

On the Hautus test for exact observability of normal semigroups

joint work with Birgit Jacob

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Setup

For background, see [5]. Throughout, let

- X, Y Hilbert spaces,
- T exponentially stable C_0 -semigroup on X with generator A ,
- $C : X_1 \rightarrow Y$ admissible observation operator, where $X_1 := \text{dom}(A)$ with $\|v\|_1 := \|Av\|$.

Exact observability

(A, C) *exactly observable* $\Leftrightarrow \exists m_o \in (0, \infty) : \forall v \in X_1 :$

$$m_o \|v\|^2 \leq \int_0^\infty \|CT(\tau)v\|^2 d\tau. \quad (\text{EO})$$

Hautus test

Finite dimensions [5]: $A \in \mathbb{C}^{n \times n}$, $C \in \mathbb{C}^{m \times n}$, $T(t) = e^{tA}$. *Finite-dimensional Hautus test*:

$$(A, C) \text{ exactly observable} \Leftrightarrow \forall z \in \mathbb{C}_- : \text{rank} \begin{pmatrix} zI - A \\ C \end{pmatrix} = n.$$

Infinite dimensions [4]: (A, C) satisfies *Hautus test* $\Leftrightarrow \exists m_h \in (0, \infty) \forall z \in \mathbb{C}_-, v \in X_1 :$

$$m_h \|v\|^2 \leq \frac{1}{|\text{Re } z|^2} \|(zI - A)v\|^2 + \frac{1}{|\text{Re } z|} \|Cv\|^2. \quad (\text{HT})$$

In finite dimensions both Hautus tests are equivalent.

Modified Russell–Weiss conjecture

We have: (A, C) exactly observable $\Rightarrow T$ similar to a contraction semigroup and (A, C) satisfies Hautus test. What about the converse?

Modified Russell–Weiss conjecture [3, 4]

If (A, C) satisfies the Hautus test and T is similar to a contraction semigroup, then (A, C) is exactly observable.

