Constructive methods for robust control of PDE systems

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Outline

1. Intro: delay-robust control of PDEs - spatial & modal decomposition
2. Finite-dimensional output-feedback for Parabolic-Elliptic systems
3. Stabilization of Non-Standard Singularly Perturbed PDE systems
4. Extensions: 2D parabolic, L_2 separation,...
5. Conclusions

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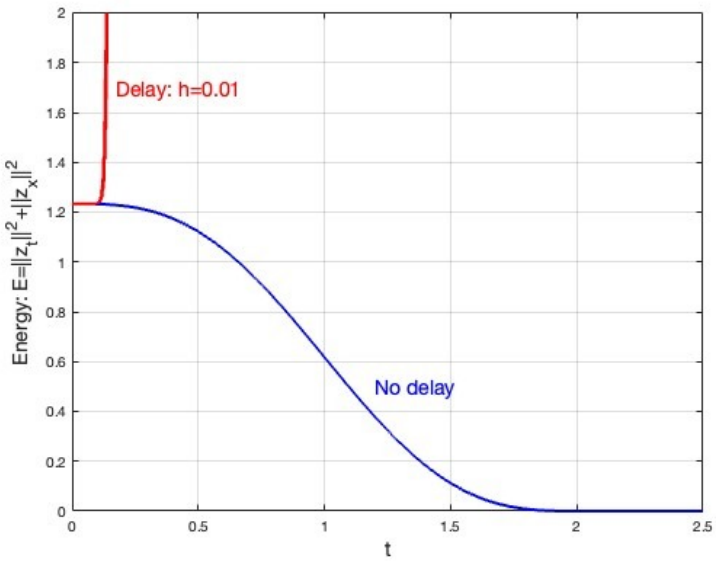
Effects of delay on stability of PDEs

Arbitrarily **small delays** may **destabilize** the PDE system [Datko, SICON'88]

- wave eq. with delay h:

$$z_{tt}(x,t) = z_{xx}(x,t), \quad x \in (0,1),$$
$$z(0,t) = 0, \quad z_x(1,t) = -z_t(1,t-h)$$

- For $h = 0$ all solutions are zero for $t \geq 2$!
- For arbitrary small $h > 0$ unbounded solutions



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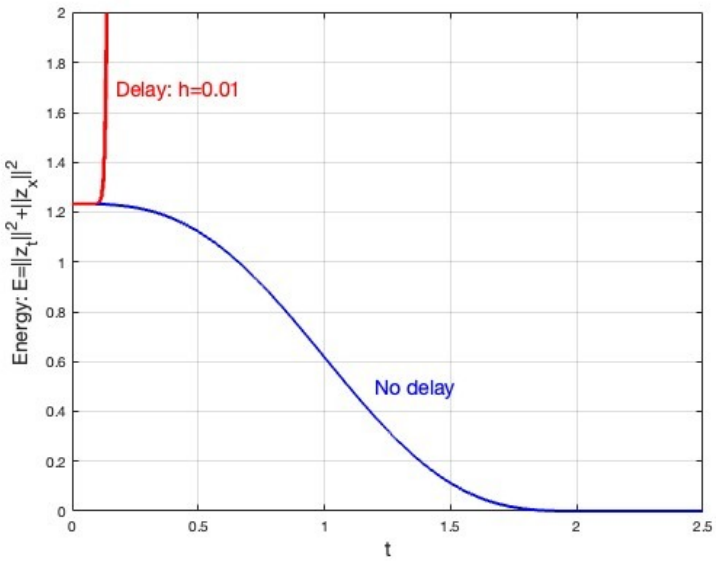


2 design methods for PDEs are robust to input/output delays:
spatial & modal decomposition

Spatial decomposition [Fridman and Blighovsky (Aut'12)]

Consider diffusion equation

$$z_t(x,t) = z_{xx}(x,t) + \phi\,(z,x,t)\,z(x,t) + \sum_{j=1}^{N} b_j(x)u_j(t),$$

$$z_x(0,t) = z_x(l,t) = 0$$

with $z\colon [0,l]\times[0,\infty)\to\mathbb{R}$ and $\quad |\phi(z,x,t)|\le q.$

