

# Lyapunov PINN for stability analysis of infinite-dimensional systems

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ENERGY AND  
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# Outline

- 1 Motivation
- 2 Theoretical results
- 3 A toy example: scalar transport equation

# Framework

System under consideration:

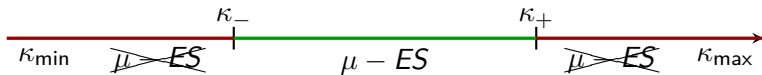
$$\dot{\phi}(t) = \mathcal{A}_\kappa \phi(t), \quad \phi(0) = \phi_0 \in \mathcal{H}, \quad (S)$$

where  $\mathcal{A}_\kappa : \mathcal{D}(\mathcal{A}_\kappa) \subset \mathcal{H} \rightarrow \mathcal{H}$  generates a  $C_0$ -semigroup  $\mathcal{T}_\kappa$  on  $\mathcal{H}$ .

## Objective

Find numerically the allowable range of parameter  $\kappa$  ensuring for a prescribed decay rate  $\mu$  the existence of  $c > 0$  such that

$$\|\mathcal{T}_\kappa(t)\phi_0\|_{\mathcal{H}} \leq ce^{-\mu t} \|\phi_0\|_{\mathcal{H}}. \quad (\mu - ES)$$



# Context

- The operator  $\mathcal{A}$  model infinite-dimensional systems such as **hyperbolic** PDEs (transport and propagation phenomena) or **parabolic** PDEs (heat or particles diffusion).
- The **control** parameter  $\kappa$  can represent a static output feedback, having for instance  $\mathcal{A}_\kappa = \mathcal{A} + \kappa \mathcal{B}\mathcal{C}$ .  
It includes proportional control of systems under delay  
 $\dot{\phi}(t) = \mathcal{A}\phi(t) + \kappa \mathcal{B}\mathcal{C}\phi(t - h)$ .
- The **model** parameter  $\kappa$  can represent an uncertainty margin, having for instance  $\mathcal{A} + \kappa \Delta \mathcal{A}$ .

# What we know in finite dimension

For  $A \in \mathbb{R}^{n \times n}$  and  $Q \succ 0$ , under the non-resonance condition at  $\mu^\dagger$ , there is a unique symmetric matrix  $P \in \mathbb{R}^{n \times n}$  solving the shifted Lyapunov matrix equation

$$PA + A^\top P + 2\mu P = -Q, \quad P = P^\top.$$

**N&S stability condition:**  $\mu - ES \iff P \succ 0$ .

- ① The **closed-form** solution

$$P = -\text{vec}^{-1} \left( ((A + \mu I)^\top \otimes I + I \otimes (A + \mu I)^\top)^{-1} \text{vec}(Q) \right).$$

- ②  $P \succ 0$  is a finite-dimensional, **numerically testable** condition.

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<sup>†</sup>For all  $i, j$ ,  $\lambda_i(A) + \lambda_j(A) + 2\mu \neq 0$ .

# What we know in infinite dimension

For  $\mathcal{A}$  on  $\mathcal{H}$  and  $Q \succ 0$ , under the non-resonance condition at  $\mu^\dagger$ , there is a unique bounded self-adjoint  $\mathcal{P} \in \mathcal{L}(\mathcal{H})$  solving the shifted Lyapunov operator equation<sup>‡</sup>

$$\mathcal{P}\mathcal{A} + \mathcal{A}^*\mathcal{P} + 2\mu\mathcal{P} = -Q, \quad \mathcal{P} = \mathcal{P}^*.$$

Assumption (uniform lower bound)

$\exists c, T > 0$  such that  $\|\mathcal{T}(t)\phi_0\|_{\mathcal{H}} \geq c\|\phi_0\|_{\mathcal{H}}$  for all  $t \in [0, T]$ .

**N&S stability condition:**  $\mu - ES \iff \exists c > 0 \mid \mathcal{P} \succeq c\mathcal{I}$ .

Contrapositive:  ~~$\mu \succ ES$~~   $\iff \mathcal{P}$  fails to be coercive.

<sup>†</sup>For all  $i, j$ ,  $\lambda_i(\mathcal{A}) + \lambda_j(\mathcal{A}) + 2\mu \neq 0$ .

<sup>‡</sup>Datko, "Extending a theorem of AM Liapunov to Hilbert space".

