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A relaxed small-gain theorem for discrete-time infinite networks

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Abstract— **This paper provides a small-gain theorem for a so-called infinite network, i.e. a network composed of infinitely many finite-dimensional systems. Such a network is mainly motivated by addressing the scalability issue in large-but-finite networks. We develop a so-called *relaxed* small-gain theorem for input-to-state stability (ISS) with respect to a closed set. It is shown that every exponentially ISS network *necessarily* satisfies the proposed small-gain condition. Finally, we truncate the infinite network to obtain a large-but-finite network for all stability properties and performance indices obtained for its infinite counterpart are preserved, if each subsystem is *individually* ISS. The effectiveness of our small-gain theorem is verified by application to an urban traffic network.**

I. Introduction

Standard tools for stability analysis and control do not scale well to the size of emerging networked systems; see e.g. [1]–[3]. Such scalability issues can be addressed by over-approximating a finite-but-large network with a network of infinitely many subsystems called *infinite network* [4]. By treating this over-approximated network, one expects to develop *scale-free* tools for analysis and control. In particular, it is envisaged that an infinite network captures the essence of its corresponding finite network, in terms of performance/stability indices; cf., e.g., a vehicle platooning application in [3]. However, this expectation has to be rigorously checked as *counter-intuitive* results for control of vehicle platoons have been already observed in [5].

This paper investigates input-to-state stability (ISS) with respect to closed sets of discrete-time infinite networks within a small-gain framework. We develop so-called *relaxed* small-gain conditions for which, in contrast with classic small-gain conditions [6], [7], not every subsystem has to be individually ISS. Indeed, subsystems may have stabilizing effects and can be individually *unstable*; see [8]–[10] for finite network examples. For the case of exponential ISS we show necessity and sufficiency of the proposed small-gain conditions. By truncation finite networks (of possibly *unknown* size) are obtained from the infinite network. We show that if each subsystem in the infinite network is individually ISS, all the stability and/or performance indices obtained for the infinite network are *preserved* for truncations. In that way, the small-gain conditions for the finite network are *independent* of network size. The effectiveness of our results is illustrated by application to a traffic network.

Related literature: ISS small-gain theory for networks of countably many finite-dimensional continuous-time systems has been studied in [11]–[14]. By [11], a network of an infinite number of ISS systems is also ISS, if the internal gains are uniformly less than identity. [1] shows that classic max-form small-gain conditions (SGCs) developed for finite-dimensional systems [6] do not guarantee the stability of infinite networks of ISS systems. To address this, [1] develops robust strong SGCs, but the small-gain criteria obtained are not tight and more investigations are needed.

For the case of linear gain operators, tight results have been achieved in [12], where sum-type SGCs for exponential ISS of infinite networks are developed. In [12] each subsystem is assumed to be exponentially ISS and a spectral small-gain condition is presented. This work is extended to exponential ISS with respect to closed sets in [13]. This approach covers a wide range of stability/stabilization problems including incremental stability, consensus type problems, ISS of time-varying systems in a *unified* setting [13]. In all of the above works, a Lyapunov small-gain approach is used. Inspired by [7], a trajectory-based formulation for infinite networks has been developed in [14].

In classic ISS small-gain theorems, it is required that each subsystem is *individually* ISS. To address this obvious conservatism, the notion of finite-step Lyapunov functions [15], [16] can be merged with a Lyapunov-based small-gain methodology, which leads to so-called *relaxed SGCs* [8]–[10]. A finite-step Lyapunov function need not decay every single time step, but only after a finite number of steps. In a network setting, this property allows for the consideration of potentially stabilizing effects of subsystems on each other. Relaxed SGCs are shown to be *necessary* and *sufficient* and are applicable to networks with unstable subsystems. In [17] relaxed SGCs for ISS of finite-dimensional networks with respect to closed sets have been developed.

In this work, we extend the results in [17] to infinite networks. Such a generalization leads to several *nontrivialities* as this calls for a careful choice of the infinite-dimensional state space, and developing direct and converse Lyapunov theorems in an infinite-dimensional setting. In the infinite networks context, our work is close to [11] as we also assume the internal ISS gains to be less than identity. However, the SGCs in [11] are only sufficient, while here we show the *necessity* of our formulation in case of exponential ISS.

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II. Preliminaries

A. Notation

We write $\mathbb{N}$ for the set of positive integers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For vector norms on (in)finite-dimensional vector spaces, we write $|\cdot|$. For the sets of comparison functions $\mathcal{K}, \mathcal{K}_\infty$, and $\mathcal{KL}$, see [18]. For $\alpha, \gamma \in \mathcal{K}$ we write $\alpha < \gamma$ if $\alpha(s) < \gamma(s)$ for all $s > 0$. Composition of functions is denoted by the symbol $\circ$ and repeated composition of, e.g., a function γ is defined inductively by $\gamma^1 := \gamma$, $\gamma^{i+1} := \gamma \circ \gamma^i$.

B. System description

We study the interconnection of countably many systems, each given by a finite-dimensional difference equation. Using $\mathbb{N}$ as the index set (by default), the subsystem i is written as

$$\Sigma_i : \quad x_i^+ = f_i(x_i, \bar{x}_i, u_i). \tag{1}$$

The family $(\Sigma_i)_{i\in\mathbb{N}}$ comes together with sequences $(n_i)_{i\in\mathbb{N}}$, $(p_i)_{i\in\mathbb{N}}$ of positive integers and finite index sets $I_i \subset \mathbb{N}\backslash\{i\}$, $i \in \mathbb{N}$, so that the following assumptions hold.

