

Exponential Output-to-State Stability of Abstract Linear Systems with Bounded Output Operators

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System Setup

$$\dot{x}(t) = Ax(t), \quad y(t) = Cx(t), \quad (A, C)$$

- A generates a C_0 -semigroup $T(t)$ on a Banach space X .
- $C \in \mathcal{L}(X, Y)$, with Y a Banach space.
- Mild solution: $\phi(t, x) = T(t)x$, $x \in X$, $t \geq 0$.

Key Definitions

- **Exponential Detectability:**
 $\exists L \in \mathcal{L}(Y, X)$ s.t. $A + LC$ generates an exponentially stable semigroup $T_{LC}(t)$.
- **eOSS Lyapunov function in dissipation form:**
 - (i) $\exists k_1, k_2, p > 0$ s.t. $k_1 \|x\|_X^p \leq V(x) \leq k_2 \|x\|_X^p$.
 - (ii) $\forall x \in X, t \mapsto V(\phi(t, x)), t \geq 0$, is continuous.
 - (iii) $\exists \alpha > 0, \sigma \in \mathcal{K}_\infty$ s.t. $\dot{V}(x) \leq -\alpha \|x\|_X^p + \sigma(\|Cx\|_Y)$.
- **eOSS Lyapunov function in implication form:**
 - (i) and (ii) hold, and (iii) is replaced with (iii')
 - $\exists \tilde{\alpha} > 0, \chi \in \mathcal{K}_\infty$ s.t. $\forall x \in X$,
 $\|x\|_X \geq \chi(\|Cx\|_Y) \implies \dot{V}(x) \leq -\tilde{\alpha} \|x\|_X^p$.

- **Exponential output-to-state stability (eOSS):**

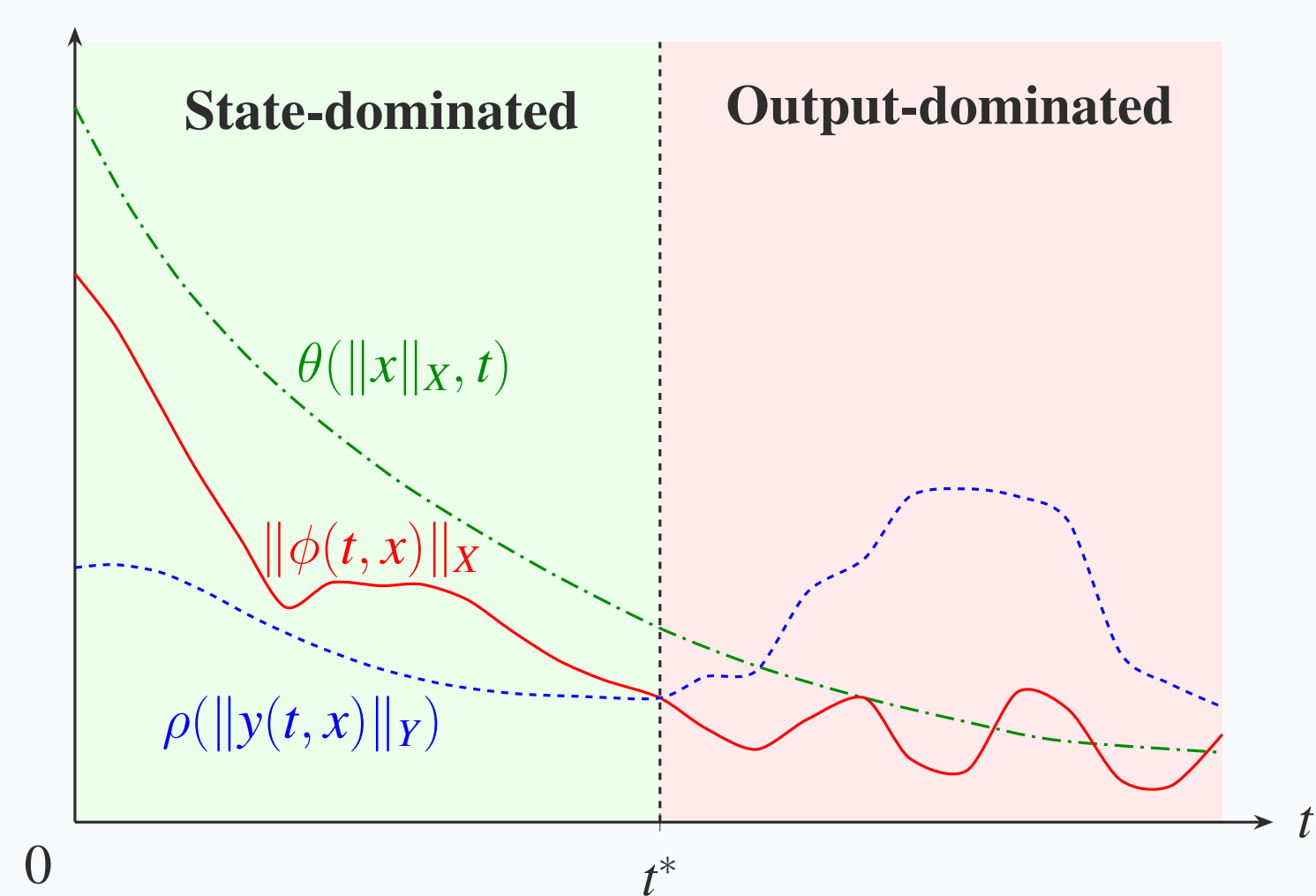
$$\exists M, \mu, k > 0 \text{ s.t. } \forall x \in X, t \geq 0, \\ \|\phi(t, x)\|_X \leq Me^{-\mu t} \|x\|_X + k \max_{s \in [0, t]} \|y(s, x)\|_Y.$$

