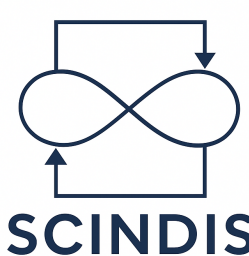


Asynchronous exponential growth of an age-structured population model with unbounded birth process



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DIMENSIONAL SYSTEMS

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We study an age-structured population in which the newborns at time t depend on the population at time $t - r$. The solution has the asynchronous exponential growth property: the total population grows like $e^{\gamma_0 t}$ and the age distribution converges to an explicit profile. The delay makes the birth term an unbounded boundary operator, so the usual compactness arguments fail; we replace it by finite-rank operators, show that the corresponding convolutions form a Cauchy sequence in operator norm, and deduce that the essential spectral radius is small. This settles a question left open by Piazzera and Tonetto (2005).

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Introduction

Many structured population models can be written as an abstract boundary problem on a Banach space X ,

$$\dot{X}(t) = A_m X(t), \quad P X(t) = \Phi X(t), \quad X(0) = x_0,$$

where A_m is a differential operator with a large domain, P is a boundary trace and Φ carries the birth law. For an age-structured population, $X = L^1(0, a_m)$, $A_m f = -f' - \mu f$ on $W^{1,1}(0, a_m)$ with μ the mortality, and $Pf = f(0)$. The choice of Φ is what decides how hard the problem is.

• **Bounded birth law.** $\Phi f = \int_0^{a_m} \beta(a) f(a) da$, the Lotka–McKendrick model. Here $\Phi \in \mathcal{L}(X, \mathbb{C})$. In our companion paper [1] we proved that the essential spectral radius of the solution semigroup equals that of the unperturbed one, for any bounded Φ .

• **Distributed delay.** Φ acts on the history $u_t(\theta) = u(t + \theta)$, $\Phi u_t = \int_0^{a_m} \int_{-r}^0 \beta(s, a) u(t + s, a) ds da$. Piazzera [2] obtained exponential stability and Piazzera and Tonetto [3] the asynchronous exponential growth, and asked what happens when the delay is concentrated at one instant.

• **Discrete delay.** $\Phi u_t = K u_t := \int_0^{a_m} \beta(a) u(t - r, a) da$. Mei and Peng [4] proved exponential stability. Asynchronous exponential growth was not known. The operator K evaluates the history at $\theta = -r$, so it is unbounded on $L^1(-r, 0; X)$ and the result of [1] cannot be used.

A C_0 -semigroup $(T(t))_{t \geq 0}$ with generator A has **asynchronous exponential growth** with intrinsic growth constant $\lambda_0 \in \mathbb{R}$ if $e^{-\lambda_0 t} T(t)$ converges in operator norm to a nonzero finite-rank projection P_0 . Webb [5, Proposition 2.3] characterised this property: it holds if and only if

(i) $\omega_{\text{ess}}(A) < \lambda_0$, (ii) λ_0 is the only point of $\sigma(A)$ on the line $\text{Re } \lambda = \omega_0(A)$, (iii) λ_0 is a simple pole of $R(\lambda, A)$.

For population models (ii) and (iii) can be read off a characteristic equation; condition (i), the essential growth bound, is where the difficulty lies.

Objective

Prove the asynchronous exponential growth property for the third case,

$$\begin{cases} \partial_t u + \partial_a u = -\mu(a) u, & t \geq 0, a \in [0, a_m), \\ u(t, 0) = \int_0^{a_m} \beta(a) u(t - r, a) da, \\ u(s, \cdot) = u_0(s, \cdot), & -r \leq s \leq 0, \end{cases}$$

with $0 < a_m \leq \infty$, $\mu \geq 0$ locally bounded, $\mu(a) \rightarrow \mu_\infty$, $\beta \in L^\infty$ and $u_0 \in L^1$; identify the limit profile; and check the result numerically.

The difficulty is that $\tilde{K}$ is unbounded and $T_{0,-1}$ acts on an extrapolation space, so the semigroup is not eventually compact and the classical criteria [8, 9], which need norm continuity of $t \mapsto B\tilde{\mathcal{C}}T_0(t)$, do not apply. We approximate the convolution by operators that are visibly compact and pass to the limit in the operator norm, where compactness is preserved.