(i) The state vector x_i of Σ_i is an element of $\mathbb{R}^{n_i}$.
(ii) The vector $\bar{x}_i$ consists of the state vectors x_j, $j \in I_i$.
(iii) The external input vector u_i is an element of $\mathbb{R}^{p_i}$.
(iv) The right-hand side is a continuous function $f_i : \mathbb{R}^{n_i} \times \mathbb{R}^{N_i} \times \mathbb{R}^{p_i} \to \mathbb{R}^{n_i}$, where $N_i := \sum_{j\in I_i} n_j$.

In system (1), we consider $\bar{x}_i(\cdot)$ as an *internal input* and $u_i(\cdot)$ as an *external input*. To define the overall network, let $x := (x_i)_{i\in\mathbb{N}}$, $u := (u_i)_{i\in\mathbb{N}}$ and the right-hand side

$$f(x,u) := (f_1(x_1,\bar{x}_1,u_1), f_2(x_2,\bar{x}_2,u_2), \ldots). \tag{2}$$

The overall system is then formally written as

$$\Sigma : \quad x^+ = f(x,u). \tag{3}$$

The state space X of Σ is the Banach space of bounded sequences $(x_i)_{i\in\mathbb{N}}$ with $x_i \in \mathbb{R}^{n_i}$. For this, we first fix a norm on each $\mathbb{R}^{n_i}$ and define $\ell^\infty(\mathbb{N},(n_i)) := \{x = (x_i)_{i\in\mathbb{N}} : x_i \in \mathbb{R}^{n_i},\ \sup_{i\in\mathbb{N}} |x_i| < \infty\}$, with the norm $|x|_\infty := \sup_{i\in\mathbb{N}} |x_i|$.

Similarly, we consider the *external input space* $U := \ell^\infty(\mathbb{N},(p_i))$, where we fix norms on $\mathbb{R}^{p_i}$ that we simply denote by $|\cdot|$ again. By the space $\mathcal{U}$ of admissible *external input functions*, we mean all the sequences $u : \mathbb{N}_0 \to U$ where $u(k) \in \ell^\infty(\mathbb{N},(p_i))$ for all $k \in \mathbb{N}_0$. The space $\mathcal{U}$ is equipped with the norm $\|u\|_\infty := \sup_{k\geq 0} |u(k)|_\infty$. For initial values $\xi \in X$ and inputs $u \in \mathcal{U}$, $x(\cdot,\xi,u)$ denotes the corresponding solution to (3). In the sequel we will write $\Sigma = \Sigma(f,X,U)$ if we want to make the data explicit.

We say that Σ is *well-posed*, if f is well-defined as a map from $X \times U$ to X. In this case, for all $\xi \in X$, $u \in \mathcal{U}$ the solution of (3) exists on $\mathbb{N}_0$, that is Σ is forward complete. The following gives a complete characterization of well-posed systems on $\ell^\infty(\mathbb{N},(n_i))$. The (simple) proof is omitted.

Lemma II.1 *Consider systems Σ_i as in* (1) *satisfying the standard assumptions (i)-(iv). The following are equivalent:*

(i) The induced system $\Sigma(f,X,U)$ is well-posed.
(ii) There exist $C > 0$ and $\kappa \in \mathcal{K}_\infty$ such that for all $i \in \mathbb{N}$ and all $x_i \in \mathbb{R}^{n_i}, \overline{x}_i \in \mathbb{R}^{N_i}, u_i \in \mathbb{R}^{p_i}$:

$$|f_i(x_i,\overline{x}_i,u_i)| \leq C + \kappa(|x_i|) + \kappa(|\overline{x}_i|) + \kappa(|u_i|).$$

C. Distances in sequence spaces

Let $X = \ell^\infty(\mathbb{N},(n_i))$. Consider nonempty closed sets $\mathcal{A}_i \subset \mathbb{R}^{n_i}$, $i \in \mathbb{N}$. For $x_i \in \mathbb{R}^{n_i}$ the distance of x_i to the set $\mathcal{A}_i$ is $|x_i|_{\mathcal{A}_i} := \inf_{y_i\in\mathcal{A}_i} |x_i - y_i|$. Further define the set

$$\mathcal{A} := \{x \in X : x_i \in \mathcal{A}_i,\ i \in \mathbb{N}\} = X \cap \prod_{i\in\mathbb{N}} \mathcal{A}_i. \tag{4}$$

If $\mathcal{A} \neq \varnothing$, we define the distance from any $x \in X$ to $\mathcal{A}$ as

$$|x|_\mathcal{A} := \inf_{y\in\mathcal{A}} |x-y|_\infty = \inf_{y\in\mathcal{A}} \sup_{i\in\mathbb{N}} |x_i - y_i|. \tag{5}$$

Lemma II.2 *Let $X = \ell^\infty(\mathbb{N},(n_i))$. Assume that $\mathcal{A}$ in by* (4) *is nonempty. Then for any $x \in X$ there is an $y^* \in \mathcal{A}$ with*

$$|x|_\mathcal{A} = \sup_{i\in\mathbb{N}} |x_i|_{\mathcal{A}_i} = |x - y^*|_\infty. \tag{6}$$

The proof is omitted for reasons of space. We stress that the statement is only applicable for $\mathcal{A} \subset X = \ell^\infty(\mathbb{N},(n_i))$, while in general $\prod_{i\in\mathbb{N}} \mathcal{A}_i$ may contain only unbounded sequences. Also note that if $\mathcal{A} = \{0\}$, then $|x|_{\{0\}} = |x|_\infty$.

III. Input-to-State Stability

We aim to establish the stability of the interconnected system with respect to a closed set $\mathcal{A} \subset X$ using finite-step Lyapunov functions. To this end, we introduce the notions of input-to-state stability (ISS) and of $\mathcal{K}$-boundedness with respect to $\mathcal{A}$. In this section we assume that Σ is well-posed.

