

Set stability of infinite networks: ISS small-gain theory and its applications

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Abstract: Motivated by a paradigm shift towards a *hyper-connected* world, we develop a *computationally tractable* small-gain theorem for a network of *infinitely* many subsystems, termed as infinite networks. The proposed small-gain theorem addresses exponential input-to-state stability with respect to closed sets, which enables us to analyze diverse stability problems in a unified manner. The small-gain condition, expressed in terms of the spectral radius of a gain operator collecting all the information about the internal Lyapunov gains, can be numerically checked efficiently for a large class of systems. To demonstrate broad applicability of our small-gain theorem, we apply it to the stability analysis of infinite *time-varying* networks, to consensus of *infinite-agent* systems, and to the design of *distributed observers* for infinite networks.

Keywords: Nonlinear systems, small-gain theorems, infinite-dimensional systems, input-to-state stability, Lyapunov methods, large-scale systems

1. INTRODUCTION

Emerging technologies such as the Internet of Things, Cloud computing, 5G communication system and so on are expected to encompass almost every aspect of our lives and to generate a paradigm shift towards a *hyper-connected* world composed of smart networked systems. Such advances provide us with much more *autonomy* and *flexibility* at the price of increasing *complexity* and *uncertainty*. Examples of such smart networked systems include smart grids, connected vehicles, swarm robotics, and smart cities in which the participating agents may be plugged into and out from the network at any time. Therefore, the sizes of such large networks are unknown and possibly time-varying.

Most of these smart applications are *safety-critical*. This calls for a rigorous analysis and synthesis of such systems. However, standard tools for stability analysis/stabilization of control systems do not scale well to these large-scale complex systems (Sarkar et al., 2018; Sarkar et al., 2018; Bamieh et al., 2012; Jovanović and Bamieh, 2005). A promising way to address this critical issue is to over-approximate a finite but very large network by an *infinite network*, and control this over-approximated system; see

e.g. (Curtain et al., 2009; Jovanović and Bamieh, 2005; D’Andrea and Dullerud, 2003; Bamieh et al., 2002).

Current results on stability analysis and control of infinite networks are mostly concerned with spatially invariant and/or linear systems (Curtain et al., 2009; D’Andrea and Dullerud, 2003; Bamieh et al., 2002). A striking progress in the infinite-dimensional input-to-state stability (ISS) theory (Dashkovskiy and Mironchenko, 2013; Mazenc and Prieur, 2011; Mironchenko and Wirth, 2018; Karafyllis and Krstic, 2016; Jacob et al., 2018) (see (Mironchenko and Prieur, 2019) for a recent survey on this topic) together with the powerful nonlinear small-gain criteria for stability analysis of finite networks (Jiang et al., 1996; Dashkovskiy et al., 2007, 2010) create a foundation for the development of stability conditions for infinite networks of general nature without assuming linearity and/or spatial invariance of the systems.

In (Dashkovskiy and Pavlichkov, 2020) it is shown that a countably infinite network of continuous-time input-to-state stable systems is ISS, provided that the gain functions capturing the influence of subsystems on each other are all less than identity, which is a very conservative condition. In (Dashkovskiy et al., 2019) it was shown that classic max-form strong small-gain conditions (SGCs) developed for finite networks in (Dashkovskiy et al., 2010) do not ensure stability of infinite networks, even for linear ones. To address this issue, more restrictive robust strong SGCs are developed in (Dashkovskiy et al., 2019). The small-gain theorems in (Dashkovskiy and Pavlichkov,

^{*} C. Kawan is supported by the German Research Foundation (DFG) through the grant ZA 873/4-1. A. Mironchenko is supported by the DFG through the grant MI 1886/2-1. N. Noroozi is supported by the DFG through the grant WI 1458/16-1. M. Zamani is supported in part by the DFG through the grant ZA 873/4-1 and the H2020 ERC Starting Grant AutoCPS (grant agreement No. 804639).

2020; Dashkovskiy et al., 2019) are formulated in terms of ISS Lyapunov functions and a trajectory-based small-gain theorem for infinite networks is provided in (Mironchenko, 2019).

In contrast to that, for networks consisting of exponentially ISS systems, possessing exponential ISS Lyapunov functions with linear gains, it was shown in (Kawan et al., 2019) that if the spectral radius of the gain operator is less than one, then the whole network is exponentially ISS and there is a coercive exponentially ISS Lyapunov function for the whole system. This result provides a complete and *nontrivial* generalization of (Dashkovskiy et al., 2011, Prop. 3.3) from finite networks to infinite networks. It deeply relies on the spectral theory of positive operators (Karlin, 1959). In (Kawan et al., 2019), the effectiveness of the main result has been demonstrated by application to nonlinear spatially invariant systems with sector nonlinearities and to the stability analysis of a road traffic network.

All of the above small-gain theorems for infinite networks address ISS with respect to the origin. A more general notion of the input-to-state stability with respect to a closed set covers several further stability problems such as incremental stability, robust consensus/synchronization, ISS of time-varying systems as well as variants of input-to-output stability in a unified and generalized manner (Noroozi et al., 2018). In this paper, we extend the main result of our recent work (Kawan et al., 2019) to ISS of infinite networks with respect to closed sets. This modification widely extends the applicability of the small-gain result to several control theoretic problems including the stability analysis of infinite time-varying networks, consensus of infinite agent systems, and the design of distributed observers for infinite networks which are all studied in this work.

Due to the page limitation, we omit all the proofs.

2. PRELIMINARIES

2.1 Notation

We write $\mathbb{N} = \{1, 2, 3, \dots\}$ for the set of positive integers, \mathbb{R} denotes the reals and $\mathbb{R}_+ := \{t \in \mathbb{R} : t \geq 0\}$ the nonnegative reals. For vector norms on finite- and infinite-dimensional vector spaces, we write $|\cdot|$. For associated operator norms, we use the notation $\|\cdot\|$. We write A^\top for the transpose of a matrix A (which can be finite or infinite). We use Greek letters for infinite matrices and Latin ones for finite matrices. Elements of \mathbb{R}^n are by default regarded as column vectors and we write $x^\top \cdot y$ for the Euclidean inner product of two vectors $x, y \in \mathbb{R}^n$. We use the same notation for dot products of vectors with infinitely many components. By ℓ^p , $p \in [1, \infty]$, we denote the Banach space of all real sequences $x = (x_i)_{i \in \mathbb{N}}$ with finite ℓ^p -norm $|x|_p < \infty$, where $|x|_p = (\sum_{i=1}^{\infty} |x_i|^p)^{1/p}$ for $p < \infty$ and $|x|_\infty = \sup_{i \in \mathbb{N}} |x_i|$. We write $\ell_+^p := \{x = (x_i)_{i \in \mathbb{N}} \in \ell^p : x_i \geq 0, \forall i \in \mathbb{N}\}$.