The approximation and the Cauchy sequence argument

Write the convolution by components. The age component is

$$\Gamma_2(t)\mathcal{F} = \int_0^t T_{0,-1}(t-s) B_2 \tilde{K} \pi_1(s) \mathcal{F} ds,$$

where $\pi_1(s)$ is the history part of $\mathcal{T}(s)$, and the history component $\Gamma_1(t)$ is obtained from Γ_2 in the same way. We show that $\Gamma_2(t)$ is compact; $\Gamma_1(t)$ follows.

1 Replace $\tilde{K}$ by something bounded. For $\lambda > 0$ let $K_\lambda = \lambda KR(\lambda, G_0)$. This is a bounded operator of rank one, and $K_\lambda \rightarrow \tilde{K}$ strongly by the definition of the Yosida extension. Put

$$\mathcal{L}_\lambda(t)\mathcal{F} = \int_0^t T_{0,-1}(t-s) B_2 K_\lambda \pi_1(s) \mathcal{F} ds.$$

2 Each $\mathcal{L}_\lambda(t)$ is compact. Insert one more resolvent and consider $R_\mu \mathcal{F} = \mu \int_0^t T_{0,-1}(t-s) B_2 K_\lambda R(\mu, G_0) \pi_1(s) \mathcal{F} ds$. The map $s \mapsto R(\mu, G_0) \pi_1(s)$ is continuous in operator norm (a resolvent in front of any C_0 -semigroup gives this), and K_λ has finite rank. For a bounded sequence $(\mathcal{F}_n)$ the functions $s \mapsto B_2 K_\lambda R(\mu, G_0) \pi_1(s) \mathcal{F}_n$ are then uniformly bounded and equicontinuous in the Favard space $\mathbb{F}_{-1}$, Arzelà–Ascoli gives a convergent subsequence, and the convolution with $T_{0,-1}$ is bounded from $C([0, t]; \mathbb{F}_{-1})$ to X [11]. So R_μ is compact, and $R_\mu \rightarrow \mathcal{L}_\lambda(t)$ in operator norm as $\mu \rightarrow \infty$.

Methodology

1 Product space. Let $E = L^1(-r, 0; X)$ be the history space and $\mathcal{E} = E \times X$. With $G_m F = \partial_\theta F$ and $QF = F(0)$ on E , the model becomes a boundary system on $\mathcal{E}$,

$$\dot{\omega} = \mathcal{A}_m \omega, \quad \mathcal{N} \omega = \mathcal{M} \omega, \quad \mathcal{A}_m = \begin{pmatrix} G_m & 0 \\ 0 & A_m \end{pmatrix}, \quad \mathcal{N} = \begin{pmatrix} Q & 0 \\ 0 & P \end{pmatrix}, \quad \mathcal{M} = \begin{pmatrix} 0 & \text{Id} \\ K & 0 \end{pmatrix}.$$

The operator $\mathcal{A}$ is $\mathcal{A}_m$ restricted to $\{\mathcal{N} \omega = \mathcal{M} \omega\}$. When $\mathcal{M} = 0$ one gets $T_0(t) = \text{diag}(S_0(t), T_0(t))$, a left shift on histories and a transport with survival factor $V(a, a-t) = e^{-\int_a^{a-t} \mu}$.

2 Well-posedness. We use the feedback theory of regular linear systems [6, 7]. The Dirichlet operator $\mathcal{D}_\lambda = \text{diag}(e^{\lambda \theta}, e^{-\lambda a} V(a, 0))$ solves the boundary problem $(\lambda - \mathcal{A}_m) \omega = 0$, $\mathcal{N} \omega = v$; the control operator is $B = (\lambda - \mathcal{A}_{0,-1}) \mathcal{D}_\lambda$ and the observation operator is $C = \mathcal{M}|_{D(\mathcal{A}_0)}$, with Yosida extension $\tilde{C} = \begin{pmatrix} 0 & \text{Id} \\ \tilde{K} & 0 \end{pmatrix}$, $\tilde{K} = \lim_{\lambda \rightarrow \infty} \lambda KR(\lambda, G_0)$. Then $\mathcal{A}$ generates a positive C_0 -semigroup on $\mathcal{E}$ and

$$\mathcal{T}(t)\mathcal{F} = T_0(t)\mathcal{F} + \int_0^t T_{0,-1}(t-s) B \tilde{C} \mathcal{T}(s) \mathcal{F} ds.$$

The whole question is whether this convolution is a compact operator.