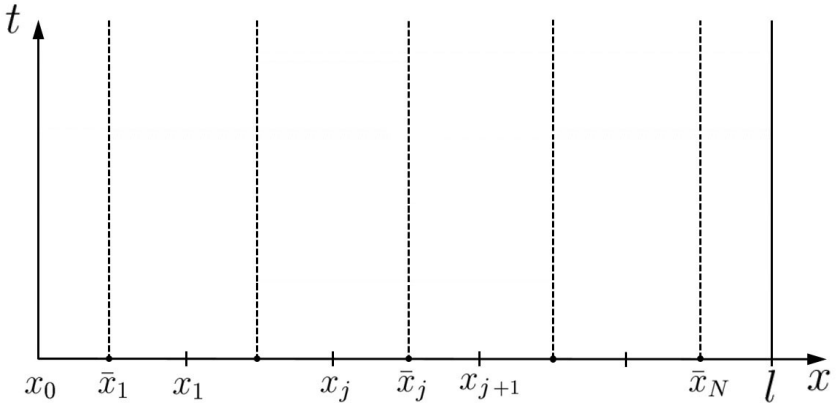
Divide $[0,l]$ into N subdomains $\Omega_j = [x_{j-1}, x_j]$:

$$0 = x_0 < x_1 < \ldots < x_N = l, \quad x_j - x_{j-1} \le \Delta.$$

Point measurements:

$$y_j(t) = z(\bar{x}_j, t), \quad \bar{x}_j = \frac{x_{j-1}+x_j}{2}.$$

Shape functions: $b_j(x) = \chi_j(x)$.



Static output-feedback: $u_j(t) = -ky_j(t) = -kz(\bar{x}_j, t)$

Point measurements [Fridman & Blighovsky, Aut12]

Static output-feedback: $u_j(t) = -ky_j(t) = -kz(\bar{x}_j, t)$.

$$\sum_{j=1}^{N} b_j(x)u_j(t) = -kz(x,t) + k\sum_{j=1}^{N} b_j(x)\underbrace{[z(x,t) - z(\bar{x}_j,t)]}_{\zeta(x,t)}$$

Choosing $V = \int_0^1 z^2 dx = \|z\|^2$,

$$\dot{V} \leq -2\int z_x^2 + 2\int [(\sigma - k)z^2 + kz \cdot \sum_{j=1}^{N} b_j\zeta]$$

To compensate $\zeta(x,t)$ we use the Wirtinger inequality:

$$-2\int z_x^2 = -2\sum_{j=1}^{N}\int_{x_{j-1}}^{x_j} z_x^2 \leq -2\frac{\pi^2}{\Delta^2}\sum_{j=1}^{N}\int_{x_{j-1}}^{x_j}\underbrace{[z(x,t) - z(\bar{x}_j,t)]^2}_{\zeta^2(x,t)}$$

Point measurements

Stability condition:

$$\begin{bmatrix} \sigma - k & \frac{k}{2} \\ \frac{k}{2} & -\frac{\pi^2}{\Delta^2} \end{bmatrix} < 0 \iff \sigma - k + \frac{k^2\Delta^2}{4\pi^2} < 0.$$

Then $k = \frac{2\pi^2}{\Delta^2}$ stabilizes if $\sigma < \frac{\pi^2}{\Delta^2}$

- If f is **locally sector bounded** and at least 1 b.c. is Dirichlet we find domain of attraction by using $V_1(t) = \|z_x(\cdot, t)\|^2$:
  1. E.g. for $f = z^2$ assume that $V_1(t) < \sigma^2 \ \forall t \geq 0$. Then by Sobolev's inequality

$$\sup_{x\in[0,1]} z^2(x,t) \leq V_1(t) < \sigma$$

  2. From the conditions for $\dot{V}_1 \leq 0$ with $|f| \leq \sigma|z|^2$ we find that $V_1(t) \leq V_1(0) < \sigma^2$ if $\|z_x(\cdot, 0)\|^2 < \sigma$

     The resulting bound on domain of attraction is conservative!