# Problem statement

Two difficulties arise

- ① The equation for  $\mathcal{P}$  is **as hard to solve as the system**.
- ② Testing coercivity of  $\mathcal{P}$  is **not directly a finite computation**.

## Technical questions

- ① How to solve (approximately) the shifted infinite-dimensional Lyapunov operator equation  $\mathcal{P}\mathcal{A} + \mathcal{A}^*\mathcal{P} + 2\mu\mathcal{P} = -Q$ ?
- ② How to test  $\mathcal{P} \succeq c\mathcal{I}$  on an infinite-dimensional operator?

Can a Lyapunov kernel, obtained by a PINN, be turned into a **certificate** of  $\mu - ES$  (or  ~~$\mu \succ ES$~~ ), and be exploited to locate conservative stability boundaries in a parameter  $\kappa$ ?

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# The key trick

Let  $\hat{\mathcal{P}}$  be *any* candidate self-adjoint operator, obtained by approximation methods (FEM, Galerkin, PINN, ...) *or not*. Define the residual

$$\hat{\mathcal{R}} := \hat{\mathcal{P}}\mathcal{A} + \mathcal{A}^*\hat{\mathcal{P}} + 2\mu\hat{\mathcal{P}} + \mathcal{Q},$$

and set

$$\boxed{\hat{\mathcal{Q}} := \mathcal{Q} - \hat{\mathcal{R}}} \implies \hat{\mathcal{P}}\mathcal{A} + \mathcal{A}^*\hat{\mathcal{P}} + 2\mu\hat{\mathcal{P}} = -\hat{\mathcal{Q}}.$$

All the *error* is absorbed into a *computable* right-hand side  $\hat{\mathcal{Q}}$ .

- ① Assuming  $\hat{\mathcal{Q}} \succ 0$ , the **N&S stability condition** now applies *verbatim* to  $\hat{\mathcal{P}}$ :  $\mu - ES \iff \exists c > 0 \mid \hat{\mathcal{P}} \succeq c\mathcal{I}$ .

# The need of a coercivity margin

In practice, at **finite order**  $d$ , we project on

$V_d = \text{span}\{\psi_0, \dots, \psi_d\} \subset \mathcal{H}$  and check  $\underline{\lambda}(G_d(K)) > 0$  for

$$G_d(K) = (\langle \psi_i, K \psi_j \rangle)_{0 \leq i, j \leq d}, \quad \text{the Gram matrix.}$$

**Rayleigh–Ritz:**  $\underline{\lambda}(G_d(K)) \geq \underline{\lambda}(K)$  &  $\underline{\lambda}(G_d(K)) \xrightarrow{d \rightarrow \infty} \underline{\lambda}(K)$ .

If  $K$  is only positive without any coercivity margin, a finite  $d$  cannot certify it: small negative directions may hide outside  $V_d$ .

- ② A strict **coercivity margin** on  $\hat{Q}$  and  $\hat{P}$  is what makes a finite-order numerical test conclusive.

# Certification theorems

Choose  $d^*$  suff. large so that it absorbs the coercivity margin.

## Sufficient certificate of stability

If  $G_{d^*}(\hat{Q}) \succ 0$  and  $G_{d^*}(\hat{P}) \succ 0$ , then  $(S)$  is  $\mu - ES$ .

## Sufficient certificate of instability

If  $G_{d^*}(\hat{Q}) \succ 0$  and  $\exists d \in \mathbb{N} \mid G_d(\hat{P}) \preceq 0$ , then  $(S)$  is *not*  $\mu - ES$ .

| $\mu - ES$                 | <i>not</i> $\mu - ES$      | Inconclusive             |
|----------------------------|----------------------------|--------------------------|
| $G_{d^*}(\hat{Q}) \succ 0$ | $G_{d^*}(\hat{Q}) \succ 0$ | $G_d(\hat{Q}) \preceq 0$ |
| $G_{d^*}(\hat{P}) \succ 0$ | $G_d(\hat{P}) \preceq 0$   | —                        |

**Table 1:** N&S stability condition based on  $(\hat{Q}, \hat{P})^\dagger$

<sup>†</sup>Bajodek et al., "Identifying Lyapunov functionals with a PINN : application to a hyperbolic transport equation".

# Choosing the coercivity margin

**On  $\mathcal{Q}$ : easy, it's a choice.**

- $\mathcal{Q}$  is a design parameter of the Lyapunov equation.
- Any coercive kernel works:  $\mathcal{Q} = \mathcal{I}$  with margin 1.