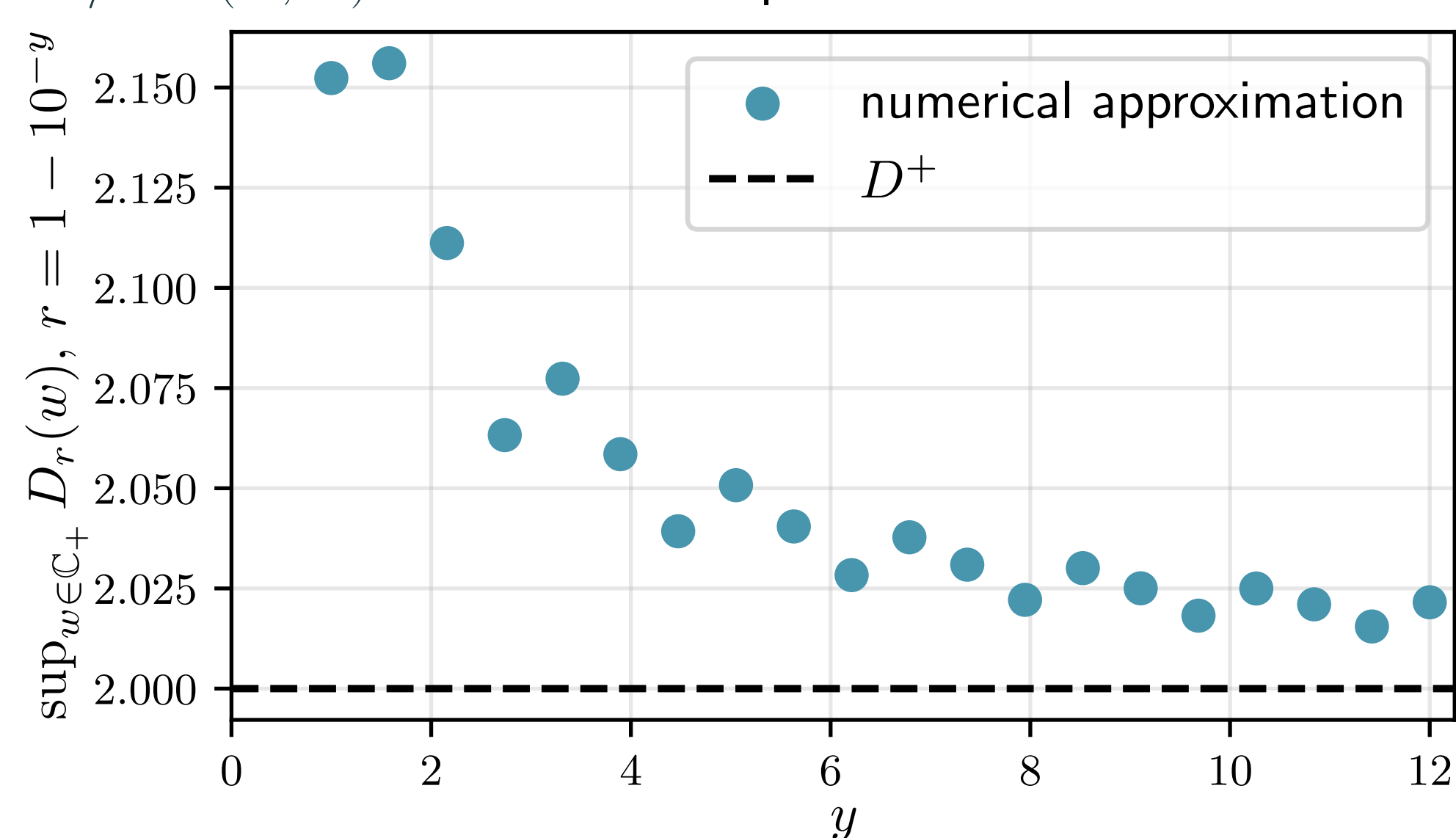
Counterexample to modified Russell–Weiss conjecture

Counterexample [1]

- $X := \ell^2(\mathbb{N})$, $Y := L^2([0, \infty))$,
- A the unique normal operator on X : $Ae_n = -\bar{\mu}_n e_n$, $\forall n \in \mathbb{N}$, where $(\mu_n)_{n \in \mathbb{N}}$ is an enumeration of $\{e^{k(1+ij\pi)} : j \in \mathbb{Z}, k \in \mathbb{N}\}$,
- C the unique bounded linear operator $X_1 \rightarrow Y$: $(Ce_n)(s) = 2 \text{Re } \mu_n e^{-\bar{\mu}_n s}$.
- A normal, $\sup_{z \in \sigma(A)} \text{Re } z \leq -1 \Rightarrow A$ generates exp. stable C_0 -semigr. of contractions,
- $(\mu_n)_{n \in \mathbb{N}}$ is *separated*, i.e. $\inf_{n \neq m} \rho(\mu_n, \mu_m) > 0$, where $\rho(z, w) := |(z - w)/(z + \bar{w})|$,
- $(\mu_n)_{n \in \mathbb{N}}$ *separated*, $\|Ce_n\|^2 = 2 \text{Re } \mu_n \Rightarrow (A, C)$ satisfies (HT) by [4],
- using Bergman space theory [2] and the separation of $(\mu_n)_{n \in \mathbb{N}}$ we show that C is admissible and that (A, C) is exactly observable $\Leftrightarrow D^+ < 1/2$ with

$$D^+ := \limsup_{r \uparrow 1} \sup_{w \in \mathbb{C}_+} D_r(w), \quad D_r(w) := \frac{1}{-\log(1-r)} \sum_{\substack{n \in \mathbb{N}, \\ 1/2 < w_n < r}} -\log(w_n), \quad w_n := \rho(w, \mu_n).$$

But $D^+ = 2 > 1/2$ so (A, C) is a counterexample to the modified Russell–Weiss conjecture!



