

A Backstepping–KKL observer for a cascade of a nonlinear ODE with a linear PDE

SCINDIS 2026

Lucas Brivadis lucas.brivadis@cnrs.fr

Collaborators: Adam Braun, Jean Auriol, and Florent Koudohode



A remark on Backstepping and KKL observers

Backstepping observer design

Consider a linear system with measured output:

$$\begin{cases} \dot{x} = Fx \\ y = Hx \end{cases}$$

To design a **Luenberger** observer

$$\dot{\hat{x}} = F\hat{x} - K(H\hat{x} - y)$$

with error system ($\bar{x} = \hat{x} - x$)

$$\dot{\bar{x}} = (F - KH)\bar{x},$$

look for **a map** T and **a gain** K such that

$$T(F - KH) = AT \quad (\text{Sylvester equation})$$

with A generating an exponentially stable semigroup.

$$\dot{\bar{x}} = (F - KH)\bar{x} \xrightarrow[\bar{z}=T\bar{x}]{T} \dot{\bar{z}} = A\bar{z}$$

Backstepping observer design

Consider a linear system with measured output:

$$\begin{cases} \dot{x} = Fx \\ y = Hx \end{cases}$$

To design a **Luenberger** observer

$$\dot{\hat{x}} = F\hat{x} - K(H\hat{x} - y)$$

with error system ($\bar{x} = \hat{x} - x$)

$$\dot{\bar{x}} = (F - KH)\bar{x},$$

look for **a map** T and **a gain** K such that

$$T(F - KH) = AT \quad (\text{Sylvester equation})$$

with A generating an exponentially stable semigroup.

$$\dot{\bar{x}} = (F - KH)\bar{x} \xrightarrow[\bar{z}=T\bar{x}]{T} \dot{\bar{z}} = A\bar{z}$$

Example: 1D heat equation

Heat equation:

$$\begin{cases} \partial_t v = \partial_\xi^2 v + \alpha v \\ \partial_\xi v(t, 0) = 0, \quad v(t, 1) = 0 \\ y(t) = v(t, 0) \end{cases}$$

i.e. $F = \partial_\xi^2 + \alpha$, $Hv = v(0)$.

Observer:

$$\begin{cases} \partial_t \hat{v} = \partial_\xi^2 \hat{v} + \alpha \hat{v} - k(\xi)(\hat{v}(t, 0) - y) \\ \partial_\xi \hat{v}(t, 0) = -p_{10}(\hat{v}(t, 0) - y) \\ \hat{v}(t, 1) = 0 \end{cases}$$

Target: $\begin{cases} \partial_t z = \partial_\xi^2 z - \gamma z \\ \partial_\xi z(t, 0) = 0, \quad z(t, 1) = 0 \end{cases}$ i.e. $A = \partial_\xi^2 - \gamma$, $\gamma > 0$.

Transform: backstepping transformation

$$(Tv)(\xi) = v(\xi) - \int_0^\xi p(\xi, \bar{\xi}) v(\bar{\xi}) d\bar{\xi}, \quad \begin{cases} p_{\xi\xi} - p_{\bar{\xi}\bar{\xi}} = (\alpha + \gamma)p \\ \frac{d}{d\xi} p(\xi, \xi) = -\frac{\alpha + \gamma}{2} \\ p(1, \bar{\xi}) = 0 \end{cases}$$

Luenberger approach

“As a first step toward the construction of an observer it is useful to consider a slightly more general problem. Instead of requiring that the observer reconstruct the state vector itself, require only that it reconstruct some constant linear transformation of the state vector. [...]

Assuming it were possible to build a system which reconstructs some constant linear transformation T of the state vector x , it is clear that it would then be possible to reconstruct the state vector itself, provided that the transformation T were invertible. This is the approach taken in this paper.”

— D. G. Luenberger, *Observing the state of a linear system*,
IEEE Trans. Mil. Electron., 8(2):74–80, 1964.



David G. Luenberger
(1937–2026)

Luenberger approach

Step 1: For any map T satisfying

$$TF = AT + BH \quad (\text{Sylvester equation})$$

with A exponentially stable and B arbitrary, $z = Tx$ can be estimated with

$$\dot{\hat{z}} = A\hat{z} + By,$$

with error system ($\bar{z} = \hat{z} - z$)

$$\dot{\bar{z}} = A\hat{z} + By - TFx = A(\hat{z} - Tx) + By - By = A\bar{z}.$$

Step 2: Look for a **left-invertible** map T as above.

In that case, $\hat{x} = T^{-1}\hat{z}$ satisfies

$$\dot{\hat{x}} = T^{-1}(A\hat{z} + By) = T^{-1}AT\hat{x} + T^{-1}By = F\hat{x} - T^{-1}B(H\hat{x} - y).$$

Luenberger approach

Step 1: For any map T satisfying

$$TF = AT + BH \quad (\text{Sylvester equation})$$

with A exponentially stable and B arbitrary, $z = Tx$ can be estimated with

$$\dot{\hat{z}} = A\hat{z} + By,$$

with error system ($\bar{z} = \hat{z} - z$)

$$\dot{\bar{z}} = A\hat{z} + By - TFx = A(\hat{z} - Tx) + By - By = A\bar{z}.$$

Step 2: Look for a **left-invertible** map T as above.

In that case, $\hat{x} = T^{-1}\hat{z}$ satisfies

$$\dot{\hat{x}} = T^{-1}(A\hat{z} + By) = T^{-1}AT\hat{x} + T^{-1}By = F\hat{x} - T^{-1}B(H\hat{x} - y).$$

Luenberger approach

Step 1: For any map T satisfying

$$TF = AT + BH \quad (\text{Sylvester equation})$$

with A exponentially stable and B arbitrary, $z = Tx$ can be estimated with

$$\dot{\hat{z}} = A\hat{z} + By,$$

with error system ($\bar{z} = \hat{z} - z$)

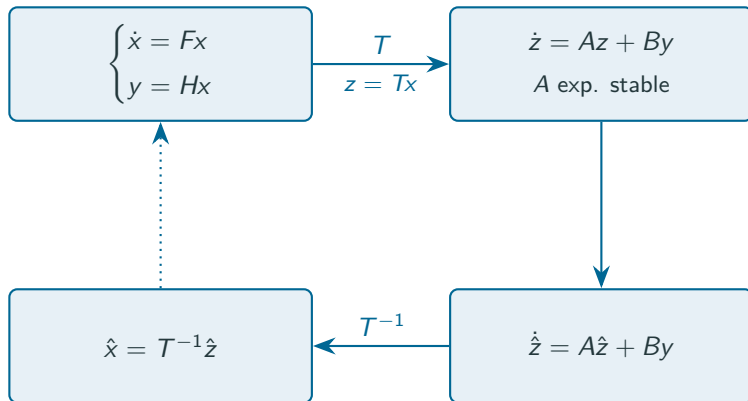
$$\dot{\bar{z}} = A\hat{z} + By - TFx = A(\hat{z} - Tx) + By - By = A\bar{z}.$$

Step 2: Look for a **left-invertible** map T as above.