A more general class of ℓ^p -spaces is defined as follows. Let $p \in [1, \infty]$, let $(n_i)_{i \in \mathbb{N}}$ be a sequence of positive integers and fix a norm $|\cdot|_i$ on \mathbb{R}^{n_i} for every $i \in \mathbb{N}$. Then

$$\ell^p(\mathbb{N}, (n_i)) := \left\{ x = (x_i)_{i \in \mathbb{N}} : x_i \in \mathbb{R}^{n_i}, \sum_{i=1}^{\infty} |x_i|_i^p < \infty \right\}$$

equipped with the norm

$$|x|_p := \left(\sum_{i=1}^{\infty} |x_i|_i^p \right)^{\frac{1}{p}}$$

is a separable Banach space (see e.g. (Dunford and Schwartz, 1957)). Usually, we drop the index i from the norm. If all n_i are identical, say $n_i \equiv n$, we also write $\ell^p(\mathbb{N}, n)$. Similarly, $\ell^\infty(\mathbb{N}, (n_i))$ can be defined.

We write $L^\infty(\mathbb{R}_+, \mathbb{R}^n)$ for the Banach space of essentially bounded measurable functions from \mathbb{R}_+ to \mathbb{R}^n . If X is a Banach space, we write $r(T)$ for the spectral radius of a bounded linear operator $T : X \rightarrow X$. The notation $C^0(X, Y)$ stands for the set of all continuous mappings $f : X \rightarrow Y$ between metric spaces X and Y . Given a metric space X , we write $\text{int } A$ for the interior of a subset $A \subset X$. The right upper (resp. lower) Dini derivative of a function $\gamma : \mathbb{R} \rightarrow \mathbb{R}$ at $t \in \mathbb{R}$ is denoted by $D^+\gamma(t)$ (resp. $D_+\gamma(t)$); see (Kawan et al., 2019) for their definitions. We will consider $\mathcal{K}, \mathcal{K}_\infty$, and \mathcal{KL} comparison functions, see (Khalil, 2002, Chapter 4.4) for definitions.

2.2 Infinite interconnections

We study interconnections of countably many systems, each given by a finite-dimensional ordinary differential equation (ODE). Using \mathbb{N} as the index set (by default), the i th subsystem is written as

$$\Sigma_i : \dot{x}_i = f_i(x_i, \bar{x}, u_i). \quad (1)$$

The family $(\Sigma_i)_{i \in \mathbb{N}}$ comes together with a number $p \in [1, \infty]$ and sequences $(n_i)_{i \in \mathbb{N}}, (m_i)_{i \in \mathbb{N}}$ of positive integers so that the following holds with $X := \ell^p(\mathbb{N}, (n_i))$ for a specified sequence of norms on the spaces \mathbb{R}^{n_i} :

- The state vector x_i of Σ_i is an element of \mathbb{R}^{n_i} .
- The internal input vector \bar{x} is an element of X .
- The external input vector u_i is an element of \mathbb{R}^{m_i} .
- The right-hand side $f_i : \mathbb{R}^{n_i} \times X \times \mathbb{R}^{m_i} \rightarrow \mathbb{R}^{n_i}$ is a continuous function.
- Unique local solutions of the ODE (1) exist for all initial states $x_{i0} \in \mathbb{R}^{n_i}$ and all continuous $\bar{x}(\cdot)$ and locally essentially bounded $u_i(\cdot)$ (which are regarded as time-dependent inputs). We denote the corresponding solution by $\phi_i(\cdot, x_{i0}, (\bar{x}, u_i))$.

The values of the function f_i can be independent of certain components of the input vector \bar{x} . We write I_i for the set of indices $j \in \mathbb{N}$ so that $f_i(x_i, \bar{x}, u_i)$ is non-constant with respect to the component x_j of \bar{x} , and w.l.o.g. we assume that $i \notin I_i$ (note that f_i depends on x_i explicitly).

In the ODE (1), we consider $\bar{x}(\cdot)$ as an *internal input* and $u_i(\cdot)$ as an *external input*. The interpretation is that the subsystem Σ_i is affected by a certain set of neighbors, indexed by I_i , and its external input. We note that the set I_i does *not* have to be finite, implying that subsystem i can be connected to infinitely many other subsystems.

To define the interconnection of the subsystems Σ_i , we consider the state vector $x = (x_i)_{i \in \mathbb{N}} \in X = \ell^p(\mathbb{N}, (n_i))$, the input vector $u = (u_i)_{i \in \mathbb{N}} \in \ell^q(\mathbb{N}, (m_i))$ for some $q \in [1, \infty]$ and the right-hand side $f(x, u) := (f_1(x_1, \bar{x}, u_1), f_2(x_2, \bar{x}, u_2), \dots)$. The interconnection is then written as

$$\Sigma : \dot{x} = f(x, u). \quad (2)$$

The class of admissible control functions is defined as

$$\mathcal{U} := \{u : \mathbb{R}_+ \rightarrow U : u \text{ is strongly measurable and}$$

essentially bounded}, (3)

and we equip this space with the sup-norm

$$\|u\|_{q,\infty} := \operatorname{ess\,sup}_{t \geq 0} |u(t)|_q.$$

A continuous mapping $\xi : I \rightarrow X$, defined on an interval $I = [0, T_*)$ with $T_* \in (0, \infty]$, is called a *solution* of the infinite-dimensional ODE (2) with initial value $x^0 \in X$ for the external input $u \in \mathcal{U}$ provided that the two conditions

$$f(\xi(t), u(t)) \in X \quad \text{and} \quad \xi(t) = x^0 + \int_0^t f(\xi(s), u(s)) ds$$

hold for all $t \in I$, where the integral is the Bochner integral, see e.g. (Arendt et al., 2011).

If for each $x^0 \in X$ and $u \in \mathcal{U}$ a unique (local) solution exists, we say that the system is *well-posed* and write $\phi(\cdot, x^0, u)$ for any such solution. As usual, we consider the maximal extension of $\phi(\cdot, x^0, u)$ and write $I_{\max}(x^0, u)$ for its interval of existence. We say that the system is *forward complete* if $I_{\max}(x^0, u) = \mathbb{R}_+$ for all $(x^0, u) \in X \times \mathcal{U}$.

We note that (Kawan et al., 2019, Thm. 3.2) provides sufficient conditions for well-posedness of Σ .