Proof of Russell–Weiss conjecture for self-adjoint A

Theorem [1]

A self-adjoint and (A, C) satisfies the Hautus test $\Rightarrow (A, C)$ is exactly observable.

Proof sketch.

Let $d \in \mathbb{N}_+$, $\alpha \in \mathbb{C}^d \setminus \{0\}$, $\beta \in (0, \infty)^d$, $f : (0, \infty) \rightarrow \mathbb{C}$,

$$f(\tau) := \sum_{j=1}^d \alpha_j e^{-\beta_j \tau}. \quad (1)$$

Calculation shows $\exists M \in (0, \infty) : \forall v \in X_1 :$

$$\int_0^\infty \|Cf(-\tau A)v\|^2 d\tau \leq M \int_0^\infty \|CT(\tau)v\|^2 d\tau. \quad (2)$$

Let $w \in X_1$. Insertion of $z = -\tau^{-1}$ and $v = f(-\tau A)w$ in (HT) for $\tau \in (0, \infty)$, integration with respect to $\tau^{-1} d\tau$, the spectral theorem, and Tonelli's theorem show

$$c_1 m_h \|w\|^2 \leq c_2 \|w\|^2 + \int_0^\infty \|Cf(-\tau A)w\|^2 d\tau, \quad (3)$$

where

$$c_1 := \int_0^\infty \tau^{-1} |f(\tau)|^2 d\tau, \quad c_2 := \int_0^\infty \tau^{-1} (1 - \tau)^2 |f(\tau)|^2 d\tau. \quad (4)$$

We can choose d, α, β such that

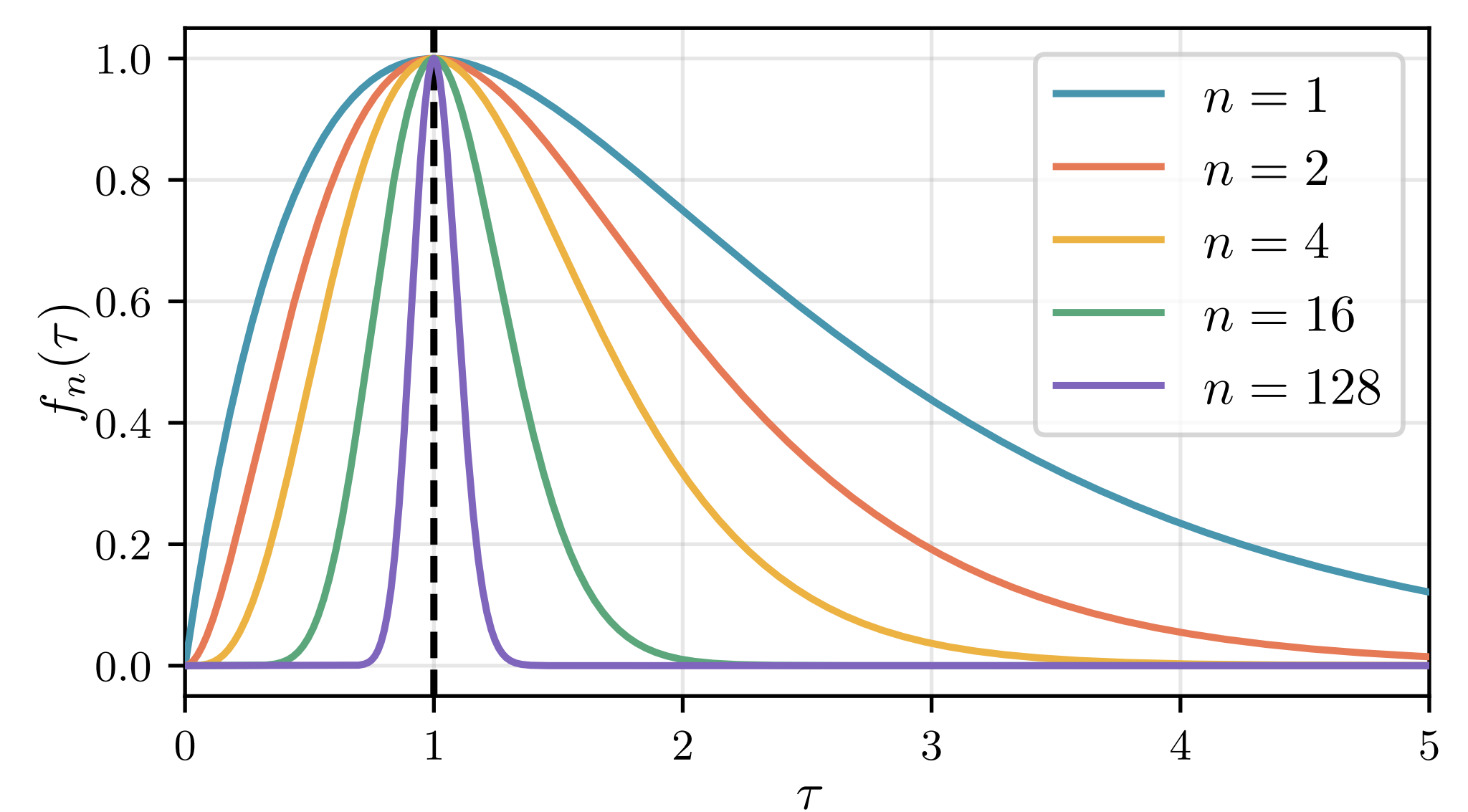
$$0 < c_1 < \infty, \quad c_2 < \infty, \quad 0 < c_1 m_h - c_2. \quad (5)$$