**On  $\mathcal{P}$ : less obvious.**

- $\mathcal{P}$  is not chosen; its margin is inherited from the system.
- Instability test: always valid — it suffices to exhibit one direction along which  $\hat{\mathcal{P}}$  is not coercive.
- Stability test: under the assumption of a uniform lower bound of  $\mathcal{T}$  — one still needs to compute the coercivity margin first on  $\mathcal{P}$  and then on  $\hat{\mathcal{P}}$ .

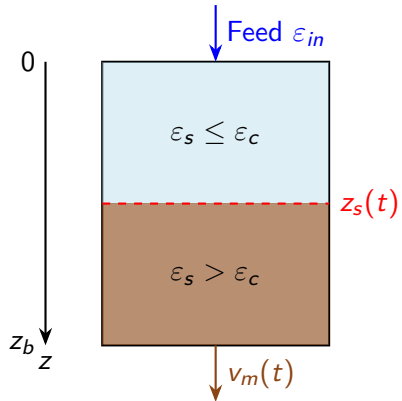
Some further work is still needed to make the stability argument quantitative. How to compute  $d^*$ ?

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# Waste water treatment

Clarifier



Model: 4 (weakly-)hyperbolic PDEs + 1 ODE<sup>‡</sup>.

Control parameter:  $v_m(t) = \kappa(z_s(t) - z_s^*)$ .

<sup>‡</sup>Amirat et al., “Insights into the stability analysis of  $2 \times 2$  linear weakly hyperbolic systems”.

# System description

A simple enough example to have a **closed-form reference solution** (see *appendix*) – ideal to validate the approach.

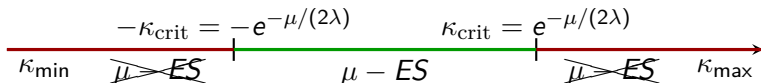
## Scalar transport equation with boundary coupling

$$\begin{cases} \partial_t \phi(t, x) + \lambda \partial_x \phi(t, x) = 0, & x \in (0, 1), t > 0, \\ \phi(t, 0) = \kappa \phi(t, 1), \end{cases}$$

On  $\mathcal{H} = L^2(0, 1)$ :  $\mathcal{A}\phi = -\partial_x \phi$ ,  $\mathcal{D}(\mathcal{A}) = \{\phi : \phi(0) = \kappa \phi(1)\}$ .

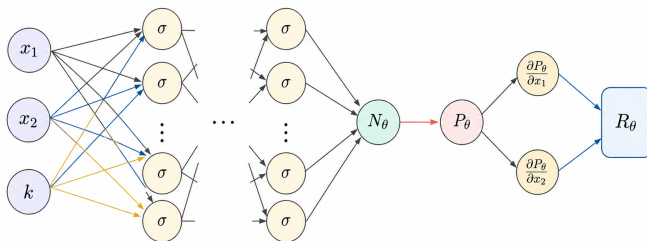
The shifted operator equation rewrites as

$$\begin{cases} \lambda(\partial_{x_1} P + \partial_{x_2} P) + \mu P + Q = 0, \\ P(1, x) = \kappa P(x, 0). \end{cases}$$



# PINN (Physics Informed Neural Network)

**What is a PINN?** "A learning model that embeds physical laws"



**Why a PINN as Lyapunov candidate?**

- No mesh, no re-assembly of matrices when  $\kappa$  changes.
- Residual  $R_\theta$  evaluated by automatic differentiation.

# Architecture taking into account $\mathcal{D}(\mathcal{A})$

The kernel is reconstructed as

$$P_{\theta}(x_1, x_2, \kappa) = \begin{cases} (1 - x_1)N_{\theta}(x_1, x_2, \kappa) + \kappa x_1(1 - x_2)N_{\theta}(x_2, 0, \kappa) \\ \quad + \kappa^2 x_1 x_2 N_{\theta}(0, 0, \kappa) & \text{if } x_1 \geq x_2, \\ P_{\theta}(x_2, x_1, \kappa) & \text{otherwise.} \end{cases}$$

with  $N_{\theta}$  an MLP (4 layers  $\times$  128 neurons, SiLU activations).

The advantages of this architecture:

- Symmetry is ensured by construction.
- Boundary condition holds by construction.