In that case, $\hat{x} = T^{-1}\hat{z}$ satisfies

$$\dot{\hat{x}} = T^{-1}(A\hat{z} + By) = T^{-1}AT\hat{x} + T^{-1}By = F\hat{x} - T^{-1}B(H\hat{x} - y).$$

Luenberger approach



Example: 1D heat equation

Heat equation:

$$\begin{cases} \partial_t v = \partial_\xi^2 v + \alpha v \\ \partial_\xi v(t, 0) = 0, \quad v(t, 1) = 0 \\ y(t) = v(t, 0) \end{cases}$$

Target system: $z = Tv$ satisfies

$$\begin{cases} \partial_t z = \partial_\xi^2 z - \gamma z + p_1(\xi) y \\ \partial_\xi z(t, 0) = p_{10} y, \quad z(t, 1) = 0 \end{cases}$$

Transform: the **same** backstepping transformation T solves
 $TF = AT + BH$, with $A = \partial_\xi^2 - \gamma$ and B acting in the domain (p_1) and
 on the boundary (p_{10}) .

Observer:
$$\begin{cases} \partial_t \hat{z} = \partial_\xi^2 \hat{z} - \gamma \hat{z} + p_1(\xi) y \\ \partial_\xi \hat{z}(t, 0) = p_{10} y, \quad \hat{z}(t, 1) = 0 \end{cases} \quad \hat{v} = T^{-1} \hat{z}$$

Instead of embedding the error system into a stable system, embed the system into a stable filter of the output.

KKL observer design

Consider a nonlinear finite-dimensional system

$$\begin{cases} \dot{x} = f(x) \\ y = h(x) \end{cases}$$

How to design a “nonlinear Luenberger” observer?

KKL observer design

Consider a nonlinear finite-dimensional system

$$\begin{cases} \dot{x} = f(x) \\ y = h(x) \end{cases}$$

How to design a “nonlinear Luenberger” observer?

$$\dot{\hat{x}} = f(\hat{x}) - k(\hat{x}, y)(h(\hat{x}) - y)$$

R. G. Sanfelice, L. Praly, Convergence of nonlinear observers on $\mathbb{R}^n$ with a Riemannian metric (I–III), IEEE TAC, 2012–2024.

KKL observer design

Consider a nonlinear finite-dimensional system

$$\begin{cases} \dot{x} = f(x) \\ y = h(x) \end{cases}$$

How to design a “nonlinear Luenberger” observer?

Step 1: For any nonlinear map T satisfying

$$\frac{\partial T}{\partial x}(x) f(x) = AT(x) + Bh(x) \quad (\text{nonlinear Sylvester equation})$$

with A exponentially stable, $z = T(x)$ can be estimated.

Step 2: Look for a left-invertible map T as above.

This is the **Kazantzis-Kravaris/Luenberger** design.

N. Kazantzis, C. Kravaris, Nonlinear observer design using Lyapunov's auxiliary theorem, SCL, 1998.

V. Andrieu, L. Praly, On the existence of a Kazantzis-Kravaris/Luenberger observer, SICON, 2006.

L. Brivadis, V. Andrieu, P. Bernard, U. Serres, Further remarks on KKL observers, SCL, 2023.

Towards nonlinear infinite-dimensional systems?

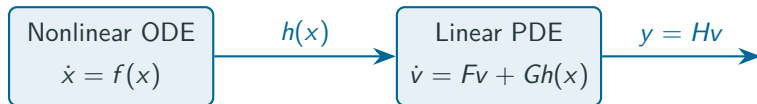
- For **linear PDEs**, backstepping and KKL observer designs are the same.
- KKL deals with **nonlinearities**.
- Can we use that to design observers for **nonlinear infinite-dimensional systems**?

This work is a first attempt to go in that direction.

Cascade system and existence of T

Cascade system

Consider a cascade of the form:



$$\begin{cases} \dot{x} = f(x) \\ \dot{v} = Fv + Gh(x) \\ y = Hv \end{cases}$$

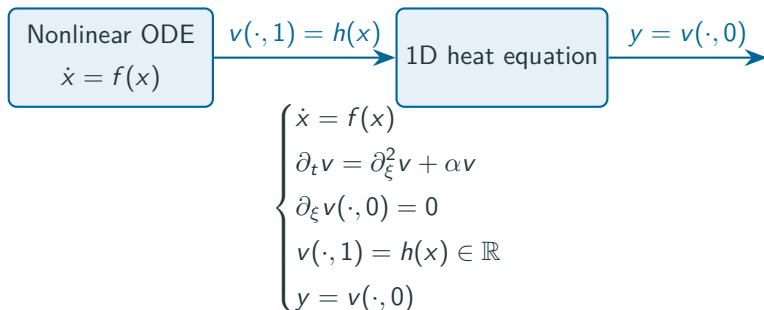
Motivation: To separate the effects of the nonlinearity and the infinite-dimensionality.

Goal: Estimate x and v from y .

Existing approaches: *T. Ahmed-Ali, F. Giri, M. Krstic, F. Lamnabhi-Lagarigue, SCL, 2015.*

Cascade system

Guiding example:



Motivation: To separate the effects of the nonlinearity and the infinite-dimensionality.

Goal: Estimate x and v from y .

Existing approaches: *T. Ahmed-Ali, F. Giri, M. Krstic, F. Lamnabhi-Lagarigue, SCL, 2015.*

Design of T

1) Start with backstepping: $(T_{\text{pde}}v)(\xi) = v(\xi) - \int_0^\xi p(\xi, \bar{\xi})v(\bar{\xi})d\bar{\xi}$:

$$\begin{cases} \partial_t v = \partial_\xi^2 v + \alpha v \\ \partial_\xi v(\cdot, 0) = 0 \\ v(\cdot, 1) = h(x) \end{cases} \xrightarrow{z_{\text{pde}} = T_{\text{pde}}v} \begin{cases} \partial_t z_{\text{pde}} = \partial_\xi^2 z_{\text{pde}} - \gamma z_{\text{pde}} + p_1(\xi) y \\ \partial_\xi z_{\text{pde}}(\cdot, 0) = p_{10} y \\ z_{\text{pde}}(\cdot, 1) = h(x) \end{cases}$$

2) Look for a KKL transform $T_{\text{ode}} : \mathcal{X} \rightarrow L^2(0, 1)$ such that:

$$\dot{x} = f(x) \xrightarrow{z_{\text{ode}} = T_{\text{ode}}(x)} \begin{cases} \partial_t z_{\text{ode}} = \partial_\xi^2 z_{\text{ode}} - \gamma z_{\text{ode}} \\ \partial_\xi z_{\text{ode}}(\cdot, 0) = 0 \\ z_{\text{ode}}(\cdot, 1) = -h(x) \end{cases}$$