- **Uniform Global Exponential Stability Modulo Output (UGESMO):**

$$\exists \rho, M, \mu > 0 \text{ s.t. } \forall x \in X, \tau \geq 0, \text{ if} \\ \|\phi(t, x)\|_X \geq \rho \|y(t, x)\|_Y, \quad \forall t \in [0, \tau],$$

then

$$\|\phi(t, x)\|_X \leq Me^{-\mu t} \|x\|_X, \quad \forall t \in [0, \tau].$$

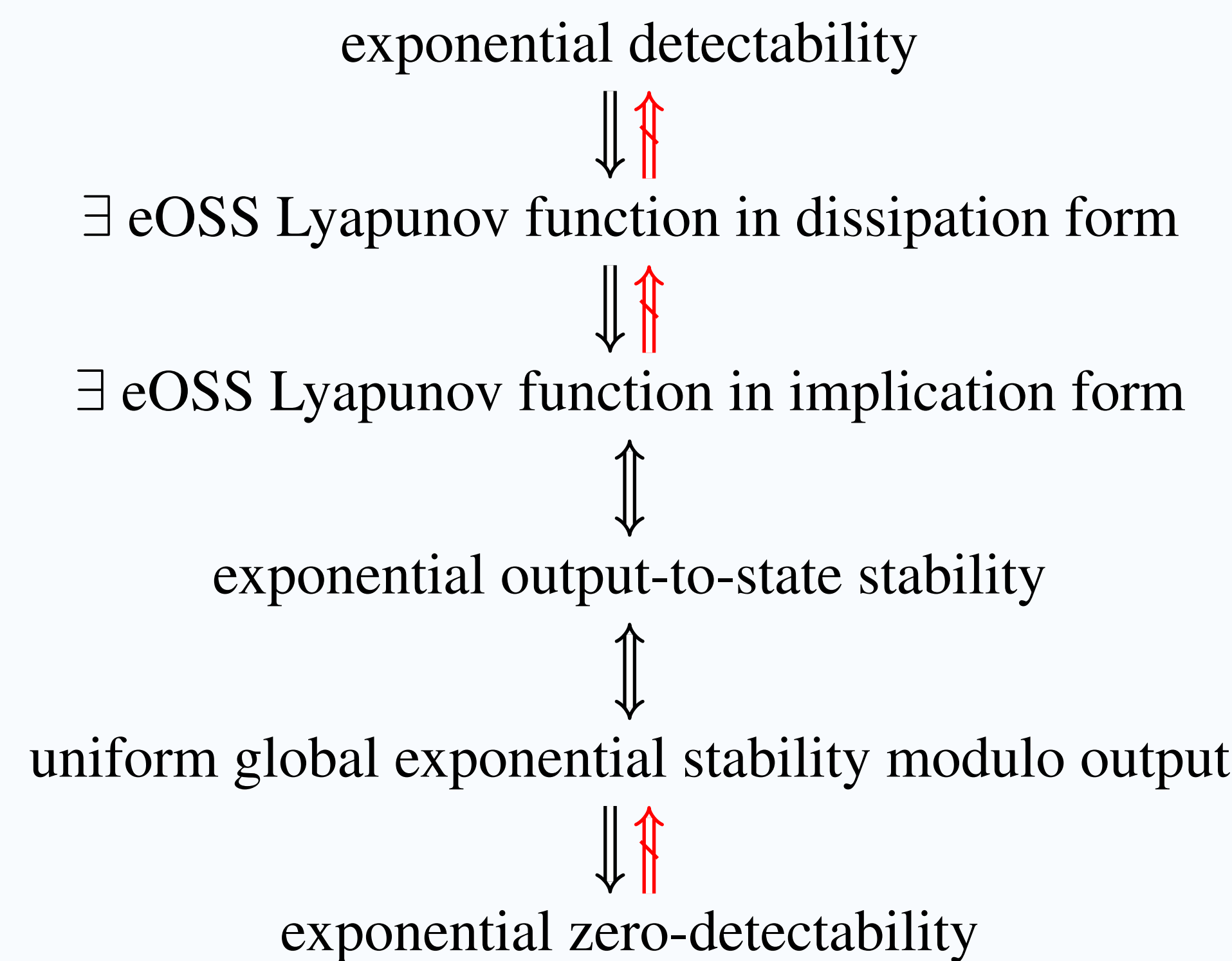


- **Exponential zero-detectability:**
 $\exists M, \omega > 0$ s.t. $\forall x \in X$,
 $y(\cdot, x) \equiv 0 \implies \|\phi(t, x)\|_X \leq Me^{-\omega t} \|x\|_X, \quad t \geq 0.$

Finite vs. Infinite Dimensions

- **Finite-dimensional nonlinear ODEs**
 $\text{OSS} \iff \exists \text{ OSS LF (diss.)} \xrightarrow{\text{regularity}} \text{OSS LF (impl.)}$
- **Finite-dimensional linear systems**
 $\text{OSS} \iff \text{detectability} \iff \text{zero-detectability}$
