with $a = \varepsilon\big(\alpha(V(x))\big)^z$, $b = \frac{c}{\varepsilon}\|u(0)\|_U^p$, $r = \frac{z+1}{z}$ and $q = z + 1$ we obtain

$$\begin{aligned} \dot{V}_{z,u}(x) &\leq -\big(\alpha(V(x))\big)^{z+1} + \Big(\varepsilon\big(\alpha(V(x))\big)^z\Big)^{\frac{z+1}{z}} \frac{z}{z+1} \\ &\qquad + \frac{1}{z+1}\Big(\frac{c}{\varepsilon}\Big)^{z+1}\|u(0)\|_U^{(z+1)p} \\ &= \Big(\frac{\varepsilon^{\frac{z+1}{z}} z}{z+1} - 1\Big)\big(\alpha(V(x))\big)^{z+1} + \frac{1}{z+1}\Big(\frac{c}{\varepsilon}\Big)^{z+1}\|u(0)\|_U^{(z+1)p}, \end{aligned}$$

which shows (ii).

Claim (i) follows from (ii) and from the first part of the proof. □

## 4. NON-COERCIVE ISS LYAPUNOV FUNCTIONS

In this section, we develop a direct non-coercive Lyapunov theorem for the $L_p$-ISS property. We are motivated by (Mironchenko and Wirth, 2018; Jacob et al., 2019a), where non-coercive Lyapunov functions have been used to study ISS w.r.t. general input spaces, with a particular emphasis on the spaces endowed with supremum norms. Here we propose the framework, which is particularly suitable for ISS analysis of systems with integrable inputs. We start by analyzing the properties of systems possessing the non-coercive ISS Lyapunov functions.

First we show that integral-to-integral ISS property naturally arises in the theory of ISS Lyapunov functions:

**Proposition 2.** *Let $p \in [1, +\infty)$. Consider a forward-complete control system $\Sigma := (X, \mathcal{U}, \phi)$, with $\mathcal{U} := L_{p,loc}(\mathbb{R}_+, U)$, where $U$ is a normed linear space. Assume that $\phi$ is continuous w.r.t. inputs.*

*Assume that there exists a non-coercive $L_p$-ISS Lyapunov function for $\Sigma$. Then $\Sigma$ is $L_p$-integral-to-integral ISS.*

**Proof.** Assume that $V$ is a non-coercive $L_p$-ISS Lyapunov function for $\Sigma$ with corresponding $\psi_2, \alpha, c$. Pick any $u \in C(\mathbb{R}_+, U)$ and any $x \in X$.

Since $\Sigma$ is forward complete, the trajectory $\phi(\cdot,x,u)$ exists for all times and by (15) we have the following inequality for the derivative of $y(t) := V\big(\phi(t,x,u)\big)$ for any $t>0$:

$$D^+y(t) \le -\alpha(\|\phi(t,x,u)\|_X) + c\|u(t)\|_U^p. \tag{18}$$

and $y(0)=V(x)$ due to the identity axiom of $\Sigma$.

In view of the continuity axiom of $\Sigma$, for fixed $x,u$ the map $\phi(\cdot,x,u)$ is continuous, and thus $t\mapsto -\alpha(\|\phi(t,x,u)\|_X)$ is continuous as well. Define

$$G(t) := \int_0^t \alpha(\|\phi(s,x,u)\|_X)ds - c\int_0^t \|u(s)\|_U^p ds.$$

Since $u$ is continuous, $G$ is continuously differentiable, and we can rewrite the inequality (18) as

$$D^+(y(t)+G(t)) \le 0. \tag{19}$$

It follows from (Szarski, 1965, Theorem 2.1) that $t\mapsto y(t)+G(t)$ is nonincreasing. As $G(0)=0$ and $y(t)\ge 0$ for all $t\in\mathbb{R}_+$, it follows that

$$G(t) \le y(t)+G(t) \le y(0) = V(x).$$

This shows the following $L_p$-integral-to-integral ISS estimate (6) for continuous inputs.

$$\int_0^t \alpha(\|\phi(s,x,u)\|_X)ds \le \psi_2(\|x\|_X) + c\int_0^t \|u(s)\|_U^p ds.$$