2.3 Distances in sequence spaces

Let $X = \ell^p(\mathbb{N}, (n_i))$ for a certain $p \in [1, \infty)$. Consider nonempty closed sets $\mathcal{A}_i \subset \mathbb{R}^{n_i}$, $i \in \mathbb{N}$. For each $x_i \in \mathbb{R}^{n_i}$ we define the distance of x_i to the set \mathcal{A}_i by

$$|x_i|_{\mathcal{A}_i} := \inf_{y_i \in \mathcal{A}_i} |x_i - y_i|.$$

Now we define the set

$$\mathcal{A} := \{x \in X : x_i \in \mathcal{A}_i, i \in \mathbb{N}\} = X \cap (\mathcal{A}_1 \times \mathcal{A}_2 \times \dots). (4)$$

If $\mathcal{A} \neq \emptyset$, we define the distance from any $x \in X$ to \mathcal{A} as

$$|x|_{\mathcal{A}} := \inf_{y \in \mathcal{A}} |x - y|_p = \inf_{y \in \mathcal{A}} \left(\sum_{i=1}^{\infty} |x_i - y_i|^p \right)^{\frac{1}{p}}. (5)$$

Lemma 2.1. Assume that \mathcal{A} defined by (4) is nonempty. Then for any $x \in X$ it holds that

$$\sum_{i=1}^{\infty} |x_i|_{\mathcal{A}_i}^p < \infty, \quad \text{and} \quad |x|_{\mathcal{A}} = \left(\sum_{i=1}^{\infty} |x_i|_{\mathcal{A}_i}^p \right)^{\frac{1}{p}}.$$

3. EXPONENTIAL INPUT-TO-STATE STABILITY

Having a well-posed interconnection (2) with state space $X = \ell^p(\mathbb{N}, (n_i))$ and external input space $U = \ell^q(\mathbb{N}, (m_i))$ for $p, q \in [1, \infty)$, we aim to study the stability of the interconnected system with respect to a closed set $\mathcal{A} \subset X$. For this purpose, we introduce the notions of input-to-state stability and exponential input-to-state stability with respect to a set \mathcal{A} .

Definition 3.1. Given a nonempty closed set $\mathcal{A} \subset X$, the system Σ is called

- *input-to-state stable (ISS) w.r.t. \mathcal{A}* if it is forward complete and there are functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for any initial state $x^0 \in X$ and any $u \in \mathcal{U}$ the corresponding solution satisfies

$$|\phi(t, x^0, u)|_{\mathcal{A}} \leq \beta(|x^0|_{\mathcal{A}}, t) + \gamma(\|u\|_{q,\infty}) \quad \text{for all } t \geq 0.$$

- *exponentially input-to-state stable (eISS) w.r.t. \mathcal{A}* if it is ISS w.r.t. \mathcal{A} with a \mathcal{KL} -function β of the form $\beta(t, r) = Me^{-at}r$ for some $a, M > 0$. \square

For any function $V : X \rightarrow \mathbb{R}$, which is continuous on $X \setminus \mathcal{A}$, we define the *orbital derivative* at $x \in X \setminus \mathcal{A}$ for the external input $u \in \mathcal{U}$ by

$$D^+V_u(x) := D^+V(\phi(t, x, u))|_{t=0},$$

where the right-hand side is the right upper Dini derivative of the function $t \mapsto V(\phi(t, x, u))$, evaluated at $t = 0$.

Exponential input-to-state stability is implied by the existence of a (power-bounded) exponential ISS Lyapunov function, which we define in a dissipative form as follows.

Definition 3.2. Let a nonempty closed set $\mathcal{A} \subset X$ be given. A function $V : X \rightarrow \mathbb{R}_+$, which is continuous on $X \setminus \mathcal{A}$, is called a *(power-bounded) eISS Lyapunov function* for Σ w.r.t. \mathcal{A} if there are constants $\underline{\omega}, \bar{\omega}, b, \kappa > 0$ and $\gamma \in \mathcal{K}_\infty$ such that

$$\underline{\omega}|x|_{\mathcal{A}}^b \leq V(x) \leq \bar{\omega}|x|_{\mathcal{A}}^b \quad \forall x \in X, (6a)$$

$$D^+V_u(x) \leq -\kappa V(x) + \gamma(\|u\|_{q,\infty}) \quad \forall x \in X \setminus \mathcal{A}, \forall u \in \mathcal{U}. (6b)$$

The function γ is sometimes called a *Lyapunov gain*. \square

Proposition 3.3. If there exists an eISS Lyapunov function for Σ w.r.t. \mathcal{A} , then Σ is eISS w.r.t. \mathcal{A} . \square

The proof follows similar steps as those in the proof of Proposition 4.4 in (Kawan et al., 2019).

4. THE GAIN OPERATOR AND ITS PROPERTIES

Our main objective is to develop conditions for input-to-state stability of the interconnection of countably many subsystems (1), depending on the ISS properties of the subsystems and the interconnection structure. Throughout this section, we assume that the infinite interconnection Σ is well-posed with state space $X = \ell^p(\mathbb{N}, (n_i))$ and external input space $U = \ell^q(\mathbb{N}, (m_i))$ for some $p, q \in [1, \infty)$.

4.1 Assumptions on the subsystems

We assume that each subsystem Σ_i , given by (1), is exponentially ISS w.r.t. a closed set \mathcal{A}_i and there exist continuous eISS Lyapunov functions w.r.t. \mathcal{A}_i with linear gains for all Σ_i . The following assumption formulates the eISS property for the subsystems.

Assumption 4.1. For each $i \in \mathbb{N}$ there exist a nonempty closed set $\mathcal{A}_i \subset \mathbb{R}^{n_i}$ and $V_i \in C^0(\mathbb{R}^{n_i}, \mathbb{R}_+)$, satisfying for certain $p, q \in [1, \infty)$ the following properties.