- **Infinite-dimensional linear systems**
The finite-dimensional exponential equivalences **fail** in general

Main Contributions

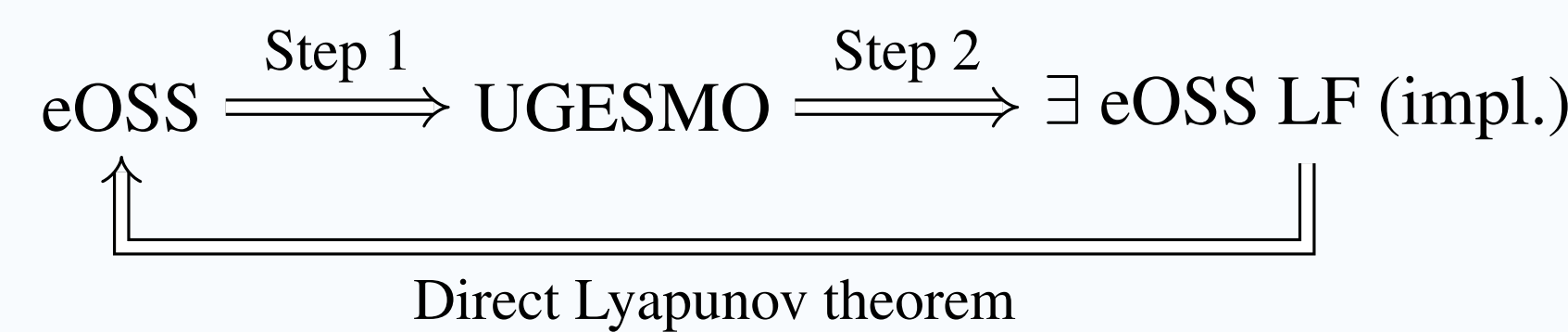


Lyapunov Criterion for eOSS

Theorem. For system (A, C) :

$$\text{eOSS} \iff \text{UGESMO} \iff \exists \text{ eOSS LF (impl.)}.$$

Proof strategy.



Key Step in the Converse Theorem: UGESMO $\implies \exists$ eOSS LF (impl.)

Let $\rho > 0$ be as in UGESMO and define

$$\mathcal{E}_\rho := \{x \in X : \|x\|_X > \rho \|Cx\|_Y\}, \quad \mathcal{D}_\rho := X \setminus \mathcal{E}_\rho.$$

For $x \in \mathcal{E}_\rho$, define the **first entrance time** into $\mathcal{D}_\rho$

$$\lambda(x) := \inf\{t \geq 0 : T(t)x \in \mathcal{D}_\rho\}$$

with $\lambda(x) = \infty$ if the trajectory never enters $\mathcal{D}_\rho$.