As we assume the continuous dependence of $\phi$ w.r.t. inputs, we obtain by density of $C(\mathbb{R}_+,U)\cap L_p(\mathbb{R}_+,U)$ in $L_p(\mathbb{R}_+,U)$ the $L_p$-integral-to-integral ISS of $\Sigma$. □

Next, we show how $L_p$-ISS can be inferred from $L_p$-integral-to-integral ISS. We exploit the following lower estimate of $\mathcal{K}$-functions, which is easy to check:

**Lemma 2.** *For any $\alpha\in\mathcal{K}$ and any $a,b\ge 0$ it holds that*

$$\alpha(a+b) \ge \frac{1}{2}\alpha(a) + \frac{1}{2}\alpha(b). \tag{20}$$

In the next proposition we relate $L_p$-integral-to-integral ISS to $L_p$-ULIM property.

**Proposition 3.** *Let $p\in[1,+\infty)$. Consider a forward-complete control system $\Sigma := (X,\mathcal{U},\phi)$, with $\mathcal{U} := L_{p,loc}(\mathbb{R}_+,U)$, where $U$ is a normed linear space. Assume that $\phi$ is continuous w.r.t. inputs.*

*If $\Sigma$ is $L_p$-integral-to-integral ISS, then $\Sigma$ is $L_q$-ULIM for $q\ge p$. Furthermore, the functions $\gamma$ and $\tau$ in the definition of ULIM can be chosen independently on $q$.*

**Proof.** As $\Sigma$ is $L_p$-integral-to-integral ISS, there are $\alpha,\psi\in\mathcal{K}_\infty$ and $c>0$ so that the following holds for all $t\ge 0$, $x\in X$ and $u\in\mathcal{U}$:

$$\int_0^t \alpha(\|\phi(s,x,u)\|_X)ds \le \psi(\|x\|_X) + c\int_0^t \|u(s)\|_U^p ds. \tag{21}$$

Furthermore, by Hölder's inequality it holds for any $q>p$ and for any $u\in L_q(\mathbb{R}_+,U)$ that

$$\begin{aligned}
&\int_0^t \alpha(\|\phi(s,x,u)\|_X)ds\\
&\le \psi(\|x\|_X) + c\Big(\int_0^t 1^{\frac{q}{q-p}}ds\Big)^{\frac{q-p}{p}}\Big(\int_0^t (\|u(s)\|_U^p)^{\frac{q}{p}}ds\Big)^{\frac{p}{q}}\\
&= \psi(\|x\|_X) + ct^{\frac{q-p}{p}}\|u\|^p_{L_q([0,t],U)}\\
&\le \psi(\|x\|_X) + c(1+t)\|u\|^p_{L_q(\mathbb{R}_+,U)}.
\end{aligned} \tag{22}$$

Define $\gamma(r) := \alpha^{-1}\big(4cr^p\big)$, $r\in\mathbb{R}_+$ and

$$\tau(r,\varepsilon) := \max\Big\{2(\psi(r)+1)(\alpha(\varepsilon))^{-1},1\Big\}$$

for any $r,\varepsilon>0$.

Assume that $\Sigma$ is not $L_q$-ULIM with these $\gamma$ and $\tau$. Then there are some $\varepsilon>0$, $r>0$, $x\in\overline{B_r}$ and $u\in L_q(\mathbb{R}_+,U)$ so that $\|\phi(t,x,u)\|_X > \varepsilon + \gamma(\|u\|_{L_q(\mathbb{R}_+,U)})$ for all $t\in[0,\tau(r,\varepsilon)]$.

Via Lemma 2 we have for these $\varepsilon,x,u$ and all $t\in[0,\tau(r,\varepsilon)]$ that:

$$\begin{aligned}
\int_0^t \alpha(\|\phi(s,x,u)\|_X)ds &\ge \int_0^t \alpha\big(\varepsilon+\gamma(\|u\|_{L_q(\mathbb{R}_+,U)})\big)ds\\
&\ge \int_0^t \frac{1}{2}\alpha(\varepsilon) + 2c\|u\|^p_{L_q(\mathbb{R}_+,U)}ds\\
&= \frac{t}{2}\alpha(\varepsilon) + 2ct\|u\|^p_{L_q(\mathbb{R}_+,U)}.
\end{aligned}$$