Result

Characteristic equation. Set $\xi(\lambda) = e^{-\lambda r} \int_0^{a_m} \beta(a) e^{-\lambda a} \Pi(a) da - 1$ with $\Pi(a) = e^{-\int_a^0 \mu}$. Then $\sigma(\mathcal{A}) = \{\xi = 0\}$ and $\gamma_0 = s(\mathcal{A}) = \omega_0(\mathcal{A})$ is the unique real zero of ξ [4]. Its sign is the sign of $\xi(0)$.

Theorem. Assume $\beta \in L^q(0, a_m)$ with $\|\beta\|_q \leq \kappa$ for all $1 < q < \infty$, and either $a_m < \infty$ and $\xi(0) \geq 0$, or $a_m = \infty$ and $\mu_\infty > 0$. Then there is a rank-one projection $\mathcal{P}$ on $\mathcal{E}$ such that

$$\lim_{t \rightarrow \infty} \|e^{-\gamma_0 t} \mathcal{T}(t) - \mathcal{P}\| = 0, \quad \mathcal{P}(\mathcal{E}) = \ker(\gamma_0 - \mathcal{A}).$$

Webb's conditions. Condition (i) is the essential-radius estimate below. For (ii), taking moduli in $\xi(\lambda) = 0$ shows that γ_0 is the only zero of ξ on the line $\text{Re } \lambda = \gamma_0$ ($\beta \geq 0$, positive on an interval). For (iii), γ_0 is a simple zero of ξ , since $\xi'(\gamma_0) = -r - e^{-\gamma_0 r} \int_0^{a_m} a \beta(a) e^{-\gamma_0 a} \Pi(a) da < 0$, and $\ker(\gamma_0 - \mathcal{A})$ is one-dimensional, so $P_0 = \mathcal{P}$ has rank one.