Sampled-data control of heat eq: point measurements

[Fridman and Blighovsky (Aut12)]:

$$u_j(t) = -Kz(\bar{x}_j, t_k),\ t \in [t_k, t_{k+1}) \Rightarrow$$

The closed-loop system has a form

$$z_t = z_{xx} + \phi z - Kz(x, t_k) + K \sum_{j=1}^{N} b_j \underbrace{\int_{\bar{x}_j}^{x} z_\xi(\xi, t_k) d\xi}_{\zeta(x, t_k)}$$

For sampled-data control we need H^1-stability $\Rightarrow$ Lyapunov functional:

$$\begin{aligned} V(t) \quad &= p_1 \int_0^l z^2(x,t)dx + p_3 \int_0^l z_x^2(x,t)dx \\ &+ r(t_{k+1} - t) \int_0^l \int_{t_k}^t e^{2\delta_0(s-t)} z_s^2(x,s) ds dx, \\ &t \in [t_k, t_{k+1}),\ p_1, p_3, r > 0. \end{aligned}$$

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- To compensate $\zeta(x, t_k)$ we employ Halanay and Wirtinger

Halanay's inequality

Lemma

Let $0 < \delta_1 < \delta_0$ *&* $V : [-h, \infty) \to [0, \infty)$ *be abs. continuous s.t.*

$$\dot{V}(t) \leq -2\delta_0 V(t) + 2\delta_1 \sup_{-h \leq \theta \leq 0} V(t+\theta),\ t \geq 0.$$

Then

$$V(t) \leq e^{-2\delta t} \sup_{-h \leq \theta \leq 0} V(\theta),\ t \geq 0,$$

where δ *is a unique solution of* $\delta = \delta_0 - \delta_1 e^{2\delta h}$.

By employing Halanay, we have

$$-2\delta_1 \sup_{-h \leq \theta \leq 0} V(t+\theta) \leq \quad -2\delta_1 p_3 \int z_x^2(x, t_k) \overbrace{\leq}^{Wirtinger}$$
$$-2\delta_1 p_3 \tfrac{\pi^2}{\Delta^2} \sum_{j=1}^{N} \int_{x_{j-1}}^{x_j} \underbrace{[z(x, t_k) - z(\bar{x}_j, t_k)]^2}_{\zeta^2(x, t_k)}$$

Spatial decomposition - extensions

- Averaged measurements, ND semilinear heat, H_∞ control - [N. Bar Am & EF, Aut '14]
- 1D Pointlike measurements [Jun-Wei Wang & Wu, JFI14]
- 1D Event-triggered, 2D under pointlike measurements - [A. Selivanov & EF, Aut '16,18]
- Regional stabilization for KSE (1D, 2D), KdVB - [W. Kang & EF, Aut '18,19; TAC '22]
- Event-triggered control of Korteweg–de Vries equation [W. Kang & Baudouin & EF, Aut'21]
- Damped wave, beam - [M. Terushkin & EF, Aut '19; SCL '20]
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Drawback: many actuators covering (almost) all domain & many sensors.

Challenges:

- Few actuators & sensors $\rightarrow$ *moving actuator via switching* [W. Kang & EF& Liu, Aut '23]

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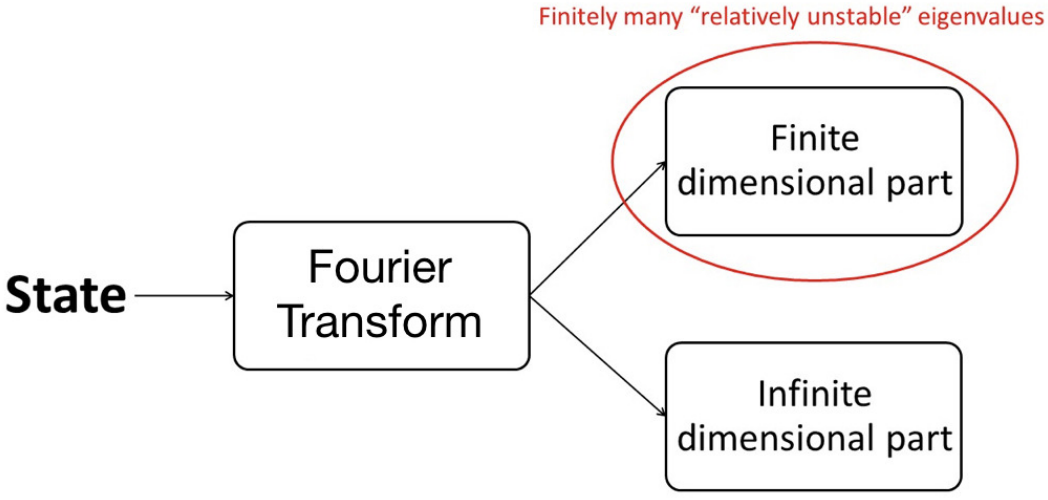
[Karafyllis & Krstic, Aut'18] introduced sampled-data boundary control for heat eq via modal decomposition - state-feedback