Then (2) and (3) combine to yield (EO), because $\forall v \in X_1 :$

$$\underbrace{(c_1 m_h - c_2)}_{>0} \|v\|^2 \leq \int_0^\infty \|Cf(-\tau A)v\|^2 d\tau \leq M \int_0^\infty \|CT(\tau)v\|^2 d\tau. \quad \square$$

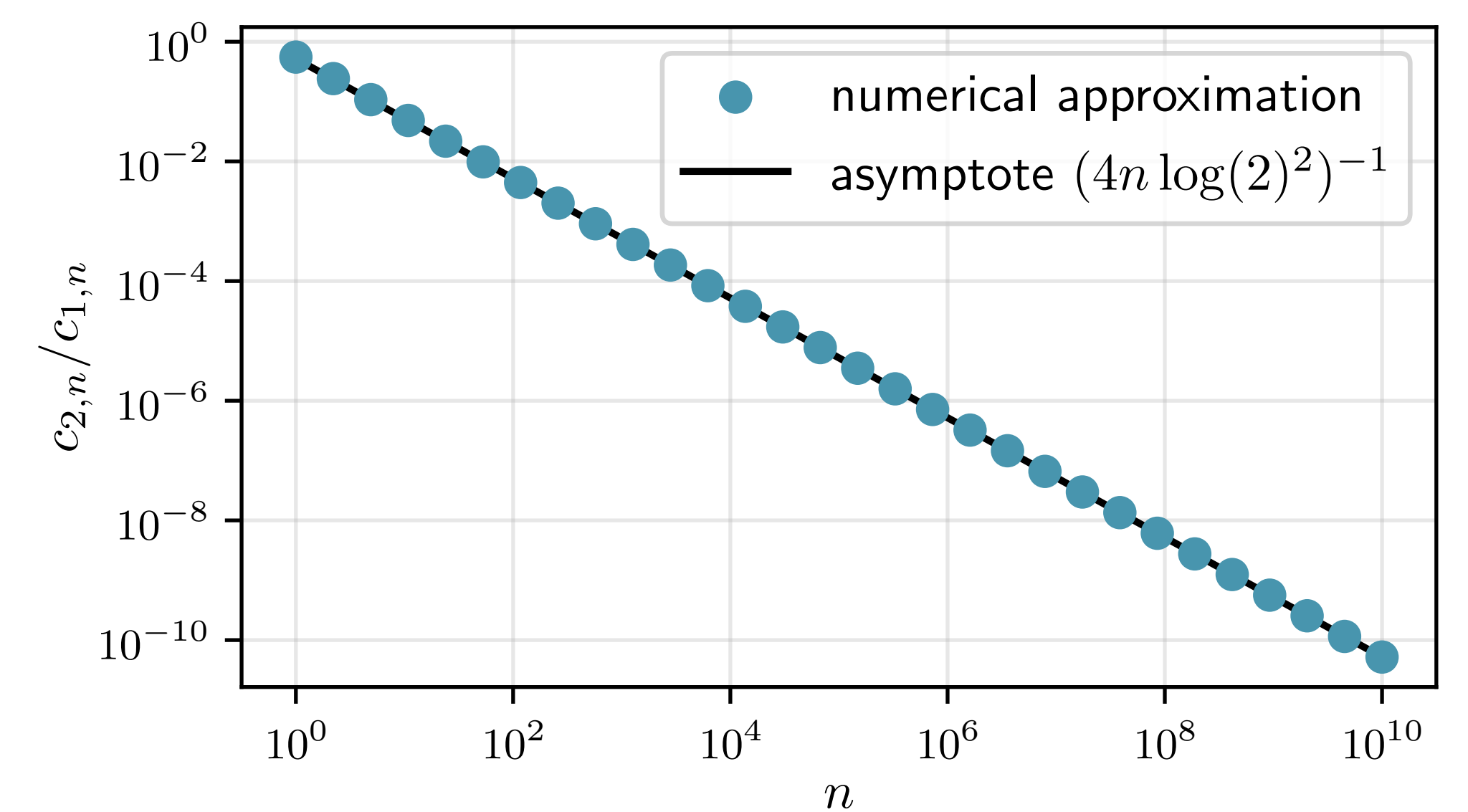
How do we choose d, α, β in the above? For $n \in \mathbb{N}_+$ define $f_n : (0, \infty) \rightarrow \mathbb{C}$ by

$$f_n(\tau) := (4 \cdot 2^{-\tau} (1 - 2^{-\tau}))^n.$$



For $n \in \mathbb{N}_+$ define $c_{1,n}, c_{2,n}$ by (4) with $f := f_n$.

- f_n is of the form (1),
- $\forall n \in \mathbb{N}_+ : 0 < c_{1,n} < \infty, c_{2,n} < \infty$,
- $\lim_{n \rightarrow \infty} c_{2,n} c_{1,n}^{-1} = 0$.



Because $\lim_{n \rightarrow \infty} c_{2,n} c_{1,n}^{-1} = 0$:

$$\exists N \in \mathbb{N}_+ : 0 < c_{1,N} m_h - c_{2,N}.$$

$\Rightarrow$ (5) is satisfied if we choose $f := f_N$ in the proof sketch.

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[2] H. Hedenmalm, B. Korenblum, and K. Zhu. *Theory of Bergman spaces*. Vol. 199. Grad. Texts Math. Springer, 2000.

[3] B. Jacob and H. Zwart. "A Hautus test for infinite-dimensional systems". In: *Unsolved problems in mathematical systems and control theory*. Ed. by V. D. Blondel and A. Megretski. Princeton University Press, 2004, pp. 251–255.

[4] D. L. Russell and G. Weiss. "A general necessary condition for exact observability". *SIAM Journal on Control and Optimization* 32.1 (1994), pp. 1–23.

[5] M. Tucsnak and G. Weiss. *Observation and control for operator semigroups*. Birkhäuser Adv. Texts, Basler Lehrbüch. Birkhäuser, 2009.



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