3) Define $z = T(x, v) = T_{\text{ode}}(x) + T_{\text{pde}}(v)$. Then

$$\begin{cases} \partial_t z = \partial_\xi^2 z - \gamma z + p_1(\xi) y \\ \partial_\xi z(\cdot, 0) = p_{10} y \\ z(\cdot, 1) = h(x) - h(x) = 0 \end{cases}$$

Design of T

1) Start with backstepping: $(T_{\text{pde}}v)(\xi) = v(\xi) - \int_0^\xi p(\xi, \bar{\xi})v(\bar{\xi})d\bar{\xi}$:

$$\begin{cases} \partial_t v = \partial_\xi^2 v + \alpha v \\ \partial_\xi v(\cdot, 0) = 0 \\ v(\cdot, 1) = h(x) \end{cases} \xrightarrow{z_{\text{pde}} = T_{\text{pde}}v} \begin{cases} \partial_t z_{\text{pde}} = \partial_\xi^2 z_{\text{pde}} - \gamma z_{\text{pde}} + p_1(\xi) y \\ \partial_\xi z_{\text{pde}}(\cdot, 0) = p_{10} y \\ z_{\text{pde}}(\cdot, 1) = h(x) \end{cases}$$

2) Look for a KKL transform $T_{\text{ode}} : \mathcal{X} \rightarrow L^2(0, 1)$ such that:

$$\dot{x} = f(x) \xrightarrow{z_{\text{ode}} = T_{\text{ode}}(x)} \begin{cases} \partial_t z_{\text{ode}} = \partial_\xi^2 z_{\text{ode}} - \gamma z_{\text{ode}} \\ \partial_\xi z_{\text{ode}}(\cdot, 0) = 0 \\ z_{\text{ode}}(\cdot, 1) = -h(x) \end{cases}$$

3) Define $z = T(x, v) = T_{\text{ode}}(x) + T_{\text{pde}}(v)$. Then

$$\begin{cases} \partial_t z = \partial_\xi^2 z - \gamma z + p_1(\xi) y \\ \partial_\xi z(\cdot, 0) = p_{10} y \\ z(\cdot, 1) = h(x) - h(x) = 0 \end{cases}$$

Design of T

1) Start with backstepping: $(T_{\text{pde}}v)(\xi) = v(\xi) - \int_0^\xi p(\xi, \bar{\xi})v(\bar{\xi})d\bar{\xi}$:

$$\begin{cases} \partial_t v = \partial_\xi^2 v + \alpha v \\ \partial_\xi v(\cdot, 0) = 0 \\ v(\cdot, 1) = h(x) \end{cases} \xrightarrow{z_{\text{pde}} = T_{\text{pde}}v} \begin{cases} \partial_t z_{\text{pde}} = \partial_\xi^2 z_{\text{pde}} - \gamma z_{\text{pde}} + p_1(\xi) y \\ \partial_\xi z_{\text{pde}}(\cdot, 0) = p_{10} y \\ z_{\text{pde}}(\cdot, 1) = h(x) \end{cases}$$

2) Look for a KKL transform $T_{\text{ode}} : \mathcal{X} \rightarrow L^2(0, 1)$ such that:

$$\dot{x} = f(x) \xrightarrow{z_{\text{ode}} = T_{\text{ode}}(x)} \begin{cases} \partial_t z_{\text{ode}} = \partial_\xi^2 z_{\text{ode}} - \gamma z_{\text{ode}} \\ \partial_\xi z_{\text{ode}}(\cdot, 0) = 0 \\ z_{\text{ode}}(\cdot, 1) = -h(x) \end{cases}$$

3) Define $z = T(x, v) = T_{\text{ode}}(x) + T_{\text{pde}}(v)$. Then

$$\begin{cases} \partial_t z = \partial_\xi^2 z - \gamma z + p_1(\xi) y \\ \partial_\xi z(\cdot, 0) = p_{10} y \\ z(\cdot, 1) = h(x) - h(x) = 0 \end{cases}$$

Design of T_{ode}

Existence assumption: There exists $\gamma_0 \geq 0$ such that, for every $x_0 \in \mathcal{X}_0$, $x(t) = \varphi(t, x_0)$ is well defined for all $t \leq 0$, and there exists $C > 0$ for which

$$e^{\gamma_0 t} |h(x(t))| \leq C, \quad \forall t \leq 0.$$

Design of T_{ode}

Existence assumption: There exists $\gamma_0 \geq 0$ such that, for every $x_0 \in \mathcal{X}_0$, $x(t) = \varphi(t, x_0)$ is well defined for all $t \leq 0$, and there exists $C > 0$ for which

$$e^{\gamma_0 t} |h(x(t))| \leq C, \quad \forall t \leq 0.$$

Remarks:

- OK if h is continuous and $x(t)$ remains in a compact set for $t \leq 0$.
- OK if f is linearly bounded and h is polynomially bounded.

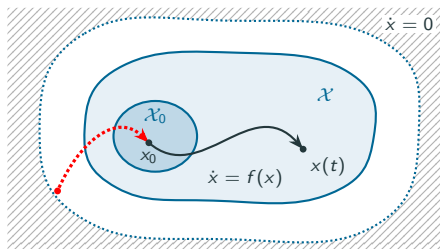
Design of T_{ode}

Existence assumption: There exists $\gamma_0 \geq 0$ such that, for every $x_0 \in \mathcal{X}_0$, $x(t) = \varphi(t, x_0)$ is well defined for all $t \leq 0$, and there exists $C > 0$ for which

$$e^{\gamma_0 t} |h(x(t))| \leq C, \quad \forall t \leq 0.$$

Remarks:

- OK if the trajectories of interest remain in a bounded set $\mathcal{X}$ (with $\gamma_0 = 0$).



Design of T_{ode}

Theorem: Under the existence assumption, for any $\gamma > \gamma_0$, there exists $T_{\text{ode}} : \mathcal{X} \rightarrow L^2(0, 1)$ such that:

$$\dot{x} = f(x) \xrightarrow{z_{\text{ode}} = T_{\text{ode}}(x)} \begin{cases} \partial_t z_{\text{ode}} = \partial_\xi^2 z_{\text{ode}} - \gamma z_{\text{ode}} \\ \partial_\xi z_{\text{ode}}(\cdot, 0) = 0 \\ z_{\text{ode}}(\cdot, 1) = -h(x) \end{cases}$$

A. Braun, L. Brivadis, J. Auriol, A Backstepping-KKL observer for a cascade of a nonlinear ODE with a heat equation, *arXiv:2510.04941*, 2025.