Define

$$W(x) := \int_0^{\lambda(x)} \|T(s)x\|_X ds.$$

Then

$$\dot{W}(x) = -\|x\|_X, \quad x \in \mathcal{E}_\rho.$$

For sufficiently large ω_1 , define the **equivalent norm**

$$N(x) := \max_{s \geq 0} e^{-\omega_1 s} \|T(s)x\|_X, \quad x \in X.$$

Choose a continuous **cutoff function**

$$\theta : [0, 1] \rightarrow [0, 1]$$

which vanishes near $\mathcal{D}_\rho$ and equals 1 sufficiently inside $\mathcal{E}_\rho$.

For sufficiently large $K > 0$, define

$$V(x) := \begin{cases} N(x) + K\theta\left(\frac{\|x\|_X}{\|x\|_X + \|Cx\|_Y}\right) W(x), & x \in \mathcal{E}_\rho, \\ N(x), & x \in \mathcal{D}_\rho. \end{cases}$$

Why this works: W provides decay, N ensures coercivity, and θ glues the two constructions continuously.

Boundary continuity: For $t_n \rightarrow t$, split according to whether $T(t_n)x \in \mathcal{D}_\rho$ or $\mathcal{E}_\rho$.

Detectability $\implies \exists$ eOSS LF (diss.)

Proposition. Let (A, C) be **exponentially detectable**. For sufficiently small $\gamma > 0$, define

$$V^\gamma(x) := \max_{s \geq 0} \|e^{\gamma s} T_{LC}(s)x\|_X.$$

Then V^γ is an **eOSS Lyapunov function in dissipation form** for system (A, C) .

Counterexamples

- **Exponential zero-detectability $\not\Rightarrow$ eOSS**

$X = L^2(0, 1)$, $Y = \mathbb{C}$, with

$$(Af)(s) = isf(s), \quad \|T(t)f\|_X = \|f\|_X, \quad Cf = \int_0^1 f(s) ds.$$

- **$\exists$ eOSS LF (impl.) $\not\Rightarrow \exists$ eOSS LF (diss.)**

$X = C_{2\pi}(\mathbb{R}, \mathbb{C})$, $Y = \mathbb{C}$, with the left translation group

$$(T(t)x)(s) = x(s+t), \quad Cx = x(0).$$

- **$\exists$ eOSS LF (diss.) $\not\Rightarrow$ exponential detectability**

$X = c_0$, $Y = \ell^\infty$, with

$$\dot{x}(t) = 0, \quad y(t) = Cx(t), \quad C : c_0 \hookrightarrow \ell^\infty.$$

Equivalence under Decomposition

Assumption^[1].

- (i) $X = X_u \oplus X_s$ with $\dim X_u < \infty$,
- (ii) X_u and X_s are closed and $T(t)$ -invariant,
- (iii) $T(t)|_{X_s}$ is exponentially stable.

Theorem. Under the above assumption, for system (A, C) ,

$$\text{exp. detectability} \iff \text{eOSS} \iff \text{exp. zero-detectability}.$$

Example: Parabolic PDE

Let $a > 0$ and $c, q \in \mathbb{R}$. Consider

$$x_t(z, t) = ax_{zz}(z, t) + cx(z, t),$$

$$y(t) = \int_0^1 x(z, t) dz,$$

$$x_z(0, t) = -qx(0, t), \quad x(1, t) = 0.$$

Proposition. Let $X = L^2(0, 1)$ and $Y = \mathbb{R}$. The system is **exponentially detectable and eOSS** for all $a > 0$, $c, q \in \mathbb{R}$.

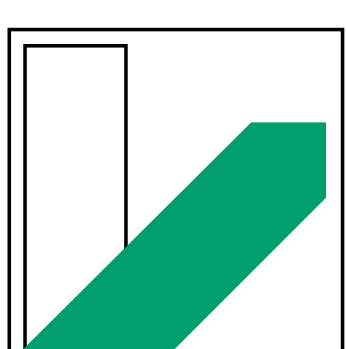
Outlook: Nonlinear Systems

For a broad class of nonlinear infinite-dimensional systems:

- $\text{OSS} \iff \text{UGASMO}$.
- $\text{OSS} \iff \exists \text{ OSS Lyapunov function (impl.)}$.
- Illustrated by a nonlinear reaction-diffusion example.

References

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- [1] R. Curtain and H. Zwart. *Introduction to Infinite-Dimensional Systems Theory: A State-Space Approach*, Springer, 2020.



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