Our objective - finite-dim output-feedback via modal decomposition

Modal decomposition



- Popular in 80s - [Curtain, TAC '82], [Sakawa, SICON'83], [Balas, JMAA '88].
- Popular again because of
  - *robustness to sampling/delay*:
    state-feedback [Karafyllis & Krstic, Aut'18],
    finite-dimensional observer [Selivanov & EF, TAC'19]
  - easy for *input delay compensation*:
    state-feedback [Prieur & Trelat, TAC'18; Lhachemi et al, Aut'19]

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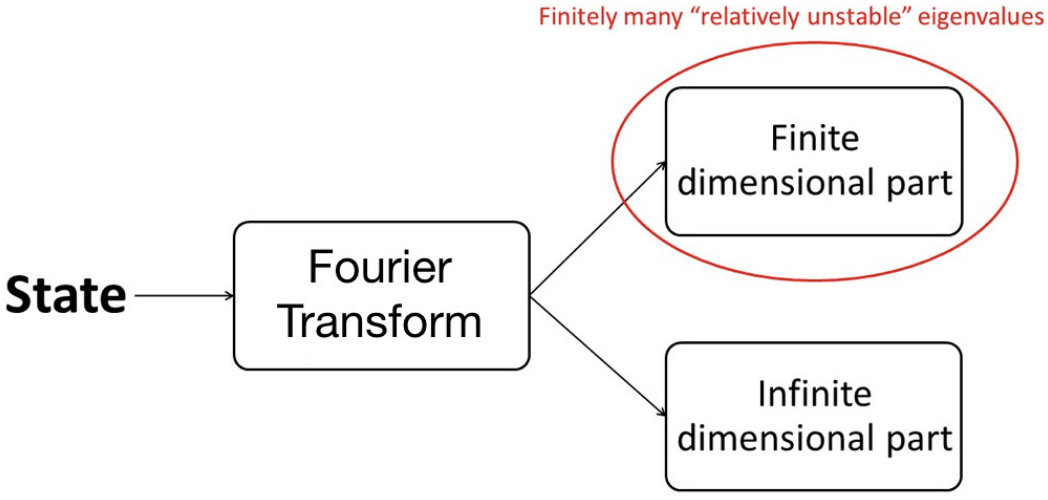

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  - easy for *input delay compensation*:
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- Delayed boundary observer-based control
  [Katz & Fridman & Selivanov, TAC'21] - PDE observer (separation).

Constructive LMI approach to finite-dimensional output-feedback

Rami Katz (Tel Aviv University)

A LMI-based method was introduced for finite-dimensional observer-based control of parabolic PDEs via **modal decomposition** [Katz & EF, Aut'20]

- $\rightarrow$ Observer dimension, ISS, L^2-gain, delays & domains of attraction are found from LMIs.
- $\rightarrow$ LMIs are proved to be asymptotically feasible.
- $\rightarrow$ Feasibility for N implies feasibility for $N+1$.