- There are constants $\underline{\alpha}_i, \bar{\alpha}_i > 0$ so that for all $x_i \in \mathbb{R}^{n_i}$

$$\underline{\alpha}_i|x_i|_{\mathcal{A}_i}^p \leq V_i(x_i) \leq \bar{\alpha}_i|x_i|_{\mathcal{A}_i}^p. (7)$$
- There are constants λ_i, γ_{ij} ($j \in I_i$), $\gamma_{iu} > 0$ so that the following holds: for each $x_i \in \mathbb{R}^{n_i} \setminus \mathcal{A}_i$, $u_i \in L^\infty(\mathbb{R}_+, \mathbb{R}^{m_i})$, each internal input $\bar{x} \in C^0(\mathbb{R}_+, X)$ and for almost all t in the maximal interval of existence of $\phi_i(t) := \phi_i(t, x_i, (\bar{x}, u_i))$ one has

$$D^+(V_i \circ \phi_i)(t) \leq -\lambda_i V_i(\phi_i(t)) + \sum_{j \in I_i} \gamma_{ij} V_j(x_j(t)) + \gamma_{iu} \|u_i(t)\|^q, (8)$$

where we denote the components of \bar{x} by $x_j(\cdot)$.¹

- For all t in the maximal interval of the existence of ϕ_i one has $D_+(V_i \circ \phi_i)(t) < \infty$. \square

¹ At this point, the right-hand side of (8) is not necessarily finite. However, this requirement is not needed here.

Note that if V_i is continuously differentiable, Assumption 4.1 can be written in a simpler form, see (Kawan et al., 2019) for details.

We furthermore assume that the following uniformity conditions hold for the constants introduced above.

Assumption 4.2. (a) There are constants $\underline{\alpha}, \bar{\alpha} > 0$ so that for all $i \in \mathbb{N}$

$$\underline{\alpha} \leq \underline{\alpha}_i \leq \bar{\alpha}_i \leq \bar{\alpha}. \quad (9)$$

(b) There is a constant $\underline{\lambda} > 0$ so that for all $i \in \mathbb{N}$

$$\underline{\lambda} \leq \lambda_i. \quad (10)$$

(c) There is a constant $\bar{\gamma}_u > 0$ so that for all $i \in \mathbb{N}$

$$\gamma_{iu} \leq \bar{\gamma}_u. \quad (11)$$

□

In order to formulate a small-gain condition, we further introduce the following infinite nonnegative matrices by collecting the coefficients from (8)

$$\Lambda := \text{diag}(\lambda_1, \lambda_2, \lambda_3, \dots), \quad \Gamma := (\gamma_{ij})_{i,j \in \mathbb{N}},$$

where we put $\gamma_{ij} := 0$ whenever $j \notin I_i$. We also introduce the infinite matrix

$$\Psi := \Lambda^{-1}\Gamma = (\psi_{ij})_{i,j \in \mathbb{N}}, \quad \psi_{ij} = \frac{\gamma_{ij}}{\lambda_i}. \quad (12)$$

We make the following assumption, which is equivalent to Γ being a bounded operator from ℓ^1 to ℓ^1 .

Assumption 4.3. The matrix $\Gamma = (\gamma_{ij})$ satisfies

$$\|\Gamma\|_{1,1} = \sup_{j \in \mathbb{N}} \sum_{i=1}^{\infty} \gamma_{ij} < \infty, \quad (13)$$

where the double index on the left-hand side indicates that we consider the operator norm induced by the ℓ^1 -norm both on the domain and codomain of the operator Γ . □

Under Assumptions 4.3 and 4.2(b) (see also (Kawan et al., 2019, Lem. V.7)), the matrix Ψ acts as a linear operator on ℓ^1 by

$$(\Psi x)_i = \sum_{j=1}^{\infty} \psi_{ij} x_j \quad \text{for all } i \in \mathbb{N}.$$

We call $\Psi : \ell^1 \rightarrow \ell^1$ the *gain operator* associated with the decay rates λ_i and coefficients γ_{ij} .

Moreover, clearly Ψ is a positive operator with respect to the standard positive cone $\ell^1_+ := \{x = (x_1, x_2, \dots) \in \ell^1 : x_i \geq 0, \forall i \in \mathbb{N}\}$ in ℓ^1 . Also recall from (Kawan et al., 2019, Lem. V.10) the following lemma which uses positive operator theory to deduce the existence of a positive vector μ that can be used to construct an eISS Lyapunov function for the interconnected system.

Lemma 4.4. Assume that $r(\Psi) < 1$ and that there exists a constant $\bar{\lambda} > 0$ such that $\lambda_i \leq \bar{\lambda}$ for all $i \in \mathbb{N}$. Then the following statements hold:

(i) There exist a vector $\mu = (\mu_i)_{i \in \mathbb{N}} \in \text{int } \ell^1_+$ and a constant $\lambda_\infty > 0$ so that

$$\frac{[\mu^\top(-\Lambda + \Gamma)]_i}{\mu_i} \leq -\lambda_\infty \quad \text{for all } i \in \mathbb{N}. \quad (14)$$

(ii) For every $\rho > 0$ we can choose the vector μ and the constant λ_∞ so that

$$\lambda_\infty \geq (1 - r(\Psi))\underline{\lambda} - \rho. \quad (15)$$

□

5. SMALL-GAIN THEOREM

In this section, we prove that the interconnected system Σ is exponentially ISS under the given assumptions, provided that the spectral radius of the gain operator satisfies $r(\Psi) < 1$. By Proposition 3.3, our objective is reduced to finding an eISS Lyapunov function for the interconnection Σ , which is accomplished by the following *small-gain theorem*, which is the main result of the paper.

Theorem 5.1. Consider the infinite interconnection Σ , composed of the subsystems Σ_i , $i \in \mathbb{N}$, with fixed $p, q \in [1, \infty)$. Suppose that the following hold.

- (i) Σ is well-posed as a system with state space $X = \ell^p(\mathbb{N}, (n_i))$, space of input values $U = \ell^q(\mathbb{N}, (m_i))$, and the external input space \mathcal{U} , as defined in (3).
- (ii) Each Σ_i admits a continuous eISS Lyapunov function V_i w.r.t. a nonempty closed set $\mathcal{A}_i \subset \mathbb{R}^{n_i}$ so that Assumptions 4.1 and 4.2 are satisfied.
- (iii) The operator $\Gamma : \ell^1 \rightarrow \ell^1$ is bounded, i.e., Assumption 4.3 holds.
- (iv) The spectral radius of Ψ satisfies $r(\Psi) < 1$.

Consider the set $\mathcal{A} := X \cap (\mathcal{A}_1 \times \mathcal{A}_2 \times \dots)$. Then Σ admits an eISS Lyapunov function w.r.t. \mathcal{A} of the form

$$V(x) = \sum_{i=1}^{\infty} \mu_i V_i(x_i), \quad V : X \rightarrow \mathbb{R}_+ \quad (16)$$

for some $\mu = (\mu_i)_{i \in \mathbb{N}} \in \ell^\infty$ satisfying $\underline{\mu} \leq \mu_i \leq \bar{\mu}$ with constants $\underline{\mu}, \bar{\mu} > 0$. In particular, the function V has the following properties.