Definition III.1 *Let $\mathcal{A} \subset X$ be nonempty and closed. The function $f : X \times U \to X$ is called* $\mathcal{K}$-bounded with respect to $\mathcal{A}$, *if there are $\kappa_1, \kappa_2 \in \mathcal{K}$ such that for all $\xi \in X$, $\mu \in U$*

$$|f(\xi,\mu)|_\mathcal{A} \leq \kappa_1(|\xi|_\mathcal{A}) + \kappa_2(|\mu|_\infty). \tag{7}$$

Definition III.2 *Let $\varnothing \neq \mathcal{A} \subset X$ be closed. System Σ is said to be* input-to-state stable (ISS) with respect to $\mathcal{A}$, *if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for any initial state $\xi \in X$ and any $u \in \mathcal{U}$ the corresponding solution to* (3) *satisfies*

$$|x(k,\xi,u)|_\mathcal{A} \leq \max\left\{\beta(|\xi|_\mathcal{A},k), \gamma(\|u\|_\infty)\right\} \ \forall k \in \mathbb{N}_0. \tag{8}$$

Σ *is called* exponentially input-to-state stable (eISS) in $\mathcal{A}$, *if there are constants $C \geq 1, \rho \in [0,1)$ and $\gamma \in \mathcal{K}$ such that* (8) *holds with $\beta(r,k) = C\rho^k r$.*

As observed in [17], [19], every ISS system is *necessarily* $\mathcal{K}$-bounded, see Proposition III.4.

The underlying idea is to formulate stability properties from a subsequence of state trajectories. In particular, we aim to answer the question: whether one can conclude ISS by only looking at solutions every M time steps, with $M \in \mathbb{N}$. We define the iterates of f inductively, by $f^1 := f$, and $f^{k+1} : X \times U^{k+1} \to X$ by $f^{k+1}(x,(u_0,\ldots,u_k)) = f\big(f^k\big(x,(u_0,\ldots,u_{k-1})\big),u_k\big)$. The M-iterate system is then $\Sigma_M := \Sigma(f^M, X, U^M)$. The following lemma establishes that Σ_M is ISS if and only if the trajectories of Σ satisfy an ISS-like estimate every M time steps.

Lemma III.3 *Given a nonempty closed set $\mathcal{A} \subset X$ and $M \in \mathbb{N}$, Σ_M is ISS with respect to $\mathcal{A}$, if and only if there are $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for all $\xi \in X$, $u \in \mathcal{U}$ the corresponding solution to* (3) *satisfies for all $k \in \mathbb{N}_0$*

$$|x(Mk, \xi, u)|_{\mathcal{A}} \le \max\{\beta(|\xi|_{\mathcal{A}}, k), \gamma(\|u\|_\infty)\}. \quad (9)$$

The following proposition shows a useful criterion for ISS of discrete-time systems. It was shown for finite-dimensional systems in [19, Remark 4.2, Corollary 4.3] using the converse ISS Lyapunov theorems. One can show the result also by a direct method, which we omit due to the space reasons.

Proposition III.4 *Let $\mathcal{A} \subset X$ be closed.*

(i) *Σ is ISS with respect to $\mathcal{A}$ $\Leftrightarrow$ there is $M \in \mathbb{N}$ such that Σ_M is ISS with respect to $\mathcal{A}$, and f is $\mathcal{K}$-bounded with respect to $\mathcal{A}$.*

(ii) *Σ is eISS with respect to $\mathcal{A}$ $\Leftrightarrow$ there is $M \in \mathbb{N}$ such that Σ_M is eISS in $\mathcal{A}$, and f is $\mathcal{K}$-bounded with respect to $\mathcal{A}$ with a linear κ_1.*

Now we introduce *finite-step* ISS Lyapunov functions.

Definition III.5 *Let a nonempty closed set $\mathcal{A} \subset X$ be given. A continuous function $V : X \to \mathbb{R}_+$ is called a* finite-step ISS Lyapunov function *for Σ with respect to $\mathcal{A}$ if there exist $M \in \mathbb{N}$, $\underline{\omega}, \overline{\omega}, \alpha \in \mathcal{K}_\infty$ with $\alpha < \mathrm{id}$ and $\gamma \in \mathcal{K}$ such that*

$$\underline{\omega}(|\xi|_{\mathcal{A}}) \le V(\xi) \le \overline{\omega}(|\xi|_{\mathcal{A}}), \quad (10a)$$

$$V(x(M, \xi, u)) \le \max\{\alpha(V(\xi)), \gamma(\|u\|_\infty)\}, \quad (10b)$$

hold for all $\xi \in X$ and $u \in \mathcal{U}$. The function V is called a (power-bounded) finite-step eISS *Lyapunov function for Σ with respect to $\mathcal{A}$ if there are $M \in \mathbb{N}$, constants $\underline{w}, \overline{w}, b > 0$, $\kappa \in [0, 1)$ and $\gamma \in \mathcal{K}$ such that*

$$\underline{w}|\xi|_{\mathcal{A}}^b \le V(\xi) \le \overline{w}|\xi|_{\mathcal{A}}^b, \quad (11a)$$

$$V(x(M, \xi, u)) \le \max\{\kappa V(\xi), \gamma(\|u\|_\infty)\}, \quad (11b)$$

hold for all $\xi \in X$ and $u \in \mathcal{U}$. If inequality (10b) *(resp.* (11b)*) holds with $M = 1$, then we drop the term "finite-step" and simply speak of an ISS (resp. eISS) Lyapunov function.*

Note that every ISS Lyapunov function is a finite-step ISS Lyapunov function. However, a finite-step Lyapunov function need not decay at each time step, but only every M time steps. This relaxation is useful in the analysis and design of control systems; see e.g. [8], [9], [15], [17], [20] for finite-step Lyapunov function-based analysis and synthesis of *finite*-dimensional control systems. In particular, for large-scale networks, the use of finite-step Lyapunov functions enables us to introduce small-gain conditions which are not only sufficient, but also necessary for the verification of ISS of the network (see Theorem IV.4 below).