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Control of parabolic-elliptic systems [Wang & EF, ACC'25/L-CSS'24]

Pengfei Wang (post-doc, Tel Aviv University; As. Prof. Nanjing Uni)

Applications:

- Models of lithium-ion batteries [Consiglieri, Bol. Soc. Mat. Mex.'20]
- Biological transport networks [Li, Results in Mathematics'20]
- Chemotaxis phenomena [Guo & Zhang, SCL'14]
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Existing results:

- Backstepping [Alalabi & Morris, Aut'25] - State-feedback & output feedback – PDE observer
- Modal decomposition [Parada, Master Thesis'20, Parada, Cerpa & Morris, preprint'24] – State-feedback – Assumptions on simple eigenvalues

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Objective:

- Finite-dimensional output-feedback control for parabolic-elliptic systems.
- **Challenge** - multiple eigenvalues.

Parabolic-elliptic systems with boundary input

$$z_t(x,t) = z_{xx}(x,t) - \rho z(x,t) + \alpha v(x,t)$$
$$0 = v_{xx}(x,t) - \gamma v(x,t) + \beta z(x,t)$$
$$z(x,0) = z_0(x), \ x \in (0,1)$$

where $\alpha\beta \neq 0, \quad \gamma \neq -(n-0.5)^2\pi^2$

Boundary

$$z(0,t) = \mathbf{u}_1(t), \quad z_x(1,t) = \mathbf{u}_2(t)$$
$$v(0,t) = v_x(1,t) = 0$$

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Measurement:

$$y(t) = \begin{bmatrix} z(x_1^*,t) & z(x_2^*,t) \end{bmatrix}^T, \quad x_1^* \neq x_2^* \in (0,1]$$

- To manage with multiple eigenvalues we consider 2 inputs and 2 outputs

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- To manage with multiple eigenvalues we consider 2 inputs and 2 outputs
- For $g \in L^2(0,1)$, $v = F_\gamma(g) := (\gamma I - \partial_{xx})^{-1} g$ is the solution to

$$-v_{xx} + \gamma v = g \quad , \quad v(0) = v'(1) = 0$$

- F_γ is linear, continuous, and self-adjoint [Brezis, Springer'11].

$$\begin{array}{l} z_t(x,t) = z_{xx}(x,t) - \rho z(x,t) + \alpha v(x,t) \\ 0 = v_{xx}(x,t) - \gamma v(x,t) + \beta z(x,t) \\ z(0,t) = \mathbf{u}_1(t),\ z_x(1,t) = \mathbf{u}_2(t) \\ v(0,t) = v_x(1,t) = 0 \end{array} \quad \overset{v=F_\gamma(\beta z)}{\Longleftrightarrow} \quad \begin{array}{l} z_t = z_{xx} - \rho z + \alpha\beta F_\gamma(z) \\ z(0,t) = \mathbf{u}_1(t), \quad z_x(1,t) = \mathbf{u}_2(t) \end{array}$$

Lifting transformation

Lifting transformation ([Karafyllis, IJC'21] for heat equtions)

$$w(x,t) = z(x,t) - \psi(x)\mathbf{u}(t), \quad \mathbf{u}(t) = \begin{bmatrix} \mathbf{u}_1(t) \\ \mathbf{u}_2(t) \end{bmatrix}$$
$$\psi(x) = [\psi_1(x), \psi_2(x)], \quad \psi_1(x) = \cos(\pi x) \; \psi_2(x) = -\frac{1}{\pi}\sin(\pi x)$$

$$\psi_1''(x) = -\mu\psi_1(x), \quad \psi_1(0) = 1, \quad \psi_1'(1) = 0 \; (\mu = \pi^2)$$
$$\psi_2''(x) = -\mu\psi_2(x), \quad \psi_2(0) = 0 \quad \psi_2'(1) = 1$$

$$\Downarrow$$

$$w_t = w_{xx} - \rho w + \alpha\beta F_\gamma(w) - \psi[\Xi_0\mathbf{u}(t) + \dot{\mathbf{u}}(t)] + \alpha\beta[F_\gamma(\psi_1), F_\gamma(\psi_2)]\mathbf{u}(t),$$
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where

$$\Xi_0 = \begin{bmatrix} \mu+\rho & 0 \\ 0 & \mu+\rho \end{bmatrix}$$

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where

$$\Xi_0 = \begin{bmatrix} \mu+\rho & 0 \\ 0 & \mu+\rho \end{bmatrix}$$

$$\Downarrow$$

$$\begin{cases} \dot{\mathbf{u}}(t) = -\Xi_0\mathbf{u}(t) + \mathbf{v}(t), \; t > 0, \quad \mathbf{u}(0) = 0 \\ w_t = w_{xx} - \rho w + \alpha\beta F_\gamma(w) - \psi\mathbf{v}(t) + \alpha\beta[F_\gamma(\psi_1), \; F_\gamma(\psi_2)]\mathbf{u}(t) \\ w(0,t) = 0, \; w_x(1,t) = 0 \end{cases}$$