Design of T_{ode}

Proof: Consider the following **steady-state** problem:

For a given $u : (-\infty, 0] \rightarrow \mathbb{R}$,

$$\begin{cases} \partial_t w^u(t, \xi) = \partial_\xi^2 w^u(t, \xi) - \gamma w^u(t, \xi) \\ \partial_\xi w^u(t, 0) = 0 \\ w^u(t, 1) = u(t) \end{cases} \quad \forall t \leq 0$$

Lemma 1: For every $u \in C^0((-\infty, 0], \mathbb{R})$ such that $t \mapsto e^{\gamma t} u(t)$ is bounded, the steady-state problem admits a unique solution w^u such that $\|w^u(t, \cdot)\|_{L^2} \leq Ce^{-\gamma t}$ for all $t \leq 0$.

Lemma 2: Set $T_{\text{ode}}(x) := w^{u_x}(0, \cdot)$, where $u_x(t) := -h(\varphi(t, x))$ for all $t \leq 0$. Then

$$\dot{x} = f(x) \xrightarrow{z_{\text{ode}} = T_{\text{ode}}(x)} \begin{cases} \partial_t z_{\text{ode}} = \partial_\xi^2 z_{\text{ode}} - \gamma z_{\text{ode}} \\ \partial_\xi z_{\text{ode}}(\cdot, 0) = 0 \\ z_{\text{ode}}(\cdot, 1) = -h(x) \end{cases} \quad \forall t \geq 0$$

Design of T_{ode}

Proof: Consider the following **steady-state** problem:

For a given $u : (-\infty, 0] \rightarrow \mathbb{R}$,

$$\begin{cases} \partial_t w^u(t, \xi) = \partial_\xi^2 w^u(t, \xi) - \gamma w^u(t, \xi) \\ \partial_\xi w^u(t, 0) = 0 \\ w^u(t, 1) = u(t) \end{cases} \quad \forall t \leq 0$$

Lemma 1: For every $u \in C^0((-\infty, 0], \mathbb{R})$ such that $t \mapsto e^{\gamma t} u(t)$ is bounded, the steady-state problem admits a unique solution w^u such that $\|w^u(t, \cdot)\|_{L^2} \leq C e^{-\gamma t}$ for all $t \leq 0$.

Lemma 2: Set $T_{\text{ode}}(x) := w^{u_x}(0, \cdot)$, where $u_x(t) := -h(\varphi(t, x))$ for all $t \leq 0$. Then

$$\dot{x} = f(x) \xrightarrow{z_{\text{ode}} = T_{\text{ode}}(x)} \begin{cases} \partial_t z_{\text{ode}} = \partial_\xi^2 z_{\text{ode}} - \gamma z_{\text{ode}} \\ \partial_\xi z_{\text{ode}}(\cdot, 0) = 0 \\ z_{\text{ode}}(\cdot, 1) = -h(x) \end{cases} \quad \forall t \geq 0$$

Design of T_{ode}

Proof: Consider the following **steady-state** problem:

For a given $u : (-\infty, 0] \rightarrow \mathbb{R}$,

$$\begin{cases} \partial_t w^u(t, \xi) = \partial_\xi^2 w^u(t, \xi) - \gamma w^u(t, \xi) \\ \partial_\xi w^u(t, 0) = 0 \\ w^u(t, 1) = u(t) \end{cases} \quad \forall t \leq 0$$

Lemma 1: For every $u \in C^0((-\infty, 0], \mathbb{R})$ such that $t \mapsto e^{\gamma t} u(t)$ is bounded, the steady-state problem admits a unique solution w^u such that $\|w^u(t, \cdot)\|_{L^2} \leq C e^{-\gamma t}$ for all $t \leq 0$.

Lemma 2: Set $T_{\text{ode}}(x) := w^{u_x}(0, \cdot)$, where $u_x(t) := -h(\varphi(t, x))$ for all $t \leq 0$. Then

$$\dot{x} = f(x) \xrightarrow{z_{\text{ode}} = T_{\text{ode}}(x)} \begin{cases} \partial_t z_{\text{ode}} = \partial_\xi^2 z_{\text{ode}} - \gamma z_{\text{ode}} \\ \partial_\xi z_{\text{ode}}(\cdot, 0) = 0 \\ z_{\text{ode}}(\cdot, 1) = -h(x) \end{cases} \quad \forall t \geq 0$$

Injectivity of T

Injectivity of T

Step 1: Injectivity of T_{pde} : granted by backstepping design ✓

Step 2: Injectivity of T_{ode} ?

Step 3: Injectivity of T ?

Step 4: Left-inversion for observer design?

Injectivity of T

Step 1: Injectivity of T_{pde} : granted by backstepping design ✓

Step 2: Injectivity of T_{ode} ?

Step 3: Injectivity of T ?

Step 4: Left-inversion for observer design?

Injectivity of T

Step 1: Injectivity of T_{pde} : granted by backstepping design ✓

Step 2: Injectivity of T_{ode} ?

Step 3: Injectivity of T ?

Step 4: Left-inversion for observer design?

Injectivity of T

Step 1: Injectivity of T_{pde} : granted by backstepping design ✓

Step 2: Injectivity of T_{ode} ?

Step 3: Injectivity of T ?

Step 4: Left-inversion for observer design?

Injectivity of T_{ode}

Differential observability assumption: There exists $m \in \mathbb{N}$ such that the map

$$\begin{aligned} H : \mathcal{X} &\longrightarrow \mathbb{R}^m \\ x &\longmapsto (h(x), L_f h(x), \dots, L_f^{m-1} h(x)) \end{aligned}$$

is injective.

Theorem: Under differential observability, T_{ode} is injective over $\mathcal{X}$.

A. Braun, L. Brivadis, J. Auriol, A Backstepping-KKL observer for a cascade of a nonlinear ODE with a heat equation, arXiv:2510.04941, 2025.

Injectivity of T_{ode}

Differential observability assumption: There exists $m \in \mathbb{N}$ such that the map

$$\begin{aligned} H : \mathcal{X} &\longrightarrow \mathbb{R}^m \\ x &\longmapsto (h(x), L_f h(x), \dots, L_f^{m-1} h(x)) \end{aligned}$$

is injective.

Theorem: Under differential observability, T_{ode} is injective over $\mathcal{X}$.

A. Braun, L. Brivadis, J. Auriol, A Backstepping-KKL observer for a cascade of a nonlinear ODE with a heat equation, arXiv:2510.04941, 2025.