# Loss function to be minimized

Only the **interior PDE residual** remains to minimize:

$$R_\theta = \lambda(\partial_{x_1} P_\theta + \partial_{x_2} P_\theta) + \mu P_\theta + Q, \mathcal{L}_{\text{EDP}}(\theta) = \sum |R_\theta(x_1, x_2, \kappa)|^2.$$

**Collocation points:** Sampled uniformly on  $[0, 1]^2 \times [\kappa_{\min}, \kappa_{\max}]$ .

**Training:** Adam (uniformly resampled batches) and L-BFGS refinement (fixed batch).

**Output**<sup>§</sup>: The PINN Lyapunov candidate  $\hat{\mathcal{P}} := P_\theta$  and its the Lyapunov equation's right hand-side  $\hat{\mathcal{Q}} := Q_\theta = Q - R_\theta$ .

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<sup>§</sup>Universal approximation theorem ensures the convergence of the solution towards  $(P, Q)$  for sufficiently large MLP, i.e. the coercivity margin of  $P, Q$  reports to  $P_\theta, Q_\theta$ .

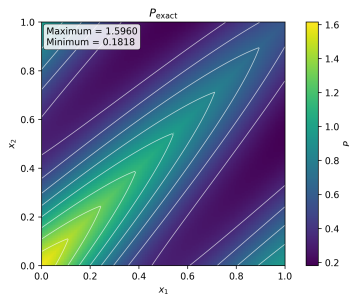
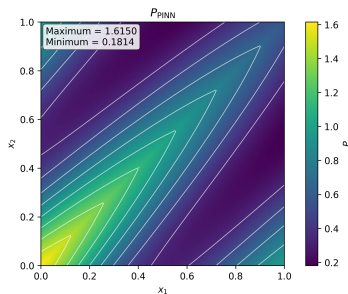
# Numerical results: training on stable / unstable ranges

Setup:  $\lambda = 2$ ,  $\mu = 1$ , Gaussian  $Q$  ( $\sigma = 0.3$ )  $\Rightarrow \kappa_{\text{crit}} \approx 0.7788$ .

Two disjoint training ranges, avoiding the neighborhood of  $\kappa_{\text{crit}}$ :

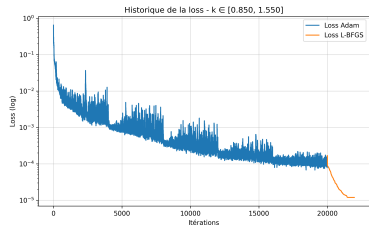
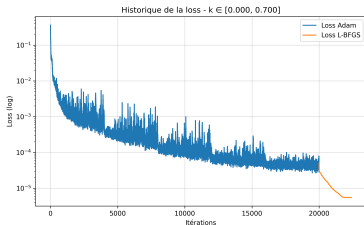
$$\kappa \in \underbrace{[0, 0.7]}_{\mu\text{-ES}} \quad \text{and} \quad \kappa \in \underbrace{[0.85, 1.55]}_{\cancel{\mu\text{-ES}}}$$

Comparison PINN vs Exact -  $k=0.600$  ( $\lambda = 2.0$ ,  $\mu = 1.0$ ,  $\sigma = 0.3$ )

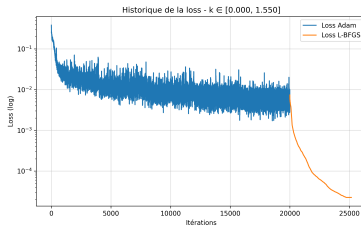


Excellent agreement between  $P_\theta$  and the exact kernel  $P$  on both ranges.

# Convergence of the loss function



Convergence of the loss function towards  $10^{-5}$  on both ranges.



Extra test for the training range  $[0, 1.55]$ .

# Adaptive search algorithm

**Idea:** instead of a single global training over  $[\kappa_{\min}, \kappa_{\max}]$  (which mixes stable/unstable regimes and degrades training), warm-start a sequence of trainings with an adaptive step  $\Delta\kappa$ .