Modal decomposition

Operator

$$\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset L^2(0,1) \to L^2(0,1)$$

$$\mathcal{A}w = \partial_{xx} w - \rho w + \alpha\beta F_\gamma(w)$$
$$\mathcal{D}(\mathcal{A}) = \{w \in H^2(0,1), w(0) = w'(1) = 0\}$$

generates an analytic C_0-semigroup on $L_2(0,1)$ [Alalabi & Morris, Aut'25]

Its eigenvalues and eigenfunctions:

$$\bar{\lambda}_n = -\rho - \lambda_n + \frac{\alpha\beta}{\lambda_n + \gamma}, \quad \lambda_n = (n - 0.5)^2\pi^2$$
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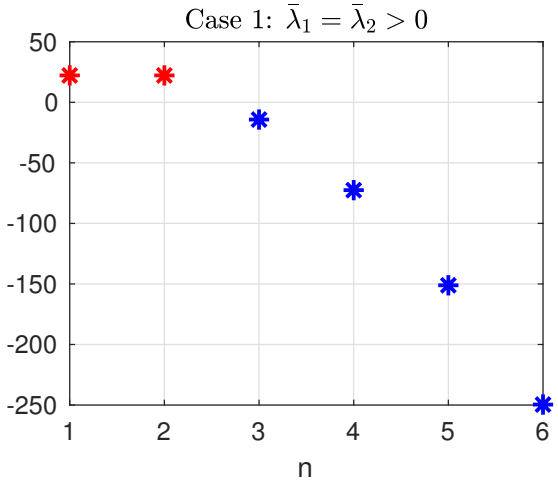
- $\{\phi_n\}_{n=1}^{\infty}$ is a basis of L^2:

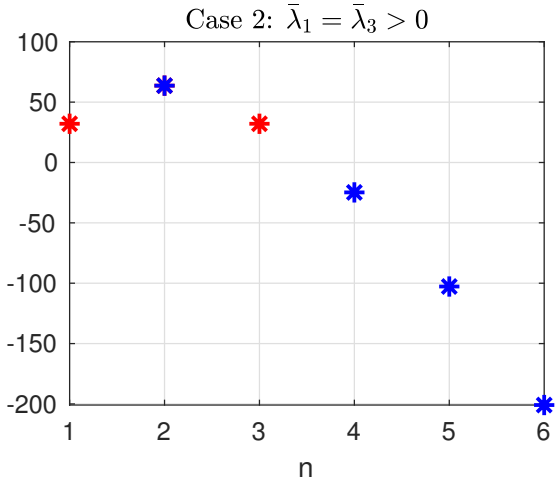
$$w(x,t) \overset{L^2}{=} \sum_{n=1}^{\infty} w_n(t)\phi_n(x), \quad w_n(t) = \langle w(\cdot,t), \phi_n \rangle \Rightarrow$$

$$\dot{w}_n(t) = \bar{\lambda}_n w_n(t) - \frac{\alpha\beta}{\lambda_n + \gamma} b_n \mathbf{u}(t) + b_n \mathbf{v}(t), \ n \in \mathbb{N} \qquad b_n = \left[\frac{\phi_n'(0)}{\mu - \lambda_n}, \frac{\phi_n(1)}{\mu - \lambda_n}\right]$$