Proof of injectivity of T_{ode}

Let $x_a, x_b \in \mathcal{X}$ be such that $T_{\text{ode}}(x_a) = T_{\text{ode}}(x_b)$, i.e.

$$w^{u_{x_a}}(0, \xi) = w^{u_{x_b}}(0, \xi), \quad \forall \xi \in [0, 1].$$

- By induction, using $\partial_t w = \partial_\xi^2 w - \gamma w$:

$$\partial_t^k w^{u_{x_a}}(0, \xi) = \partial_t^k w^{u_{x_b}}(0, \xi), \quad \forall k \in \{0, \dots, m-1\}.$$

- At $\xi = 1$, since $w^u(t, 1) = u(t)$ and $u_x(t) = -h(\varphi(t, x))$:

$$L_f^k h(x_a) = L_f^k h(x_b), \quad \forall k \in \{0, \dots, m-1\}.$$

- Hence $H(x_a) = H(x_b)$, and differential observability gives $x_a = x_b$.

Injectivity of T

Definition:

$$\mathcal{A} := \left\{ v : [0, 1] \rightarrow \mathbb{R} \text{ analytic} \mid \frac{d^{2k+1}v}{d\xi^{2k+1}}(0) = 0, \forall k \in \mathbb{N} \right\}$$

Facts:

- $v(t) \in \mathcal{A}$ for all $t > 0$ (parabolic regularization);
- $T_{\text{ode}}(x) \in \mathcal{A}$ for all $x \in \mathcal{X}$.

Lemma: $T_{\text{pde}}(\mathcal{A}) \cap \mathcal{A} = \{0\}$.

Proof. Let $v, z \in \mathcal{A}$ with $z = T_{\text{pde}}v$. At $\xi = 0$: $z'(0) = v'(0) + p_{10} v(0)$, with $z'(0) = v'(0) = 0$ and $p_{10} \neq 0$, so $v(0) = 0$. By induction, all derivatives of v and z vanish at 0. By analyticity, $v = z = 0$. □

Injectivity of T

Definition:

$$\mathcal{A} := \left\{ v : [0, 1] \rightarrow \mathbb{R} \text{ analytic} \mid \frac{d^{2k+1}v}{d\xi^{2k+1}}(0) = 0, \forall k \in \mathbb{N} \right\}$$

Facts:

- $v(t) \in \mathcal{A}$ for all $t > 0$ (parabolic regularization);
- $T_{\text{ode}}(x) \in \mathcal{A}$ for all $x \in \mathcal{X}$.

Lemma: $T_{\text{pde}}(\mathcal{A}) \cap \mathcal{A} = \{0\}$.

Proof. Let $v, z \in \mathcal{A}$ with $z = T_{\text{pde}}v$. At $\xi = 0$: $z'(0) = v'(0) + p_{10} v(0)$, with $z'(0) = v'(0) = 0$ and $p_{10} \neq 0$, so $v(0) = 0$. By induction, all derivatives of v and z vanish at 0. By analyticity, $v = z = 0$. □

Injectivity of T

Theorem: T is injective over $\mathcal{X} \times \mathcal{A}$.

Proof. Let $(x_a, v_a), (x_b, v_b) \in \mathcal{X} \times \mathcal{A}$ with $T(x_a, v_a) = T(x_b, v_b)$. Then

$$T_{\text{ode}}(x_a) - T_{\text{ode}}(x_b) = T_{\text{pde}}(v_b - v_a) \in \mathcal{A} \cap T_{\text{pde}}(\mathcal{A}) = \{0\}.$$

Hence $T_{\text{ode}}(x_a) = T_{\text{ode}}(x_b)$, so $x_a = x_b$ (injectivity of T_{ode}), and $T_{\text{pde}}(v_b - v_a) = 0$, so $v_a = v_b$ (invertibility of T_{pde}). □

A. Braun, L. Brivadis, J. Auriol, A Backstepping-KKL observer for a cascade of a nonlinear ODE with a heat equation, arXiv:2510.04941, 2025.

Left-inversion of T

Finite-dimensional KKL: If $\mathcal{X}$ is **compact** and T is continuous and injective, then $T^{-1} : T(\mathcal{X}) \rightarrow \mathcal{X}$ is **uniformly continuous**, and it extends to a uniformly continuous map on $\mathbb{R}^m$ (**McShane extension**). Hence

$$|\hat{z}(t) - z(t)| \rightarrow 0 \implies |T^{-1}(\hat{z}(t)) - x(t)| \rightarrow 0.$$

E. J. McShane, Extension of range of functions, Bull. Amer. Math. Soc., 1934.

Proposition: There is no continuous left-inverse

$$T^{-1} : (T(\mathcal{X} \times \mathcal{A}), \|\cdot\|_{L^2}) \rightarrow \mathbb{R}^n \times L^2(0, 1).$$

Proof. $\mathcal{A}$ is dense in L^2 , and so is $T_{\text{pde}}(\mathcal{A})$. Take $x_a \neq x_b$ and $v_j \in \mathcal{A}$ with $T_{\text{pde}} v_j \rightarrow T_{\text{ode}}(x_a) - T_{\text{ode}}(x_b)$. Then

$$T(x_b, v_j) \xrightarrow{L^2} T(x_a, 0), \quad \text{but } x_b \not\rightarrow x_a.$$

□

Left-inversion of T

Finite-dimensional KKL: If $\mathcal{X}$ is **compact** and T is continuous and injective, then $T^{-1} : T(\mathcal{X}) \rightarrow \mathcal{X}$ is **uniformly continuous**, and it extends to a uniformly continuous map on $\mathbb{R}^m$ (**McShane extension**). Hence

$$|\hat{z}(t) - z(t)| \rightarrow 0 \implies |T^{-1}(\hat{z}(t)) - x(t)| \rightarrow 0.$$

E. J. McShane, Extension of range of functions, Bull. Amer. Math. Soc., 1934.

Proposition: There is no continuous left-inverse

$$T^{-1} : (T(\mathcal{X} \times \mathcal{A}), \|\cdot\|_{L^2}) \rightarrow \mathbb{R}^n \times L^2(0, 1).$$

Proof. $\mathcal{A}$ is dense in L^2 , and so is $T_{\text{pde}}(\mathcal{A})$. Take $x_a \neq x_b$ and $v_j \in \mathcal{A}$ with $T_{\text{pde}} v_j \rightarrow T_{\text{ode}}(x_a) - T_{\text{ode}}(x_b)$. Then

$$T(x_b, v_j) \xrightarrow{L^2} T(x_a, 0), \quad \text{but} \quad x_b \not\rightarrow x_a.$$

□

Left-inversion of T : a stronger topology

Idea: Actually, the convergence $\hat{z} - z \rightarrow 0$ is **stronger than just L^2** .