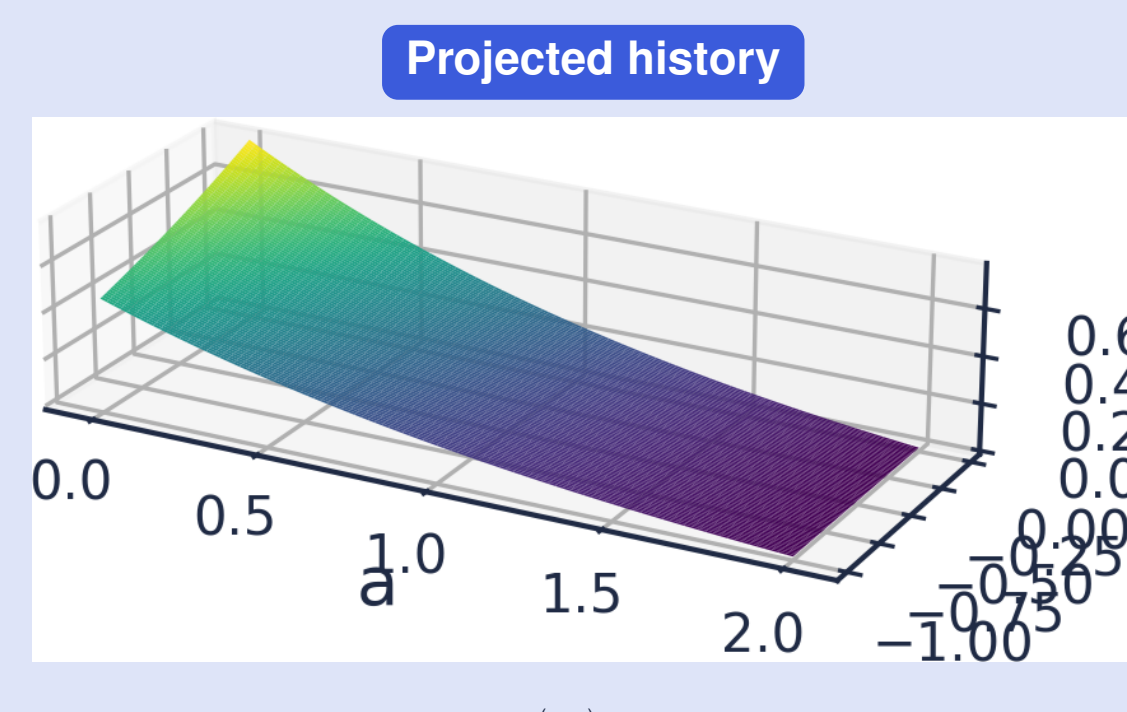
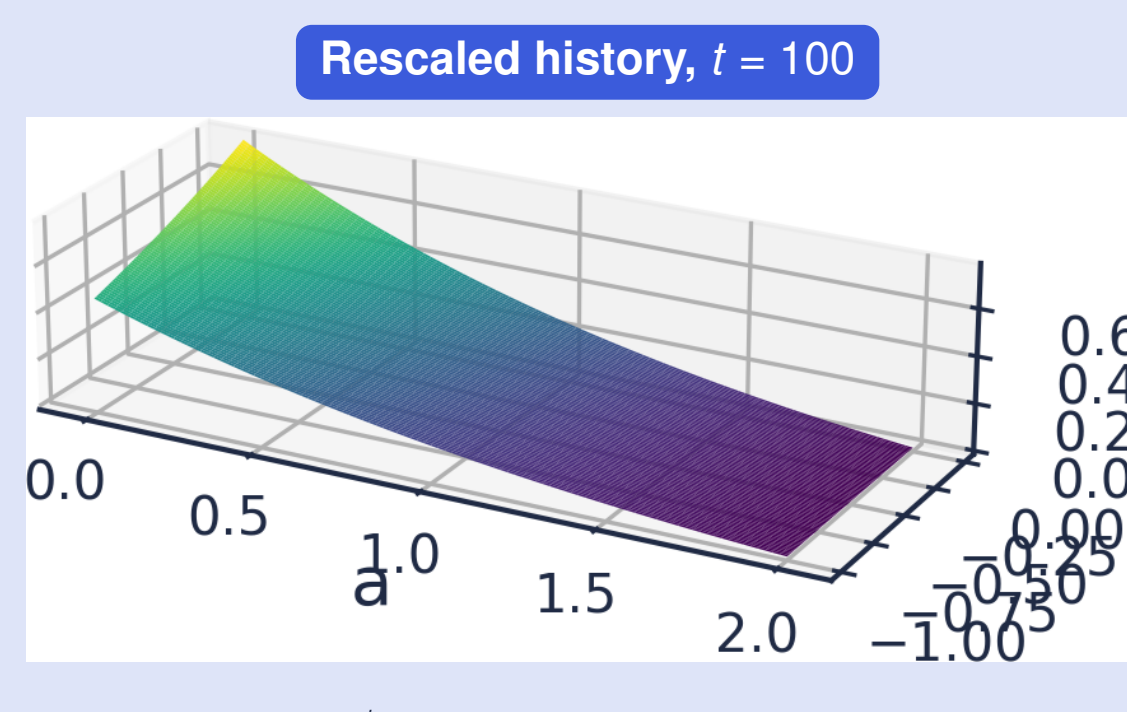
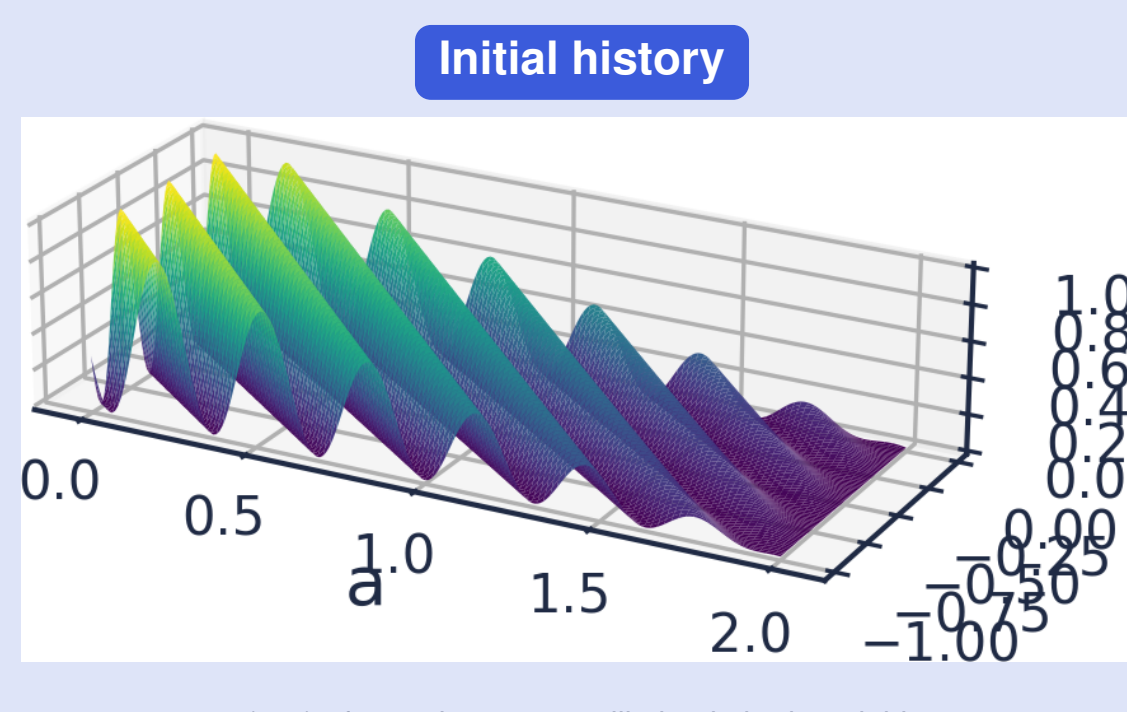
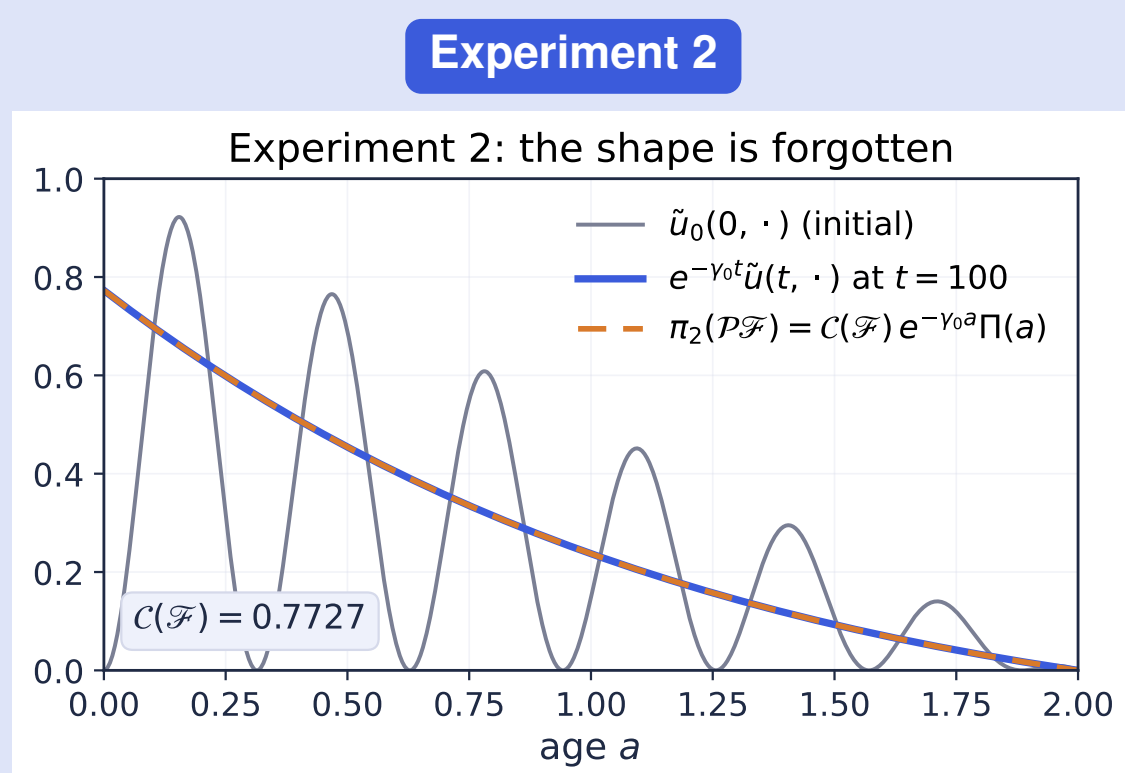
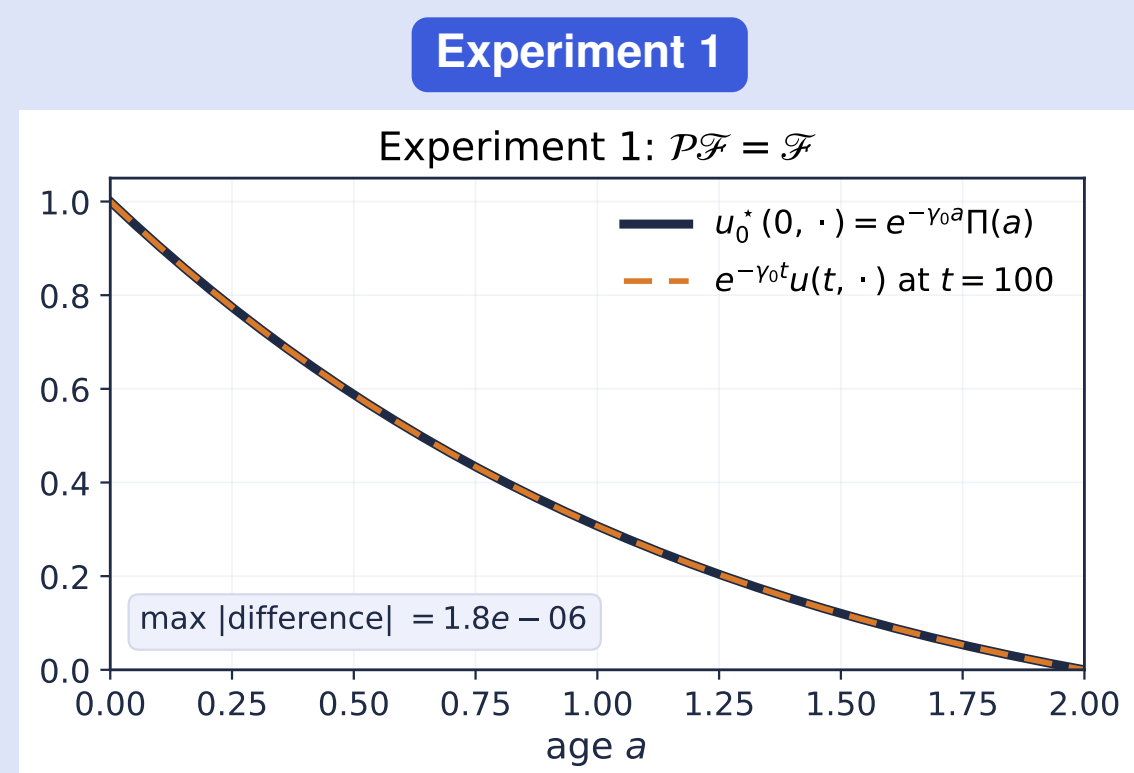
The projection. Writing $\lambda - \mathcal{A}$ in the factorised form of Nagel [10], every singular term of the resolvent carries a factor $\xi(\lambda)^{-1}$. Taking the residue at γ_0 gives