Now we show that the existence of a finite-step ISS Lyapunov function guarantees ISS of the system.

Proposition III.6 *Consider a system $\Sigma(f, X, U)$ with $\mathcal{K}$-bounded $f : X \times U \to X$. If there exists a finite-step ISS Lyapunov function for Σ with respect to $\mathcal{A}$, then Σ is ISS with respect to $\mathcal{A}$. Additionally, if κ_1 in* (7) *is a linear function, the existence of a finite-step eISS Lyapunov function for Σ with respect to $\mathcal{A}$ implies eISS of Σ with respect to $\mathcal{A}$.*

Proof: The proof boils down to the equivalence between ISS of Σ and ISS of Σ_M. A finite-step ISS Lyapunov function (with a given M) is a 1-step ISS Lyapunov function for Σ_M, which by classic direct ISS Lyapunov theorem[1] implies ISS of Σ_M. Proposition III.4 finishes the proof. ∎

Here we present a *converse* finite-step eISS Lyapunov theorem yielding an explicit formula for M in (11b).

Proposition III.7 *Let $\mathcal{A} \subset X$ be nonempty and closed. Suppose that system Σ is eISS in $\mathcal{A}$ with $\rho \in [0, 1)$, $C \ge 1$ and $\gamma \in \mathcal{K}$. Then for any function $V : X \to \mathbb{R}_+$ and constants $\underline{w}, \overline{w}, b > 0$ satisfying*

$$\underline{w}|\xi|_{\mathcal{A}}^b \le V(\xi) \le \overline{w}|\xi|_{\mathcal{A}}^b, \ \forall \xi \in X, \quad (12)$$

and for all $\kappa \in (0, 1)$ there exists an $M \in \mathbb{N}$ such that

$$V(x(M, \xi, u)) \le \max\{\kappa V(\xi), \overline{w}\gamma(\|u\|_\infty)^b\}, \quad (13)$$

for all $\xi \in X$ and $u \in \mathcal{U}$. If $M \ge \frac{1}{b}\log_\rho(\frac{\kappa \underline{w}}{C^b \overline{w}})$, then V is an exponential M-step ISS Lyapunov function for Σ.

The proof is similar to the proof of [9, Theorem 7] and is omitted. We note that the above converse Lyapunov result provides an explicit formula to compute the finite-step Lyapunov function. This proposition is an infinite-dimensional extension of Theorem 10 in [20] which has been already used for controller design, see [20, Proposition 14, Theorem 23] for more details.

IV. ISS for Infinite Networks

The main objective of this work is to develop conditions for input-to-state stability of the interconnection of countably many subsystems. We assume that each subsystem (1) is ISS in a closed set $\mathcal{A}_i$ and that for all Σ_i there exist continuous ISS Lyapunov functions with linear gains. We assume further that the interconnection Σ is well-posed. The next assumption states the ISS property for the subsystems.

Assumption IV.1 *Given $M \in \mathbb{N}$ and nonempty closed sets $\mathcal{A}_i \subset \mathbb{R}^{n_i}$, $i \in \mathbb{N}$, for each subsystem Σ_i, $i \in \mathbb{N}$ there exists a continuous function $W_i : \mathbb{R}^{n_i} \to \mathbb{R}_+$ such that*

(i) *There are $\underline{\alpha}_i, \overline{\alpha}_i \in \mathcal{K}_\infty$ so that for all $\xi_i \in \mathbb{R}^{n_i}$*

$$\underline{\alpha}_i(|\xi_i|_{\mathcal{A}_i}) \le W_i(\xi_i) \le \overline{\alpha}_i(|\xi_i|_{\mathcal{A}_i}). \quad (14)$$

(ii) *There are $\gamma_{ij} \in \mathcal{K}_\infty \cup \{0\}$ and $\gamma_{iu} \in \mathcal{K}$ so that for all $\xi \in X$, all $u \in \mathcal{U}$ the following holds*

$$W_i(x_i(M, \xi, u)) \le \max\{\sup_{j \in \mathbb{N}} \gamma_{ij}(W_j(\xi_j)), \gamma_{iu}(\|u\|_\infty)\}, \quad (15)$$

[1]The proof of a direct ISS Lyapunov theorem for infinite-dimensional systems follows similar arguments to those in [21, Lemma 3.5] and [17, Theorem 7] for finite-dimensional systems. It is not presented here.

where x_i denotes the ith component of solution x.

Note that $x_i(M)$ on the left hand side of (15) is the ith component of of the solution x of (3). However, one does not need to know the entire dynamics to compute (15) for the ith subsystem. For the sake of discussion, let $u = 0$. If $M = 1$, then the computation of $x(1)$ only requires the knowledge of f_i, ξ_i and $\overline{\xi}_i$. Now if $M = 2$, then we require not only the information from the neighbors, but also their neighbors, i.e. $\overline{\xi}_i$ and $\overline{\xi}_j$ with $j \in I_i$ and f_j with $j \in I_i$. Similar arguments hold for $M \geq 3$.

We will also need the following *uniformity* condition for the functions introduced in Assumption IV.1.