Since $v(t) \in \mathcal{A}$, the **smoothing effect** of the heat equation gives

$$\partial_{\xi}^{2k+1} \left(T_{\text{pde}}^{-1} (\hat{z}(t) - T_{\text{ode}}(x(t))) \right) (0) \rightarrow 0, \quad \forall k \in \mathbb{N}.$$

Denote $j_{\infty} w \in \mathbb{R}^{\mathbb{N}}$ the sequence of odd derivatives of w at $\xi = 0$, endowed with the **pointwise** topology, and

$$d(z_1, z_2) := \|z_1 - z_2\|_{L^2} + d_{\text{pointwise}}(j_{\infty} T_{\text{pde}}^{-1} z_1, j_{\infty} T_{\text{pde}}^{-1} z_2).$$

Theorem: If $\mathcal{X}$ is compact, then $T^{-1} : (T(\mathcal{X} \times \mathcal{A}), d) \rightarrow \mathbb{R}^n \times L^2(0, 1)$ is uniformly continuous and $d(\hat{z}(t), z(t)) \rightarrow 0$. Hence, with

$$\hat{x}(t) \in \arg \min_{\bar{x} \in \mathcal{X}} d_{\text{pointwise}}(j_{\infty} T_{\text{pde}}^{-1} \hat{z}(t), j_{\infty} T_{\text{pde}}^{-1} T_{\text{ode}}(\bar{x}))$$

and $\hat{v}(t) = T_{\text{pde}}^{-1}(\hat{z}(t) - T_{\text{ode}}(\hat{x}(t)))$,

we get $|\hat{x}(t) - x(t)| + \|\hat{v}(t) - v(t)\|_{L^2} \rightarrow 0$.

Open question: Abstract case?

A. Braun, L. Brivadis, J. Auriol, A Backstepping-KKL observer for a cascade of a nonlinear ODE with a heat equation, *arXiv:2510.04941*, 2025.

Left-inversion of T : a stronger topology

Idea: Actually, the convergence $\hat{z} - z \rightarrow 0$ is **stronger than just L^2** .

Since $v(t) \in \mathcal{A}$, the **smoothing effect** of the heat equation gives

$$\partial_{\xi}^{2k+1} \left(T_{\text{pde}}^{-1} (\hat{z}(t) - T_{\text{ode}}(x(t))) \right) (0) \rightarrow 0, \quad \forall k \in \mathbb{N}.$$

Denote $j_{\infty} w \in \mathbb{R}^{\mathbb{N}}$ the sequence of odd derivatives of w at $\xi = 0$, endowed with the **pointwise** topology, and

$$d(z_1, z_2) := \|z_1 - z_2\|_{L^2} + d_{\text{pointwise}}(j_{\infty} T_{\text{pde}}^{-1} z_1, j_{\infty} T_{\text{pde}}^{-1} z_2).$$

Theorem: If $\mathcal{X}$ is compact, then $T^{-1} : (T(\mathcal{X} \times \mathcal{A}), d) \rightarrow \mathbb{R}^n \times L^2(0, 1)$ is uniformly continuous and $d(\hat{z}(t), z(t)) \rightarrow 0$. Hence, with

$$\hat{x}(t) \in \arg \min_{\bar{x} \in \mathcal{X}} d_{\text{pointwise}}(j_{\infty} T_{\text{pde}}^{-1} \hat{z}(t), j_{\infty} T_{\text{pde}}^{-1} T_{\text{ode}}(\bar{x}))$$

and $\hat{v}(t) = T_{\text{pde}}^{-1}(\hat{z}(t) - T_{\text{ode}}(\hat{x}(t)))$,

we get $|\hat{x}(t) - x(t)| + \|\hat{v}(t) - v(t)\|_{L^2} \rightarrow 0$.

Open question: Abstract case?

A. Braun, L. Brivadis, J. Auriol, A Backstepping-KKL observer for a cascade of a nonlinear ODE with a heat equation, *arXiv:2510.04941*, 2025.

Left-inversion of T : a stronger topology

Idea: Actually, the convergence $\hat{z} - z \rightarrow 0$ is **stronger than just L^2** .

Since $v(t) \in \mathcal{A}$, the **smoothing effect** of the heat equation gives

$$\partial_{\xi}^{2k+1} \left(T_{\text{pde}}^{-1} (\hat{z}(t) - T_{\text{ode}}(x(t))) \right) (0) \rightarrow 0, \quad \forall k \in \mathbb{N}.$$

Denote $j_{\infty} w \in \mathbb{R}^{\mathbb{N}}$ the sequence of odd derivatives of w at $\xi = 0$, endowed with the **pointwise** topology, and

$$d(z_1, z_2) := \|z_1 - z_2\|_{L^2} + d_{\text{pointwise}}(j_{\infty} T_{\text{pde}}^{-1} z_1, j_{\infty} T_{\text{pde}}^{-1} z_2).$$

Theorem: If $\mathcal{X}$ is compact, then $T^{-1} : (T(\mathcal{X} \times \mathcal{A}), d) \rightarrow \mathbb{R}^n \times L^2(0, 1)$ is uniformly continuous and $d(\hat{z}(t), z(t)) \rightarrow 0$. Hence, with

$$\hat{x}(t) \in \arg \min_{\bar{x} \in \mathcal{X}} d_{\text{pointwise}}(j_{\infty} T_{\text{pde}}^{-1} \hat{z}(t), j_{\infty} T_{\text{pde}}^{-1} T_{\text{ode}}(\bar{x}))$$

and $\hat{v}(t) = T_{\text{pde}}^{-1}(\hat{z}(t) - T_{\text{ode}}(\hat{x}(t)))$,

we get $|\hat{x}(t) - x(t)| + \|\hat{v}(t) - v(t)\|_{L^2} \rightarrow 0$.

Open question: Abstract case?

A. Braun, L. Brivadis, J. Auriol, A Backstepping-KKL observer for a cascade of a nonlinear ODE with a heat equation, *arXiv:2510.04941*, 2025.

Left-inversion of T : a stronger topology

Idea: Actually, the convergence $\hat{z} - z \rightarrow 0$ is **stronger than just L^2** .

Since $v(t) \in \mathcal{A}$, the **smoothing effect** of the heat equation gives

$$\partial_{\xi}^{2k+1} \left(T_{\text{pde}}^{-1} (\hat{z}(t) - T_{\text{ode}}(x(t))) \right) (0) \rightarrow 0, \quad \forall k \in \mathbb{N}.$$

Denote $j_{\infty} w \in \mathbb{R}^{\mathbb{N}}$ the sequence of odd derivatives of w at $\xi = 0$, endowed with the **pointwise** topology, and

$$d(z_1, z_2) := \|z_1 - z_2\|_{L^2} + d_{\text{pointwise}}(j_{\infty} T_{\text{pde}}^{-1} z_1, j_{\infty} T_{\text{pde}}^{-1} z_2).$$

Theorem: If $\mathcal{X}$ is compact, then $T^{-1} : (T(\mathcal{X} \times \mathcal{A}), d) \rightarrow \mathbb{R}^n \times L^2(0, 1)$ is uniformly continuous and $d(\hat{z}(t), z(t)) \rightarrow 0$. Hence, with

$$\hat{x}(t) \in \arg \min_{\bar{x} \in \mathcal{X}} d_{\text{pointwise}}(j_{\infty} T_{\text{pde}}^{-1} \hat{z}(t), j_{\infty} T_{\text{pde}}^{-1} T_{\text{ode}}(\bar{x}))$$

and $\hat{v}(t) = T_{\text{pde}}^{-1}(\hat{z}(t) - T_{\text{ode}}(\hat{x}(t)))$,

we get $|\hat{x}(t) - x(t)| + \|\hat{v}(t) - v(t)\|_{L^2} \rightarrow 0$.