- (a) V is continuous on $X \setminus \mathcal{A}$.
- (b) There is a $\lambda_\infty > 0$ so that for all $x^0 \in X \setminus \mathcal{A}$ and $u \in \mathcal{U}$

$$D^+ V_u(x^0) \leq -\lambda_\infty V(x^0) + \bar{\mu} \bar{\gamma}_u |u|_{q,\infty}^q.$$
- (c) For all $x \in X$ the following inequalities hold:

$$\underline{\mu} \underline{\alpha} |x|_{\mathcal{A}}^p \leq V(x) \leq \bar{\mu} \bar{\alpha} |x|_{\mathcal{A}}^p. \quad (17)$$

In particular, Σ is eISS w.r.t. \mathcal{A} . □

The proof follows closely the proof of (Kawan et al., 2019, Thm. VI.1) and is based on the application of Lemma 4.4. Hence, it is omitted.

6. APPLICATIONS

In this section, we study three applications: stability analysis of time-varying interconnections, dynamic average consensus and the design of distributed observers for infinite networks.

6.1 Time-varying interconnected systems

Although our main result only considers time-invariant systems, it can also be applied to time-varying systems by transforming a time-varying system into a time-invariant one of the form (2). To see this, consider the time-varying system

$$\dot{x} = f(t, x, u), \quad (18)$$

where $x \in X$, $u \in U$ and $f : \mathbb{R} \times X \times U \rightarrow X$ is continuous with $f(t, 0, 0) = 0$ for all $t \in \mathbb{R}$. Using the same arguments as those for well-posedness of the network (2), we assume that the state space X and the input space U are chosen as $X = \ell^p(\mathbb{N}, (n_i))$ and $U = \ell^q(\mathbb{N}, (m_i))$, respectively, for fixed $p, q \in [1, \infty)$. The same class of admissible control functions as in (3) is considered here.

We assume that unique solutions exist for all initial times, initial states and admissible inputs. For any initial time $t^0 \in \mathbb{R}$, initial value $x^0 \in X$ and input $u \in \mathcal{U}$, the corresponding solution of (18) is denoted by $\phi(\cdot, t^0, x^0, u)$.

Definition 6.1. The system (18) is called *uniformly exponentially input-to-state stable (UeISS)* if it is forward complete and there are constants $a, M > 0$, and $\gamma \in \mathcal{K}$ such that for any initial time $t^0 \in \mathbb{R}$, initial state $x^0 \in X$ and external input $u \in \mathcal{U}$ the corresponding solution of (18) satisfies for all $t \geq t^0$

$$|\phi(t, t^0, x^0, u)|_p \leq M e^{-a(t-t^0)} |x^0|_p + \gamma(|u(t^0 + \cdot)|_{q, \infty}).$$

Uniformity here means that a, M do not depend on t^0 .

By adding a ‘‘clock’’, one can (see e.g. (Teel, Andrew R. and Praly, Laurent, 2000; Teel et al., 2002)) transform (18) into

$$\dot{y} = 1, \quad \dot{z} = f(y, z, u), \quad (19)$$

where $y \in \mathbb{R}$, $z \in X$, $u \in U$. We equip \mathbb{R} with an arbitrary norm $|\cdot|$ and turn $\mathbb{R} \times X$ into an ℓ^p space by putting

$$|(y, z)|_p := (|y|^p + |z|_p^p)^{1/p}.$$

Denoting the transition map of (19) by $\tilde{\phi} = \tilde{\phi}(t, (y, z), u)$, and its z -component by $\tilde{\phi}_2$, we see that the following holds:

$$\phi(t, t^0, x, u) = \tilde{\phi}_2(t - t^0, (t^0, x), u(t^0 + \cdot)) \quad \forall t \geq t^0. \quad (20)$$

The stability properties of (18) and (19) are related in the following way:

Proposition 6.2. The system (18) is UeISS if and only if (19) is eISS with respect to the closed set $\mathcal{A} = \{(y, z) \in \mathbb{R} \times X : z = 0\} = \mathbb{R} \times \{0\}$. \square

Now assume that the system (18) can be decomposed into infinitely many interconnected subsystems

$$\dot{x}_i = f_i(t, x_i, \bar{x}, u_i), \quad i \in \mathbb{N}, \quad (21)$$

with $t \in \mathbb{R}$, $x_i \in \mathbb{R}^{n_i}$, $\bar{x} \in X$ and $u_i \in \mathbb{R}^{m_i}$. Also, let $f_i: \mathbb{R} \times \mathbb{R}^{n_i} \times X \times \mathbb{R}^{m_i} \rightarrow \mathbb{R}^{n_i}$ be continuous with $f_i(t, 0, 0, 0) = 0$ for all $t \in \mathbb{R}$.

To each of the systems (21) we associate a time-invariant system by

$$\dot{z}_i = \tilde{f}_i(z_i, (y, \bar{z}), u_i) := f_i(y, z_i, \bar{z}, u_i), \quad (22)$$

where the time t now becomes an additional internal input y . Define $\mathcal{A}_0 := \mathbb{R}$ and $\mathcal{A}_i := \{0\} \subset \mathbb{R}^{n_i}$ for all $i \geq 1$. Aggregating all subsystems (22), $i \in \mathbb{N}$, and adding the clock $\dot{y} = 1$ as the 0th subsystem, we obtain an infinite network of the form (19), modeled on the state space $\ell^p(\mathbb{N}_0, (n_i))$ with $n_0 := 1$.

To enable the stability analysis of the composite system, we make the following assumption.

Assumption 6.3. For each $i \in \mathbb{N}$ there exists a continuous function $V_i: \mathbb{R}^{n_i} \rightarrow \mathbb{R}_+$, satisfying for certain $p, q \in [1, \infty)$ the following properties.

- There are constants $\underline{\alpha}_i, \bar{\alpha}_i > 0$ so that for all $z_i \in \mathbb{R}^{n_i}$

$$\underline{\alpha}_i |z_i|^p \leq V_i(z_i) \leq \bar{\alpha}_i |z_i|^p. \quad (23)$$
- There are constants $\lambda_i, \gamma_{ij}, \gamma_{iu} > 0$ so that the following holds: for each $z_i \in \mathbb{R}^{n_i}$, $u_i \in L^\infty(\mathbb{R}_+, \mathbb{R}^{m_i})$ and each internal input $(y, \bar{z}) \in C^0(\mathbb{R}_+, \mathbb{R} \times X)$ and for almost all t in the maximal interval of existence of $\phi_i(t) := \phi_i(t, z_i, (y, \bar{z}, u_i))$ one has

$$D^+(V_i \circ \phi_i)(t) \leq -\lambda_i V_i(\phi_i(t)) + \sum_{j \in \mathcal{I}_i} \gamma_{ij} V_j(z_j(t))$$

$$+ \gamma_{iu} |u_i(t)|^q, \quad (24)$$

where we denote the components of \bar{z} by $z_j(\cdot)$.