**Input:**  $\kappa_0, \Delta\kappa_0, \theta_{\text{init}}, \text{stabilityFlag}, \text{budget } T_{\max}, \rho_+ > 1 > \rho_- > 0$

**Output:**  $\kappa^*, \theta^*$

$\kappa \leftarrow \kappa_0, \Delta\kappa \leftarrow \Delta\kappa_0, \theta^* \leftarrow \theta_{\text{init}};$

**while** *elapsed time*  $< T_{\max}$  **do**

    Train (Adam then L-BFGS) at  $\kappa$ , starting from  $\theta^*$ ;

    Evaluate Gram matrices  $G_d(Q_\theta), G_d(P_\theta)$ ;

**if**  $\underline{\lambda}(G_d(Q_\theta)) > 0$  **and** *certificate matches* **stabilityFlag** **then**

        Save  $\theta^* \leftarrow \theta, \kappa^* \leftarrow \kappa; \Delta\kappa \leftarrow \rho_+ \Delta\kappa; \kappa \leftarrow \kappa + \Delta\kappa;$

**else**

        Restore  $\theta \leftarrow \theta^*; \Delta\kappa \leftarrow \rho_- \Delta\kappa; \kappa \leftarrow \kappa^* + \Delta\kappa;$

**end**

**end**

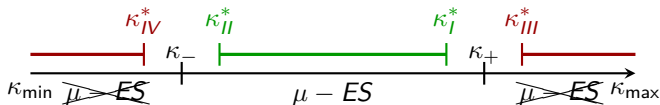
Success  $\Rightarrow$  bigger step, keep going;

Failure  $\Rightarrow$  backtrack, shrink the step.

# Four adaptive searches

- Search I ( $\kappa_0 = 0, \Delta\kappa_0 > 0$ ): certifies  $\mu - ES$  up to  $\kappa_I^*$ .
- Search II ( $\kappa_0 = 0, \Delta\kappa_0 < 0$ ): certifies  $\mu - ES$  down to  $\kappa_{II}^*$ .
- Search III (from  $\kappa_0 = 1, \Delta\kappa_0 < 0$ ): certifies  ~~$\mu - ES$~~  down to  $\kappa_{III}^*$ .
- Search IV (from  $\kappa_0 = -1, \Delta\kappa_0 > 0$ ): certifies  ~~$\mu - ES$~~  up to  $\kappa_{IV}^*$ .

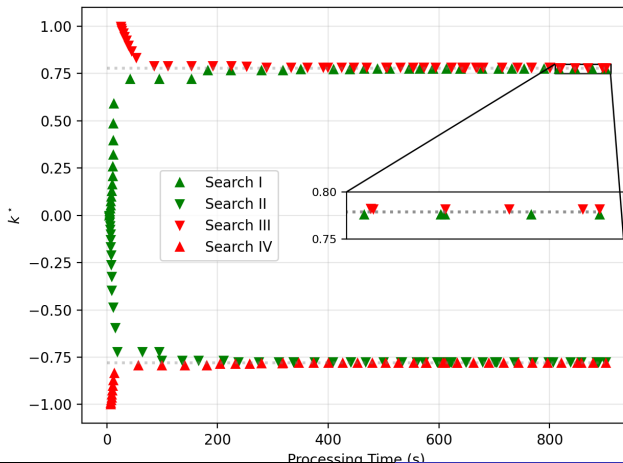
**Expected result:** Lower & upper conservative bounds for  $\kappa^\pm := \pm\kappa_{\text{crit}}$ .



# Numerical results: bracketing the stability pocket

Setup:  $\rho_+ = 1.1$ ,  $\rho_- = 0.5$ ,  $\Delta\kappa_0 = 0.1$ ,  $T_{\max} = 15$  min/search

$$[-0.7754, 0.7761] \subset (-\kappa_{\text{crit}}, \kappa_{\text{crit}}) \subset [-0.7805, 0.7817],$$



# Conclusions

## Sum-up:

- Two positivity tests on  $(\hat{\mathcal{P}}, \hat{\mathcal{Q}})$  certify stability or instability.
- Any numerically approximate Lyapunov kernel, however imperfect, can be used for N&S stability condition.
- PINNs, parametrized by the (control) parameter, enable a mesh-free adaptive search and is validated on a toy example.

## Discussion:

- The bottleneck is not wall-clock time, but the structural difficulty  $\mathcal{P} \xrightarrow{\kappa \rightarrow \kappa^\pm} \infty$ . Need for specialized architectures (rational networks) to handle near-singular kernels close to  $\kappa^\pm$ .
- The trade-off in the choice of  $\mathcal{Q}$ : a large margin is hard to beat due to sharp derivatives and large residuals; a small margin is hard to beat simply because the target is small.

# Thank you for your attention

Questions?