Open question: Abstract case?

A. Braun, L. Brivadis, J. Auriol, A Backstepping-KKL observer for a cascade of a nonlinear ODE with a heat equation, arXiv:2510.04941, 2025.

Left-inversion of T : a stronger topology

Idea: Actually, the convergence $\hat{z} - z \rightarrow 0$ is **stronger than just L^2** .

Since $v(t) \in \mathcal{A}$, the **smoothing effect** of the heat equation gives

$$\partial_{\xi}^{2k+1} \left(T_{\text{pde}}^{-1} (\hat{z}(t) - T_{\text{ode}}(x(t))) \right) (0) \rightarrow 0, \quad \forall k \in \mathbb{N}.$$

Denote $j_{\infty} w \in \mathbb{R}^{\mathbb{N}}$ the sequence of odd derivatives of w at $\xi = 0$, endowed with the **pointwise** topology, and

$$d(z_1, z_2) := \|z_1 - z_2\|_{L^2} + d_{\text{pointwise}}(j_{\infty} T_{\text{pde}}^{-1} z_1, j_{\infty} T_{\text{pde}}^{-1} z_2).$$

Theorem: If $\mathcal{X}$ is compact, then $T^{-1} : (T(\mathcal{X} \times \mathcal{A}), d) \rightarrow \mathbb{R}^n \times L^2(0, 1)$ is uniformly continuous and $d(\hat{z}(t), z(t)) \rightarrow 0$. Hence, with

$$\hat{x}(t) \in \arg \min_{\bar{x} \in \mathcal{X}} d_{\text{pointwise}}(j_{\infty} T_{\text{pde}}^{-1} \hat{z}(t), j_{\infty} T_{\text{pde}}^{-1} T_{\text{ode}}(\bar{x}))$$

and $\hat{v}(t) = T_{\text{pde}}^{-1}(\hat{z}(t) - T_{\text{ode}}(\hat{x}(t)))$,

we get $|\hat{x}(t) - x(t)| + \|\hat{v}(t) - v(t)\|_{L^2} \rightarrow 0$.

Open question: Abstract case?

A. Braun, L. Brivadis, J. Auriol, A Backstepping-KKL observer for a cascade of a nonlinear ODE with a heat equation, arXiv:2510.04941, 2025.

Numerical simulations

Example system

Harmonic oscillator with unknown frequency:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_3^2 x_1 \\ \dot{x}_3 = 0 \end{cases} \quad h(x) = x_1, \quad x_3 > 0 \text{ unknown frequency}$$

on $\mathcal{X} = \{x \in \mathbb{R}^3 \mid r \leq x_1^2 + x_2^2 \leq R, x_3^{\min} \leq x_3 \leq x_3^{\max}\}$, with $0 < r < R$.

Differential observability:

$x \mapsto H(x) = (x_1, x_2, -x_3^2 x_1, -x_3^2 x_2)$ is injective on $\mathcal{X}$.

Explicit KKL map: $T_{\text{ode}}(x)(\xi) = g_1(x_3, \xi) x_1 + g_2(x_3, \xi) x_2$, with

$$g_1 = -\operatorname{Re}(G), \quad g_2 = -\frac{1}{x_3} \operatorname{Im}(G), \quad G(x_3, \xi) = \frac{\cosh(\xi \sqrt{\gamma + ix_3})}{\cosh(\sqrt{\gamma + ix_3})}.$$

Example system

Harmonic oscillator with unknown frequency:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_3^2 x_1 \\ \dot{x}_3 = 0 \end{cases} \quad h(x) = x_1, \quad x_3 > 0 \text{ unknown frequency}$$

on $\mathcal{X} = \{x \in \mathbb{R}^3 \mid r \leq x_1^2 + x_2^2 \leq R, x_3^{\min} \leq x_3 \leq x_3^{\max}\}$, with $0 < r < R$.

Differential observability:

$x \mapsto H(x) = (x_1, x_2, -x_3^2 x_1, -x_3^2 x_2)$ is injective on $\mathcal{X}$.

Explicit KKL map: $T_{\text{ode}}(x)(\xi) = g_1(x_3, \xi) x_1 + g_2(x_3, \xi) x_2$, with

$$g_1 = -\operatorname{Re}(G), \quad g_2 = -\frac{1}{x_3} \operatorname{Im}(G), \quad G(x_3, \xi) = \frac{\cosh(\xi \sqrt{\gamma + ix_3})}{\cosh(\sqrt{\gamma + ix_3})}.$$

Example system

Harmonic oscillator with unknown frequency:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_3^2 x_1 \\ \dot{x}_3 = 0 \end{cases} \quad h(x) = x_1, \quad x_3 > 0 \text{ unknown frequency}$$

on $\mathcal{X} = \{x \in \mathbb{R}^3 \mid r \leq x_1^2 + x_2^2 \leq R, x_3^{\min} \leq x_3 \leq x_3^{\max}\}$, with $0 < r < R$.

Differential observability:

$x \mapsto H(x) = (x_1, x_2, -x_3^2 x_1, -x_3^2 x_2)$ is injective on $\mathcal{X}$.

Explicit KKL map: $T_{\text{ode}}(x)(\xi) = g_1(x_3, \xi) x_1 + g_2(x_3, \xi) x_2$, with

$$g_1 = -\operatorname{Re}(G), \quad g_2 = -\frac{1}{x_3} \operatorname{Im}(G), \quad G(x_3, \xi) = \frac{\cosh(\xi \sqrt{\gamma + ix_3})}{\cosh(\sqrt{\gamma + ix_3})}.$$

Numerical inversion of T

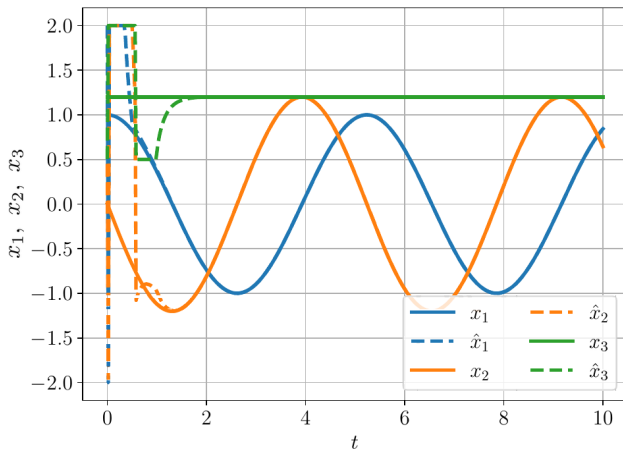
For a candidate x , set $v^x := T_{\text{pde}}^{-1}(\hat{z} - T_{\text{ode}}(x))$.