- For all t in the maximal interval of the existence of ϕ_i one has $D_+(V_i \circ \phi_i)(t) < \infty$. \square

Note that due to the inequalities (7) and $\mathcal{A}_0 = \mathbb{R}$, we necessarily have $V_0 = 0$ for the eISS Lyapunov function of the 0th subsystem. Furthermore, we can choose λ_0 as an arbitrary positive number and $\gamma_{0j} := 0$ for all $j \in \mathbb{N}$.

It follows from Theorem 5.1 that under Assumption 6.3, the infinite network of systems (21) is UeISS. This is summarized by the following corollary.

Corollary 6.4. Consider networks (18) and (19) and suppose the following:

- Assumption 6.3 holds.
- The constants in Assumption 6.3 are uniformly bounded as in Assumption 4.2.
- The operator $\Gamma: \ell^1 \rightarrow \ell^1$ is bounded, i.e., Assumption 4.3 holds.
- The spectral radius of Ψ satisfies $r(\Psi) < 1$.

Then the composite system (18) is uniformly eISS. \square

6.2 Dynamic average consensus

Let $G := (\mathcal{V}, \mathcal{E})$ be an undirected graph with the set of nodes $\mathcal{V} = \mathbb{N}$ and the set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. An edge (i, j) in an undirected infinite graph denotes that nodes j and i exchange information bidirectionally. Node j is an *input neighbor* of node i if $(j, i) \in \mathcal{E}$. We assume that each agent can only communicate with a finite number of other agents, known as neighbors. Let $\mathcal{N}_i = \{j | (j, i) \in \mathcal{E}\}$ denote the set of the input neighbors of node i .

Let $x_i \in \mathbb{R}^n$ denote the state of node $i \in \mathcal{V}$. Let each node of G be a (dynamic) agent with dynamics

$$\Sigma_i: \quad \dot{x}_i = f_i(x_i) + B u_i, \quad i \in \mathbb{N}, \quad (25)$$

where $u_i \in \mathbb{R}^m$ is the control input, the continuous function $f_i: \mathbb{R}^n \rightarrow \mathbb{R}^n$ represents the dynamics of each uncoupled node, and $B \in \mathbb{R}^{n \times m}$. We model the interconnection Σ of these systems on the state space $X := \ell^\infty(\mathbb{N}, n)$ with the external input space $U := \ell^\infty(\mathbb{N}, m)$ and assume well-posedness for the class of controls \mathcal{U} as defined before.

Note that the dynamics in (25) do not directly depend on the neighbors' states. But these states might enter the input, i.e., we can define a control law $u_i = q_i(x_i, \bar{x}_i)$, where q_i is a continuous function on $\mathbb{R}^n \times \mathbb{R}^{N_i}$, $N_i := |\mathcal{N}_i|n$, and $\bar{x}_i \in \mathbb{R}^{N_i}$ is the augmented vector of the states of the neighbors. The aim is to establish control laws, which asymptotically lead to consensus of the agents defined as follows. The agents of the network have reached *consensus* if and only if $x_i = x_j$ for all $i, j \in \mathcal{V}$. A corresponding state value is called a *consensus point*.

In several applications of distributed cooperative control, the problem of interest can be formulated as a so-called *dynamic average consensus problem* in which a group of agents cooperates to track a weighted average of locally available time-varying reference signals. To define a meaningful average of infinitely many quantities, we choose a sequence $(\alpha_i)_{i \in \mathbb{N}}$ of positive real numbers satisfying $\sum_{i=1}^{\infty} \alpha_i = 1$. One can interpret this sequence as a probability distribution on \mathbb{N} . It is of particular interest to track the following weighted average:

$$x_a := \sum_{i=1}^{\infty} \alpha_i x_i. \quad (26)$$

We observe that for every $x \in X$ we have

$$|x_a| \leq \sum_{i=1}^{\infty} \alpha_i |x_i| \leq |x|_{\infty} < \infty.$$

The interconnections of the nodes, which are produced by the control law q_i , depend on the strength of the coupling and on the state variables of the nodes. Here we consider the most popular type of coupling which is known as *diffusive coupling* (Ren et al., 2007). We assume that the coupling between the i th and j th agents is defined as a weighted difference, i.e., $a_{ij}(x_i - x_j)$. Therefore, the control input u_i is given by

$$u_i := -\sigma \sum_{j \in \mathcal{N}_i} \alpha_j a_{ij} (x_i - x_j), \quad (27)$$

where $\sigma > 0$ denotes the coupling gain between the agents and the interconnections weights a_{ij} satisfy

$$a_{ij} = a_{ji} > 0, \quad i, j \in \mathbb{N} \quad \text{and} \quad \sup_{i,j} a_{ij} = 1. \quad (28)$$

We assume that $a_{ij} = 0$ for $j \in \mathbb{N} \setminus \mathcal{N}_i$ and we note that $a_{ij} = 0$ reflects the fact that agent i does not communicate with agent j .

We aim to choose the a_{ij} 's and σ in (27) such that $x_i(t) \rightarrow x_j(t) \rightarrow x_a(t)$ for all $i, j \in \mathcal{V}$ as $t \rightarrow \infty$. *The difficulty of the dynamic average consensus problem is that each agent is normally connected to only few other agents, and therefore x_a is not available to each agent.*

Under mild assumptions on the vector fields f_i (uniform local boundedness and uniform local Lipschitz continuity), it can be shown that the average $x_a(t)$ is continuously differentiable for any solution $\phi(t)$ of Σ corresponding to a continuous control input and $\dot{x}_a(t) = \sum_{i=1}^{\infty} \alpha_i \dot{\phi}_i(t)$.

Let us define the error by

$$e_i := \alpha_i (x_i - x_a), \quad i \in \mathbb{N}.$$

One can show that $e \in \ell^1(\mathbb{N}, n)$, i.e. $|e|_1 < \infty$. The dynamics of the average is

$$\dot{x}_a = \sum_{i=1}^{\infty} \alpha_i (f_i(\alpha_i^{-1} e_i + x_a) - \sigma B \sum_{j \in \mathcal{N}_i} \alpha_j a_{ij} (\alpha_i^{-1} e_i - \alpha_j^{-1} e_j)).$$

Note that using the symmetry condition in (28) the coupling term vanishes; i.e. $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \alpha_i \alpha_j a_{ij} (\alpha_i^{-1} e_i - \alpha_j^{-1} e_j) = 0$.