Assumption IV.2 *There exist $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$, $\alpha < \mathrm{id}$ and $\overline{\gamma}_u \in \mathcal{K}$, such that*

$$\underline{\alpha} \leq \underline{\alpha}_i \leq \overline{\alpha}_i \leq \overline{\alpha}, \quad i \in \mathbb{N}. \tag{16}$$

$$\gamma_{ij} \leq \alpha, \quad i, j \in \mathbb{N}. \tag{17}$$

$$\gamma_{iu} \leq \overline{\gamma}_u, \quad i \in \mathbb{N}. \tag{18}$$

Conditions (16) and (17) are necessary to construct a *coercive* [22] finite-step ISS Lyapunov function for Σ from the W_i. Condition (17) rules out nonuniform decay rates for the solutions of (1). Without this uniformity asymptotic stability of Σ may not hold, even if the system is linear and all internal and external gains are zero. To see this, consider

$$x_i^+ = i/(i+1)x_i + u, \quad i \in \mathbb{N},$$

where $x_i, u \in \mathbb{R}$. This network is not exponentially stable in the absence of inputs. Moreover, for arbitrarily small inputs the network may exhibit unbounded state trajectories. Finally, condition (18) is also crucial for ISS of the overall system. Consider a network composed of subsystems of the form $x_i^+ = iu, x_i, u \in \mathbb{R}$, $i \in \mathbb{N}$, which is not ISS in $\{0\}$.

Now we establish that the interconnected system Σ is ISS under the given assumptions. By Proposition III.6, our aim is to find a finite-step ISS Lyapunov function for Σ. This is achieved by the following *relaxed* small-gain theorem.

Theorem IV.3 *Consider a well-posed infinite network $\Sigma = (f, X, U)$ and nonempty closed sets $\mathcal{A}_i \subset \mathbb{R}^{n_i}$, $i \in \mathbb{N}$. Let the set $\mathcal{A} = X \cap \prod_{i\in\mathbb{N}} \mathcal{A}_i$ be nonempty. Suppose that Assumptions IV.1 and IV.2 hold. Then Σ admits an M-step ISS Lyapunov function with respect to $\mathcal{A}$ of the form*

$$V(\xi) = \sup_{i\in\mathbb{N}} W_i(\xi_i), \quad V : X \to \mathbb{R}_+. \tag{19}$$

In particular, the function V has the following properties.

(a) For all $\xi \in X$ and $u \in \mathcal{U}$

$$V(x(M, \xi, u)) \leq \max\big\{\alpha(V(\xi)), \overline{\gamma}_u(\|u\|_\infty)\big\}. \tag{20}$$

(b) For every $\xi \in X$ the following inequalities hold:

$$\underline{\alpha}(|\xi|_{\mathcal{A}}) \leq V(\xi) \leq \overline{\alpha}(|\xi|_{\mathcal{A}}). \tag{21}$$

In particular, Σ is ISS with respect to $\mathcal{A}$.

Proof: From (14) and (16), we have

$$V(\xi) = \sup_{i\in\mathbb{N}} W_i(\xi_i) \leq \sup_{i\in\mathbb{N}} \overline{\omega}_i(|\xi_i|_{\mathcal{A}_i}) \leq \sup_{i\in\mathbb{N}} \overline{\omega}(|\xi_i|_{\mathcal{A}_i}).$$

It follows from the monotonicity of $\overline{\omega}$ and Lemma II.2 that

$$V(\xi) \leq \overline{\omega}(\sup_{i\in\mathbb{N}} |\xi_i|_{\mathcal{A}_i}) = \overline{\omega}(|\xi|_{\mathcal{A}}).$$

This shows that V is well-defined and gives an upper bound in (21). Similarly, it holds that $V(\xi) \geq \underline{\omega}(\sup_{i\in\mathbb{N}} |x_i|_{\mathcal{A}_i}) = \underline{\omega}(|\xi|_{\mathcal{A}})$. Given any $\xi \in X$ and $u \in \mathcal{U}$, we also have that

$$\begin{aligned} V(x(M, \xi, u)) &= \sup_{i\in\mathbb{N}} W_i(x_i(M, \xi, u)) \\ &\overset{(15),(17)}{\leq} \sup_{i\in\mathbb{N}} \max\big\{\sup_{j\in\mathbb{N}} \alpha\big(W_j(\xi_j)\big), \gamma_{iu}(\|u\|_\infty)\big\} \\ &= \max\big\{\alpha\big(V(\xi)\big), \sup_{i\in\mathbb{N}} \gamma_{iu}(\|u\|_\infty)\big\} \\ &\overset{(18)}{\leq} \max\big\{\alpha\big(V(\xi)\big), \overline{\gamma}_u(\|u\|_\infty)\big\}, \end{aligned}$$

which is identical to (20). Hence V is a finite-step Lyapunov function for $\mathcal{A}$ and by Proposition III.6, Σ is ISS in $\mathcal{A}$. ∎

For $M = 1$, Theorem IV.3 reduces to classic small-gain conditions. In this case, our result is a discrete-time counterpart of [11, Theorem 1]. There is of course conservatism in the condition (16) which demands that all coupling gains γ_{ij} be less than identity. However, as $M > 1$ subsystems may have *stabilizing* effects on each other, thus reducing the degree of conservatism. Interestingly enough, in the case of exponential ISS, we are able to establish the *necessity* of our small-gain theorem, which shows the *non*-conservatism of the proposed small-gain condition in this case.