In particular, for $t:=\tau(r,\varepsilon)$ we obtain that

$$\int_0^{\tau(r,\varepsilon)} \alpha(\|\phi(s,x,u)\|_X)ds \ge \psi_2(r) + 1 + 2c\tau(r,\varepsilon)\|u\|^p_{L_q(\mathbb{R}_+,U)}.$$

Since $\tau(r,\varepsilon)\ge 1$, and $r\ge\|x\|_X$, it follows that

$$\begin{aligned}
&\int_0^{\tau(r,\varepsilon)} \alpha(\|\phi(s,x,u)\|_X)ds\\
&\qquad \ge \psi_2(\|x\|_X) + 1 + c(1+\tau(r,\varepsilon))\|u\|^p_{L_q(\mathbb{R}_+,U)},
\end{aligned}$$

which contradicts to (22). This shows that $\Sigma$ is $L_q$-ULIM for all $q\ge p$. □

For the main result we need two auxiliary lemmas:

**Lemma 3.** *Let $p\in[1,+\infty)$. Consider a forward complete control system $\Sigma := (X,\mathcal{U},\phi)$, with $\mathcal{U} := L_{p,loc}(\mathbb{R}_+,U)$, where $U$ is a normed linear space. Assume that $\Sigma$ satisfies the $L_p$-CEP property. Then $\Sigma$ satisfies $L_q$-CEP property for all $q\in[p,+\infty]$.*

**Proof.** The proof is similar to the proof of Lemma 1. □

Using similar argumentation as in Lemma 1, we obtain a corresponding result on BRS property:

**Lemma 4.** *Let $p\in[1,+\infty)$. Consider a forward complete control system $\Sigma := (X,\mathcal{U},\phi)$, with $\mathcal{U} := L_{p,loc}(\mathbb{R}_+,U)$, where $U$ is a normed linear space. Assume that $\Sigma$ satisfies the $L_p$-BRS property. Then $\Sigma$ satisfies $L_q$-BRS property for all $q\in[p,+\infty]$.*

Having related non-coercive Lyapunov functions to ULIM property, we proceed to the ULS property.

**Proposition 4.** *Let $\Sigma=(X,\mathcal{U},\phi)$ be a forward complete control system. If $\Sigma$ is $L_p$-integral-to-integral ISS and $L_q$-CEP, for $1\le p\le q$, then $\Sigma$ is $L_r$-ULS for all $r\ge q$.*

**Proof.** The proof follows closely the proof of a direct non-coercive ISS Lyapunov theorem in (Jacob et al., 2019a), and thus it is omitted due to the reasons of space. □

For the proof of the non-coercive Lyapunov theorem we exploit the following characterization of ISS, shown in (Mironchenko and Wirth, 2018, Theorem 5)

**Theorem 2.** *Let $\Sigma=(X,\mathcal{U},\phi)$ be a forward complete control system. Then $\Sigma$ is ISS if and only if $\Sigma$ is ULIM, ULS, and BRS.*

Now we can show the main result of this section:

**Theorem 3.** (Direct non-coercive Lyapunov theorem) *Let $p \in [1, +\infty)$. Consider a forward complete control system $\Sigma := (X, \mathcal{U}, \phi)$, with $\mathcal{U} := L_{p,loc}(\mathbb{R}_+, U)$, where $U$ is a normed linear space. Assume that $\phi$ is continuous w.r.t. inputs.*

*Assume that there exists a non-coercive $L_p$-ISS Lyapunov function for $\Sigma$. If for some $q \geq p$ the system $\Sigma$ is $L_q$-BRS and $L_q$-CEP, then $\Sigma$ is $L_r$-ISS for all $r \geq q$.*

**Proof.** As there exists a non-coercive $L_p$-ISS Lyapunov function for $\Sigma$, and $\Sigma$ depends continuously on the inputs, the system $\Sigma$ is $L_p$-integral-to-integral ISS by Proposition 2. This latter property implies that $\Sigma$ has $L_r$-ULIM property for all $r \geq p$. As we assume $L_q$-BRS and $L_q$-CEP for some $q \geq p$, by Lemmas 3, 4 it follows that $\Sigma$ is $L_r$-CEP and $L_r$-BRS for all $r \geq q$. Finally, by Theorem 2 it follows that $\Sigma$ is $L_r$-ISS for all $r \geq q$. □


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