The convergence of the remaining sum follows from the estimate $\sum_{i=1}^N \sum_{j=1}^M |\alpha_j a_{ij} e_i| \leq |e|_1$ for all $N, M \in \mathbb{N}$. Hence,

$$\dot{x}_a = \sum_{i=1}^{\infty} \alpha_i f_i(\alpha_i^{-1} e_i + x_a). \quad (29)$$

The dynamics of the the errors e_i , $i \in \mathbb{N}$, is given by

$$\begin{aligned} \dot{e}_i &= \alpha_i f_i(\alpha_i^{-1} e_i + x_a) \\ &\quad - \alpha_i \sigma B \sum_{j \in \mathcal{N}_i} \alpha_j a_{ij} (\alpha_i^{-1} e_i - \alpha_j^{-1} e_j) \\ &\quad - \alpha_i \sum_{j=1}^{\infty} \alpha_j f_j(\alpha_j^{-1} e_j + x_a). \end{aligned} \quad (30)$$

We write $\hat{\Sigma}_i$ for the i th subsystem, where we start the enumeration with $i = 0$ so that x_a is the state of the 0th subsystem. The state space of the overall system will be taken to be $\hat{X} := \ell^1(\mathbb{N}_0, n)$.

Now we study the stabilization of the average and error system $\hat{\Sigma}$ w.r.t. the closed set $\mathcal{A} := \mathbb{R}^n \times \{0\} \times \{0\} \times \dots \subset \ell^1(\mathbb{N}_0, n)$. From our main result, Theorem 5.1, we can immediately conclude the following.

Theorem 6.5. Consider the interconnection $\hat{\Sigma}$ of the subsystems $\hat{\Sigma}_i$, $i \in \mathbb{N}_0$, and assume that the following assumptions hold:

- (i) The system $\hat{\Sigma}$ with state space $\hat{X} = \ell^1(\mathbb{N}_0, n)$ is well-posed.
- (ii) Each subsystem $\hat{\Sigma}_i$, $i \geq 1$, admits a continuous eISS Lyapunov function V_i (with respect to the trivial set $\{0\}$, i.e., in the usual sense) so that Assumptions 4.1 and 4.2 are satisfied.
- (iii) The operator $\Gamma : \ell^1 \rightarrow \ell^1$ is bounded, i.e., Assumption 4.3 holds.
- (iv) The spectral radius of Ψ satisfies $r(\Psi) < 1$.

Then $\hat{\Sigma}$ is eISS w.r.t. the set $\mathcal{A} = \mathbb{R}^n \times \{0\} \times \{0\} \times \dots \subset \ell^1(\mathbb{N}_0, n)$, which implies that there are $M > 0$ and $a > 0$ so that

$$|e(t)|_1 = \sum_{i=1}^{\infty} \alpha_i |\phi_i(t) - x_a(t)| \leq M e^{-at} |e(0)|_1. \quad (31)$$

□

Although Theorem 6.5 explicitly makes no assumption on the connectedness of the associated graph G , the verification of the conditions in the theorem often asks for the connectedness of G . We note that in some trivial cases; e.g. if all agents Σ_i are linear and individually asymptotically stable, all the conditions will be trivially fulfilled even without making a connectedness assumption.

Remark 6.6. Of particular interest in weighted average consensus applications is how to choose the weights α_i . A particular application of weighted average consensus is distributed cooperative spectrum sensing, in which the main objective is to develop distributed protocols for solving the cooperative sensing problem in cognitive radio systems; see e.g. (Hernandes et al., 2018; Zhang et al., 2015; Li and Guo, 2015). The weights in this case represent a ratio related to the channel conditions of each agent.

6.3 Distributed observers

We consider the problem of constructing *distributed observers for networks* of control systems. For simplicity, we set the external inputs u_i to zero and focus on the network interconnection aspect, rather than discussing the construction of individual local observers.

Our basic assumption is that in a network context, we have local observers of local subsystems. We assume that the states of these *local observers* asymptotically converge to the true state of each subsystem, given perfect knowledge of the true states of neighboring subsystems. Of course such information will be unavailable in practice, and instead each local observer will at best have the state estimates produced by other, neighboring observers available for its operation.

Distributed system to be observed: Let the distributed nominal system consist of infinitely many interconnected subsystems

$$\Sigma_i: \begin{cases} \dot{x}_i = f_i(x_i, \bar{x}_i) \\ y_i = h_i(x_i, \bar{x}_i) \end{cases}, \quad i \in \mathbb{N}. \quad (32)$$

While $x_i \in \mathbb{R}^{n_i}$ is the state of the system Σ_i , the quantity $y_i \in \mathbb{R}^{p_i}$ (for some $p_i \in \mathbb{N}$) is the output that can be measured locally and serves as an input for a state observer. We denote by \bar{x}_i the vector composed of the state variables x_j , $j \in I_i$. Although our general setting allows each subsystem to directly interact with infinitely many other subsystems, in distributed sensing normally each subsystem is only connected to a finite number of subsystems. Therefore, the set I_i is assumed to be bounded in this application. To make this observation as clear as possible, in (32), as opposed to the main body of the paper, we use the notation \bar{x}_i in place of \bar{x} . Further we assume that $f_i: \mathbb{R}^{n_i} \times \mathbb{R}^{N_i} \rightarrow \mathbb{R}^{n_i}$ and $h_i: \mathbb{R}^{n_i} \times \mathbb{R}^{N_i} \rightarrow \mathbb{R}^{p_i}$ are both continuous, where $N_i := \sum_{j \in I_i} n_j$.

Structure of the distributed observers: It is reasonable to assume that a local observer \mathcal{O}_i for a system Σ_i has access to y_i and produces an estimate \hat{x}_i of x_i for all $t \geq 0$. Moreover, we essentially need to know x_j for all $j \in I_i$ to reproduce the dynamics (32). Access to this kind of information is unrealistic, so instead we assume that it has access to the outputs y_j of neighboring subsystems and/or the estimates \hat{x}_j for $j \in I_i$ produced by neighboring observers.

This basically means that each *local observer* is represented by

$$\mathcal{O}_i: \quad \dot{\hat{x}}_i = \hat{f}_i(\hat{x}_i, y_i, \bar{y}_i, \hat{x}_i) \quad (33)$$

for some appropriate continuous function \hat{f}_i . Here \bar{y}_i (resp. \hat{x}_i) is composed of the outputs y_j (resp. state variables \hat{x}_j), $j \in I_i$.