Theorem IV.4 *Consider the infinite network Σ. Let Σ be eISS with respect to $\mathcal{A} = X \cap \prod_{i\in\mathbb{N}} \mathcal{A}_i$, with nonempty closed sets $\mathcal{A}_i \subset \mathbb{R}^{n_i}$. Then there exist continuous functions $W_i : \mathbb{R}^{n_i} \to \mathbb{R}_+$, $i \in \mathbb{N}$ and $M \in \mathbb{N}$ such that Assumptions IV.1 and IV.2 hold.*

Proof: The eISS property of Σ implies that there exist $M \in \mathbb{N}$, $\gamma \in \mathcal{K}$ and $c < 1$ such that for all $\xi \in X$, $u \in \mathcal{U}$:

$$|x(M, \xi, u)|_{\mathcal{A}} \leq \max\big\{c|\xi|_{\mathcal{A}}, \gamma(\|u\|_\infty)\big\}. \tag{22}$$

Define $W_i : \mathbb{R}^{n_i} \to \mathbb{R}_+$ for $i \in \mathbb{N}$ by $W_i(x_i) := |x_i|_{\mathcal{A}_i}$. This satisfies (14), (16) with $b = 1$, $\underline{\alpha} = \underline{\alpha}_i = \overline{\alpha}_i = \overline{\alpha} = 1$. Also

$$\begin{aligned} W_i(x_i(M, \xi, u)) &= |x_i(M, \xi, u)|_{\mathcal{A}_i} \leq |x(M, \xi, u)|_{\mathcal{A}} \\ &\overset{(22)}{\leq} \max\big\{c|\xi|_{\mathcal{A}}, \gamma(\|u\|_\infty)\big\} = \max\big\{\sup_{j\in\mathbb{N}} c|\xi_j|_{\mathcal{A}_j}, \gamma(\|u\|_\infty)\big\}. \end{aligned}$$

We can rewrite this as

$$W_i(x_i(M, \xi, u)) \leq \max\big\{\sup_{j\in\mathbb{N}} cW_j(\xi_j), \gamma(\|u\|_\infty)\big\}, \tag{23}$$

which implies that (15) is satisfied with $\gamma_{ij} = c$ and $\gamma_{iu} = \gamma$ for all $i, j \in \mathbb{N}$. Conditions (17) and (18) also hold with $\alpha = c$ and $\overline{\gamma}_u = \gamma$, which completes the proof. ∎

Note that in the proof of the necessity of small-gain theorem the individual ISS Lyapunov-like functions W_i and the corresponding gain functions are explicitly constructed. Moreover, it shows that there exists some positive integer M for which all the coupling gain functions γ_{ij} can be chosen *identically*. In the spirit of our recent work [20], we believe that one can gain from these observations for a distributed control design based on our small-gain theorem.

A. From infinite to finite networks

The underlying idea to deal with infinite networks is to develop scale-free tools for analysis and design of finite networks of arbitrary size. The question arises whether the quantitative stability indices, e.g. decay rate of solutions, obtained for an infinite network are preserved for the original network. This section addresses this question. We show that the quantitative ISS indices given by Theorem IV.3 will be preserved for any truncation of an infinite network if $M = 1$ in Assumption IV.1.

For the purpose of the truncation process, we only consider the first $n \in \mathbb{N}$ subsystems of Σ and denote the truncated system by $\Sigma^{\langle n \rangle}$. As the states x_j for $j > n$ are no longer present in $\Sigma^{\langle n \rangle}$, but *in general* may still appear in some of the equations, we interpret these x_j as additional *external* inputs. A truncation of the infinite network is represented by

$$\Sigma^{\langle n \rangle}: \quad (x^{\langle n \rangle})^+ = f^{\langle n \rangle}(x^{\langle n \rangle}, \tilde{x}, u^{\langle n \rangle}), \tag{24}$$

where the state vector $x^{\langle n \rangle} = (x_i)_{1 \le i \le n} \in \mathbb{R}^N$, $N = \sum_{i=1}^n n_i$, the input vector $u^{\langle n \rangle} = (u_i)_{1 \le i \le n}$, $P = \sum_{i=1}^n p_i$, the additional input vector $\tilde{x} := (x_i)_{i \in \mathbb{N}, i > n}$, and the dynamics $f^{\langle n \rangle} : \mathbb{R}^N \times X \times \mathbb{R}^P \to \mathbb{R}^N$ is defined accordingly. Here we identify the space of bounded sequences $(x_i)_{i \in \mathbb{N}, i > n}$ with X. Clearly the network $\Sigma^{\langle n \rangle}$ is obtained from the interconnection of the first n subsystems Σ_i of the infinite network Σ. Note that the case in which $\tilde{x}$ is set to *zero* all through the network is covered by our formulation as a special case.

In the following we establish ISS of $\Sigma^{\langle n \rangle}$ in the set $\mathcal{A}^{\langle n \rangle} := \Pi_{i=1}^n \mathcal{A}_i$ and compute the corresponding ISS gain functions under assumption that Σ is ISS.

Theorem IV.5 *Consider the infinite network Σ. Suppose that Assumption IV.1 with $M = 1$ and Assumption IV.2 are satisfied. Then the function $V^{\langle n \rangle} : \mathbb{R}^N \to \mathbb{R}_+$ defined by*

$$V^{\langle n \rangle}(\xi^{\langle n \rangle}) = \max_{1 \le i \le n} W_i(\xi_i), \tag{25}$$

satisfies for all $\xi^{\langle n \rangle} \in \mathbb{R}^N, \tilde{\xi} \in X$ and $u^{\langle n \rangle} \in \mathbb{R}^P$:

$$\underline{\alpha}(|\xi|_{\mathcal{A}^{\langle n \rangle}}) \le V^{\langle n \rangle}(\xi^{\langle n \rangle}) \le \overline{\alpha}(|\xi|_{\mathcal{A}^{\langle n \rangle}}), \tag{26}$$

$$V(f^{\langle n \rangle}(\xi^{\langle n \rangle}, \tilde{\xi}, u)) \le \max\left\{\alpha(V(\xi^{\langle n \rangle})), \alpha \circ \overline{\alpha}(|\tilde{\xi}|_\infty), \overline{\gamma}_u(|u|_\infty)\right\}. \tag{27}$$

In particular, $\Sigma^{\langle n \rangle}$ is ISS in $\mathcal{A}^{\langle n \rangle}$.