The correct x is characterized by $v^x \in \mathcal{A}$ and $v^x(0) = y$.

- Fit a polynomial $v^x(\xi) \simeq \sum_{j=0}^{2N+1} a_j(x) \xi^j$ near $\xi = 0$.
- Residual: $\mathcal{R}(x; \hat{z}, y) = (v^x(0) - y, a_1(x), a_3(x), \dots, a_{2N+1}(x))$ (odd coefficients should vanish).
- For fixed x_3 , $\mathcal{R}$ is affine in (x_1, x_2) : linear least squares.
- $\hat{x}_3(t) = \arg \min_{x_3 \in [x_3^{\min}, x_3^{\max}]} \min_{(x_1, x_2)} \|\mathcal{R}((x_1, x_2, x_3); \hat{z}(t), y(t))\|^2$.

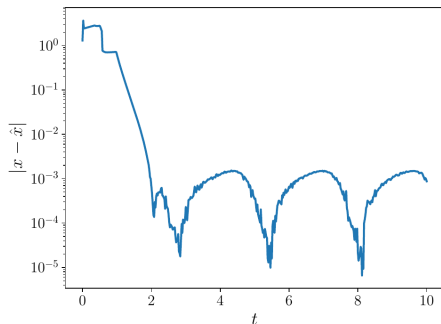
Then $\hat{v} = T_{\text{pde}}^{-1}(\hat{z} - T_{\text{ode}}(\hat{x}))$.

Simulation results: convergence

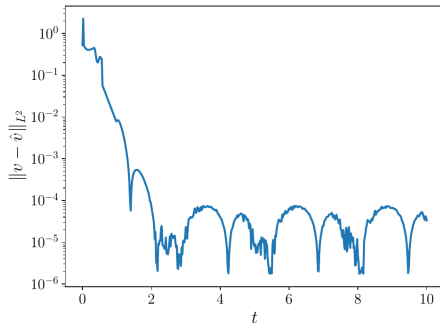


ODE state x and its estimate $\hat{x}$

Simulation results: errors



$$|x - \hat{x}|$$



$$\|v - \hat{v}\|_{L^2}$$

Influence of γ

Observer:

$$\begin{cases} \partial_t \hat{z} = \partial_\xi^2 \hat{z} - \gamma \hat{z} + p_1(\xi) y \\ \partial_\xi \hat{z}(t, 0) = p_{10} y, \quad \hat{z}(t, 1) = 0 \end{cases} \quad (\hat{x}, \hat{v}) = T^{-1}(\hat{z})$$

The gain γ influences:

- the speed of convergence of $\hat{z}$ towards z ;
- the invertibility of T .

In practice, $\gamma \approx 3$ seems to be a good compromise:

γ	t_{settle}	$ \hat{x}(t_{\text{final}}) - x(t_{\text{final}}) $	$\ \hat{v}(t_{\text{final}}) - v(t_{\text{final}})\ _{L^2}$
0.5	2.10	$2.61 \cdot 10^{-3}$	$2.03 \cdot 10^{-4}$
3	1.14	$8.73 \cdot 10^{-4}$	$3.33 \cdot 10^{-5}$
10	0.46	$1.09 \cdot 10^{-3}$	$3.86 \cdot 10^{-5}$

t_{settle} : first time after which $\|\hat{z} - T(x, v)\|_{L^2} < 10^{-3}$.

Influence of γ

Observer:

$$\begin{cases} \partial_t \hat{z} = \partial_\xi^2 \hat{z} - \gamma \hat{z} + p_1(\xi) y \\ \partial_\xi \hat{z}(t, 0) = p_{10} y, \quad \hat{z}(t, 1) = 0 \end{cases} \quad (\hat{x}, \hat{v}) = T^{-1}(\hat{z})$$

The gain γ influences:

- the speed of convergence of $\hat{z}$ towards z ;
- the invertibility of T .

In practice, $\gamma \approx 3$ seems to be a good compromise:

γ	t_{settle}	$ \hat{x}(t_{\text{final}}) - x(t_{\text{final}}) $	$\ \hat{v}(t_{\text{final}}) - v(t_{\text{final}})\ _{L^2}$
0.5	2.10	$2.61 \cdot 10^{-3}$	$2.03 \cdot 10^{-4}$
3	1.14	$8.73 \cdot 10^{-4}$	$3.33 \cdot 10^{-5}$
10	0.46	$1.09 \cdot 10^{-3}$	$3.86 \cdot 10^{-5}$

t_{settle} : first time after which $\|\hat{z} - T(x, v)\|_{L^2} < 10^{-3}$.

Conclusion

Conclusion and Perspectives

- We designed an observer for the system

$$\begin{cases} \dot{x} = f(x) \\ \partial_t v = \partial_\xi^2 v + \alpha v \\ \partial_\xi v(\cdot, 0) = 0 \\ v(\cdot, 1) = h(x) \\ y = v(\cdot, 0) \end{cases}$$

- The methodology extends to **cascades with linear PDEs**, with some remaining open questions.
- Further extensions:

$$dT(x)f(x) = AT(x) + Bh(x) \quad \text{with } x \text{ **infinite-dimensional**}$$

Conclusion and Perspectives

- We designed an observer for the system

$$\begin{cases} \dot{x} = f(x) \\ \partial_t v = \partial_\xi^2 v + \alpha v \\ \partial_\xi v(\cdot, 0) = 0 \\ v(\cdot, 1) = h(x) \\ y = v(\cdot, 0) \end{cases}$$

- The methodology extends to **cascades with linear PDEs**, with some remaining open questions.
- Further extensions:

$$dT(x)f(x) = AT(x) + Bh(x) \quad \text{with } x \text{ infinite-dimensional}$$

Conclusion and Perspectives

- We designed an observer for the system

$$\begin{cases} \dot{x} = f(x) \\ \partial_t v = \partial_\xi^2 v + \alpha v \\ \partial_\xi v(\cdot, 0) = 0 \\ v(\cdot, 1) = h(x) \\ y = v(\cdot, 0) \end{cases}$$

- The methodology extends to **cascades with linear PDEs**, with some remaining open questions.
- Further extensions:

$$dT(x)f(x) = AT(x) + Bh(x) \quad \text{with } x \text{ **infinite-dimensional**}$$

Thank you for your attention