Necessarily, the observers are coupled in the same directional sense as the original distributed subsystems. Based on the small-gain theorem introduced above, this leads us to a framework for the design of distributed observers that guarantees that an interconnection of local observers exponentially tracks the true system state. Thus we consider the composite system given by

$$\dot{x}_i = f_i(x_i, \bar{x}_i), \quad y_i = h_i(x_i, \bar{x}_i), \quad (34a)$$

$$\dot{\hat{x}}_i = \hat{f}_i(\hat{x}_i, y_i, \bar{y}_i, \hat{x}_i), \quad i \in \mathbb{N}. \quad (34b)$$

A consistency framework for the design of distributed observers: Denote by ϕ_i and $\hat{\phi}_i$ the flow maps of the x_i -subsystem and \hat{x}_i -subsystem of (34), respectively, and define

$$\mathcal{A}_i := \{(x_i, \hat{x}_i) \in \mathbb{R}^{n_i} \times \mathbb{R}^{n_i} : x_i = \hat{x}_i\}, \quad i \in \mathbb{N}.$$

Denote also by ϕ and $\hat{\phi}$ the flow maps of x -subsystem and \hat{x} -subsystem of (34), respectively.

Assumption 6.7. We assume that the sequence of local observers $\mathcal{O} = (\mathcal{O}_i)_{i \in \mathbb{N}}$ for $\Sigma = (\Sigma_i)_{i \in \mathbb{N}}$ is given. Further, there is $p \in [1, \infty)$ so that for each $i \in \mathbb{N}$ there exists a continuous function $V_i: \mathbb{R}^{n_i} \rightarrow \mathbb{R}_+$, as well as constants $\bar{\alpha}_i, \underline{\alpha}_i > 0$ and $\lambda_i, \gamma_{ij} > 0$, $j \in I_i$ such that for all $x_i, \hat{x}_i \in \mathbb{R}^{n_i}$ the following holds:

$$\underline{\alpha}_i |x_i - \hat{x}_i|^p \leq V_i(x_i, \hat{x}_i) \leq \bar{\alpha}_i |x_i - \hat{x}_i|^p. \quad (35)$$

Furthermore, we assume that dissipative estimates

$$\begin{aligned} D^+(V_i \circ (\phi_i, \hat{\phi}_i))(t) &\leq -\lambda_i V_i(\phi_i(t), \hat{\phi}_i(t)) \\ &\quad + \sum_{j \in I_i} \gamma_{ij} V_j(x_j(t), \hat{x}_j(t)) \end{aligned} \quad (36)$$

hold for all $i \in \mathbb{N}$ and for all t in the maximal interval of the existence of ϕ_i and $\hat{\phi}_i$ we have $D_+(V_i \circ (\phi_i, \hat{\phi}_i))(t) < \infty$. \square

Following our general framework, we choose the state space for the whole system as $X := \ell^p(\mathbb{N}, (n_i))$ for p as in (35).

We would like to derive conditions, which ensure that a network of local observers $\mathcal{O} = (\mathcal{O}_i)_{i \in \mathbb{N}}$ is a robust distributed observer for the whole system Σ , i.e., the error dynamics of the composite system (34) is globally exponentially stable.

Consider $X \times X$ as a Banach space with the norm $\|(x, y)\|_{X \times X} := \sqrt{|x|_p^2 + |y|_p^2}$, $(x, y) \in X \times X$ and define

$$\mathcal{A} := \{(x, \hat{x}) \in X \times X : x = \hat{x}\} = X \cap \mathcal{A}_1 \cap \mathcal{A}_2 \cap \dots \quad (37)$$

We pose the result of this subsection as a corollary, whose proof is a direct consequence of Theorem 5.1.

Theorem 6.8. Consider the infinite interconnection Σ , given by equations (32), and the corresponding composite system (34), with fixed $p \in [1, \infty)$. Suppose that the following hold.

- (i) (34) is well-posed as a system on $X \times X$, with $X = \ell^p(\mathbb{N}, (n_i))$ as a state space of Σ .
- (ii) Each Σ_i admits a continuous eISS Lyapunov function V_i so that Assumptions 6.7 and 4.2 are satisfied.
- (iii) The operator $\Gamma: \ell^1 \rightarrow \ell^1$ is bounded, i.e., Assumption 4.3 holds.
- (iv) The spectral radius of Ψ satisfies $r(\Psi) < 1$.

Then the composite system (34) admits a Lyapunov function w.r.t. \mathcal{A} as defined in (37) of the form

$$V(x, \hat{x}) = \sum_{i=1}^{\infty} \mu_i V_i(x_i, \hat{x}_i), \quad V: X \times X \rightarrow \mathbb{R}_+ \quad (38)$$

for some $\mu = (\mu_i)_{i \in \mathbb{N}} \in \ell^\infty$ satisfying $\underline{\mu} \leq \mu_i \leq \bar{\mu}$ with some constants $\underline{\mu}, \bar{\mu} > 0$. In particular, the function V has the following properties.

- (a) V is continuous on $(X \times X) \setminus \mathcal{A}$.
- (b) There is a $\lambda_\infty > 0$ so that for all $x^0 \in (X \times X) \setminus \mathcal{A}$

$$D^+ V_u(x^0) \leq -\lambda_\infty V(x^0).$$
- (c) For all $x, \hat{x} \in X$ the following inequalities hold

$$\underline{\mu} \underline{\alpha} |(x, \hat{x})|_{\mathcal{A}}^p \leq V(x, \hat{x}) \leq \bar{\mu} \bar{\alpha} |(x, \hat{x})|_{\mathcal{A}}^p. \quad (39)$$

Consequently, the error dynamics of (34) is globally exponentially stable, i.e., there is $\beta \in \mathcal{KL}$ so that the following holds for all $x, \hat{x} \in X$ and all $t \geq 0$:

$$|\phi(t, x) - \hat{\phi}(t, \hat{x})|_p \leq \beta(|x - \hat{x}|_p, t), \quad (40)$$

which in turn means that $\mathcal{O} = (\mathcal{O}_i)_{i \in \mathbb{N}}$ is a robust distributed observer for Σ . \square

7. CONCLUSIONS

We developed a small-gain theorem ensuing exponential ISS with respect to a closed set for infinite networks. The small-gain condition is given in terms of the spectral radius representing the coupling between participating subsystems, which can be very efficiently checked for a large class

of systems. We illustrated the applicability of our small-gain theorem by applying it to three different problems including stability of time-varying infinite networks at the origin, weighted average consensus, and distributed state estimation.

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