Proof: The proof of (27) follows the same arguments as those for (20) in Theorem IV.3. Here we give a sketch of the proof of (27). For each Σ_i of the network $\Sigma^{\langle n \rangle}$, we rewrite the dissipative inequality (15) for $M = 1$ as

$$\begin{aligned} W_i(f_i(\xi_i, \bar{\xi}_i, u_i)) &\le \max\Big\{\max_{1 \le j \le n} \gamma_{ij}(W_j(\xi_j)), \\ &\qquad \sup_{j \in \mathbb{N}} \gamma_{ij}(W_j(\tilde{\xi}_j)), \gamma_{iu}(|u^{\langle n \rangle}|_\infty)\Big\} \\ &\overset{(17),(18)}{\le} \max\Big\{\max_{1 \le j \le n} \alpha(W_j(\xi_j)), \alpha(\sup_{j \in \mathbb{N}} W_j(\tilde{\xi}_j)), \overline{\gamma}_u(|u^{\langle n \rangle}|_\infty)\Big\} \\ &\overset{(16)}{\le} \max\Big\{\max_{1 \le j \le n} \alpha(W_j(\xi_j)), \alpha \circ \overline{\alpha}(|\tilde{\xi}|_\infty), \overline{\gamma}_u(|u^{\langle n \rangle}|_\infty)\Big\}. \end{aligned}$$

It follows from (25) that $V^{\langle n \rangle}(f^{\langle n \rangle}(\xi^{\langle n \rangle}, \tilde{\xi}^{\langle n \rangle}, u^{\langle n \rangle})) \le$

$$\max_{1 \le j \le n} \left\{\alpha(W_j(\xi)), \alpha \circ \overline{\alpha}(|\tilde{\xi}|_\infty), \overline{\gamma}_u(|u^{\langle n \rangle}|_\infty)\right\}.$$

By [17, Theorem 7] $\Sigma^{\langle n \rangle}$ is ISS in the set $\mathcal{A}^{\langle n \rangle}$. ■

As seen from (27), the *decay* rate α is *preserved* under truncation. Moreover, if the external inputs $\tilde{x}$ are not present to $\Sigma^{\langle n \rangle}$, the input gain $\overline{\gamma}_u$ is preserved. Thus, stability/performance indices of the overall system are *independent* of network size. We further illustrate this aspect via numerical simulation below.

V. Illustrative Example

In this section, we verify the effectiveness of our small-gain theorem by application to the control of traffic networks.

We revisit an example of a traffic network composed of infinitely many cells considered in [12]. Each cell $i \in \mathbb{N}$ represents a continuous-time system Σ_i described by

$$\Sigma_i : \dot{x}_i = -\Big(\frac{v_i}{l_i} + e_i\Big)x_i + D_i\bar{x}_i + B_i u_i, \tag{28}$$

with $x_i, u_i \in \mathbb{R}$ and the following structure

- $e_i = 0, D_i = c\frac{v_{i+1}}{l_{i+1}}, \bar{x}_i = x_{i+1}, B_i = 0$ if $i \in S_1 := \{1, 3\}$;
- $e_i = 0, D_i = c\frac{v_{i+4}}{l_{i+4}}, \bar{x}_i = x_{i+4}, B_i = r > 0$ if $i \in S_2 := \{4 + 8j : j \in \mathbb{N} \cup \{0\}\}$;
- $e_i = 0, D_i = c\frac{v_{i-4}}{l_{i-4}}, \bar{x}_i = x_{i-4}, B_i = \frac{r}{2}$ if $i \in S_3 := \{5 + 8j : j \in \mathbb{N} \cup \{0\}\}$;
- $e_i = 0, D_i = c(\frac{v_{i-1}}{l_{i-1}}, \frac{v_{i+4}}{l_{i+4}})^\top, \bar{x}_i = (x_{i-1}, x_{i+4}), B_i = 0$ if $i \in S_4 := \{6 + 8j : j \in \mathbb{N} \cup \{0\}\}$;
- $e_i = e \in (0,1), D_i = c(\frac{v_{i-4}}{l_{i-4}}, \frac{v_{i+1}}{l_{i+1}})^\top, \bar{x}_i = (x_{i-4}, x_{i+1}), B_i = 0$ if $i \in S_5 := \{9 + 8j : j \in \mathbb{N} \cup \{0\}\}$;
- $e_i = 0, D_i = c(\frac{v_{i+1}}{l_{i+1}}, \frac{v_{i+4}}{l_{i+4}})^\top, \bar{x}_i = (x_{i+1}, x_{i+4}), B_i = 0$ if $i \in S_6 := \{2 + 8j : j \in \mathbb{N} \cup \{0\}\}$;
- $e_i = 0, D_i = c(\frac{v_{i-4}}{l_{i-4}}, \frac{v_{i-1}}{l_{i-1}})^\top, \bar{x}_i = (x_{i-4}, x_{i-1}), B_i = 0$ if $i \in S_7 := \{7 + 8j : j \in \mathbb{N} \cup \{0\}\}$;
- $e_i = 2e, D_i = c(\frac{v_{i-1}}{l_{i-1}}, \frac{v_{i+4}}{l_{i+4}})^\top, \bar{x}_i = (x_{i-1}, x_{i+4}), B_i = 0$ if $i \in S_8 := \{8 + 8j : j \in \mathbb{N} \cup \{0\}\}$;
- $e_i = 0, D_i = c(\frac{v_{i-4}}{l_{i-4}}, \frac{v_{i+1}}{l_{i+1}})^\top, \bar{x}_i = (x_{i-4}, x_{i+1}), B_i = 0$ if $i \in S_9 := \{11 + 8j : j \in \mathbb{N} \cup \{0\}\}$;