$$\mathcal{P} \mathcal{F} = \mathcal{C}(\mathcal{F}) \begin{pmatrix} \mathcal{D}_{\gamma_0, G_0} \mathcal{D}_{\gamma_0, A_0} 1 \\ \mathcal{D}_{\gamma_0, A_0} 1 \end{pmatrix}, \quad \mathcal{C}(\mathcal{F}) = \frac{KR(\gamma_0, G_0)F + K\mathcal{D}_{\gamma_0, G_0}R(\gamma_0, A_0)f}{-\xi'(\gamma_0)}.$$

The limit age profile is $e^{-\gamma_0 a} \Pi(a)$, with history $e^{\gamma_0(\theta - a)} \Pi(a)$, whether the population grows, stays bounded or dies out.

Analysis

We take $a_m = 2$, $r = 1$, $\beta(a) = 10a(a-m)/3^2$ and $\mu(a) = 1/(a-m)$, so that $\Pi(a) = (a-m)/a_m$ and $\gamma_0 = 0.4876$. In the first experiment the initial history is $u_0^c(s, a) = e^{\gamma_0(s-a)} \Pi(a)$, for which $\mathcal{P}\mathcal{F} = \mathcal{F}$ and $u(t, a) = e^{\gamma_0(t-a)} \Pi(a)$ is the exact solution. In the second it is the oscillating function $\tilde{u}_0(s, a) = \frac{a_m - a}{2} \sin^2(10(a+s))$.



The profile settles at the rate given by the spectrum: the second characteristic root is $\lambda_1 = 0.389 \pm 3.787i$, so the residual should decay like $e^{-(\gamma_0 - \text{Re } \lambda_1)t} = e^{-0.0985t}$, and a fit of the computed residual on $t \in [8, 70]$ gives 0.0985.

Conclusion

With a discrete delay the birth operator is unbounded and the semigroup is not eventually compact. Compact approximations forming a Cauchy sequence in operator norm give condition (i) of Webb's theorem, hence asynchronous exponential growth for every sign of the growth bound, with an explicit limit distribution confirmed numerically.

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