where, for all $i \in \mathbb{N}$, $0 < \underline{v} \le v_i \le \bar{v}$, $0 < \underline{l} \le l_i \le \bar{l}$, and $c \in (0, 0.5)$. In (28), l_i is the length of a cell in kilometers (km), and v_i is the flow speed of the vehicles in kilometers per hour (km/h). The state x_i is the density of traffic, given in vehicles per cell, for each cell i of the road. The scalars B_i represent the number of vehicles that can enter the cells through entries which are controlled by u_i, with $u_i = 1$ and $u_i = 0$ correspond to green and red light, respectively. The percentage of vehicles leaving the cells using available exits is denoted by e_i. Furthermore, c, which is a design parameter, reflects the percentage of vehicles entering cell i from the neighboring cells. Discretizing (28) in time with sampling time $T > 0$, each cell i is described by

$$\Sigma_i^d : x_i^+ = \Big(1 - T\big(\frac{v_i}{l_i} + e_i\big)\Big)x_i + TD_i\bar{x}_i + TB_iu_i. \tag{29}$$

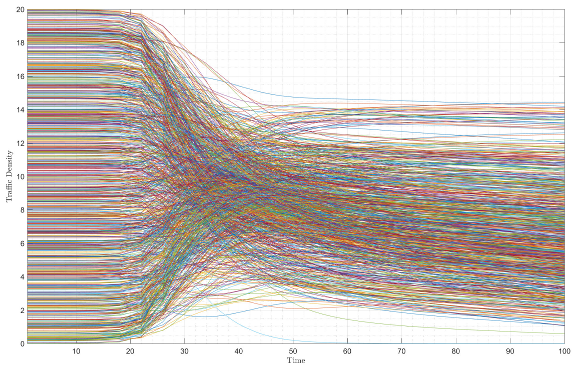

Fig. 1. State trajectories x_i for a network of 1000 cells.

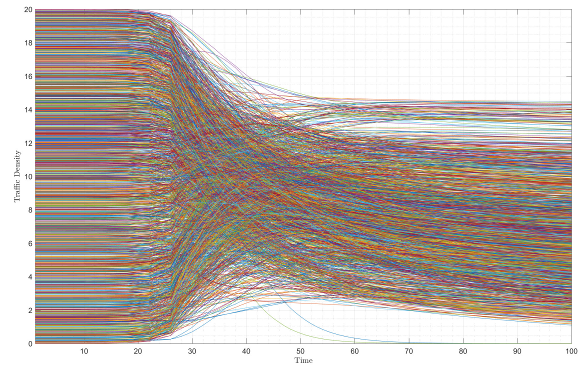

Fig. 2. State trajectories x_i for a network of 2000 cells.

To verify eISS of the network, for each subsystem Σ_i^d we take an eISS Lyapunov function of the form $V_i(x_i) = |x_i|$. The function V_i clearly satisfies (14) and (16) for all $i \in \mathbb{N}$ with $\underline{\alpha} = \underline{\alpha}_i = \overline{\alpha}_i = \overline{\alpha} = 1$. We also have

$$\begin{aligned} V_i(x_i^+) &\le \Big(1 - T\big(\frac{v_i}{l_i} + e_i\big)\Big)\,|x_i| + Tc\,\|D_i\|\,|\bar{x}_i|_\infty + TB_i\,|u_i| \\ &\le \max\big\{\gamma\,|x_i|\,, \gamma\,|\bar{x}_i|_\infty\,, \frac{1}{\varepsilon}TB_i\,|u_i|\,\big\} \\ &\le \max\big\{\gamma V_i(x_i), \gamma V_i(x_{i-1}), \gamma V_i(x_{i+1}), \frac{1}{\varepsilon}TB_i\,|u_i|\,\big\} \end{aligned}$$

where $\gamma := \Big(1 - T\big(\frac{v_i}{l_i} + e_i\big)\Big) + Tc\,\|D_i\| + \varepsilon$, $\varepsilon > 0$. This implies that (15) is satisfied with $M = 1$, $\gamma_{ij} = \gamma$ for all $j \in \{i-1, i, i+1\}$, $\gamma_{ij} = 0$ for all $j \in \mathbb{N}\backslash\{i-1, i, i+1\}$, $\gamma_{iu} = TB_i/\varepsilon$. Additionally, condition (18) holds with $\overline{\gamma}_u = Tr/\varepsilon$. Finally, we have (17) as one can *always* take T, c and ε sufficiently small such that $\alpha = 1 - T\big(\frac{\underline{v}}{\overline{l}}\big) + Tc\big(\frac{\bar{v}}{\underline{l}}\big) + \varepsilon < 1$. We note that all gain functions are linear. This together with the previous observations admits the use of Theorem IV.3 to conclude eISS of the network composed of subsystems (29).

By Theorem IV.5 the performance indices, i.e. the decay rate α and the input gain $\overline{\gamma}_u$, are preserved for any finite truncation. This is illustrated by Figures 1 and 2, for networks of 1000, respectively 2000, cells. For the simulation $u_i = 1$, the sampling period $T = 0.02$ sec. Moreover, the initial values are uniformly distributed over $[0, 20]$. From Figures 1 and 2, the overall behavior of the network remains almost identical, though the network size doubles.

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