Multiple eigenvalues for $\gamma = \frac{\pi^2}{4}$

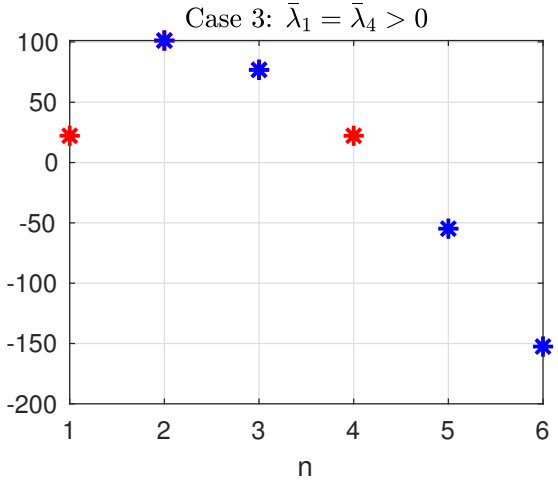
Note: the maximal multiplicity=2.

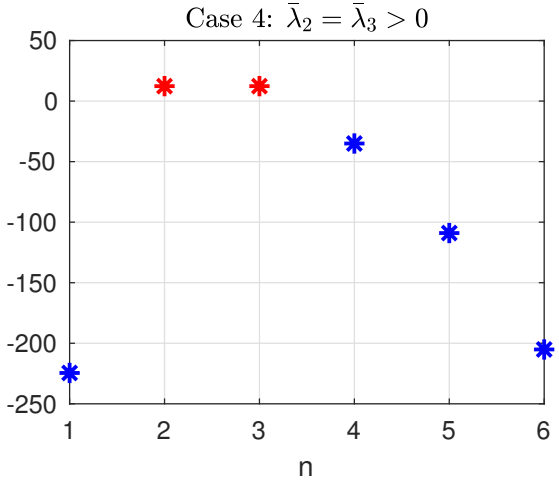




Case 1: $\alpha = -\frac{\pi^2}{2}$, $\beta = \frac{5\pi^2}{2}$, $\rho = -5\pi^2$

Case 2: $\alpha = -\frac{\pi^2}{2}$, $\beta = \frac{13\pi^2}{2}$, $\rho = -10\pi^2$





Case 3: $\alpha = -\frac{\pi^2}{2}$, $\beta = \frac{13\pi^2}{2}$, $\rho = -15\pi^2$

Case 4: $\alpha = -\frac{5\pi^2}{2}$, $\beta = \frac{5\pi^2}{2}$, $\rho = -10\pi^2$

Controllability in the presence of multiple eigenvalues

– For multiple eigenvalues: $\bar{\lambda}_{n_1} = \bar{\lambda}_{n_2}$.

- Controllability of $(\tilde{A}_0, \tilde{B}_0) \iff \operatorname{rank}\begin{bmatrix} \phi'_{n_1}(0) & \phi_{n_1}(1) \\ \phi'_{n_2}(0) & \phi_{n_2}(1) \end{bmatrix} = 2$ ✔
- Neumann $z_x(0,t) = \mathbf{u}_1(t), z_x(1,t) = \mathbf{u}_2(t)$
  controllablility $\iff \operatorname{rank}\begin{bmatrix} \phi_{n_1}(0) & \phi_{n_1}(1) \\ \phi_{n_2}(0) & \phi_{n_2}(1) \end{bmatrix} = 2$ ✗
- Dirichlet $z(0,t) = \mathbf{u}_1(t), z(1,t) = \mathbf{u}_2(t)$
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- Following 2D case [**PW** & Fridman, Aut'24, Djebour, Ramdani & Valein, JOTA'24], $z(0,t) = 0,\ z_x(1,t) = \mathbf{u}_1(t) + \mathbf{u}_2(t)$
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– For simple eigenvalues, single input and output are sufficient

– Additional input and output

- guarantee controllability and observability
- reduce the observer dimension

Controller & Observer

- Let $\delta > 0$ be a desired decay rate.

$$\bar{\lambda}_n = -\rho - \lambda_n + \frac{\alpha\beta}{\lambda_n+\gamma}, \quad \lambda_n = (n-0.5)^2\pi^2$$
$$\text{Choose } N_0: \ \bar{\lambda}_{N_0+1} + \delta < 0$$

N_0: the number of modes for the controller
$N \geq N_0$: the dimension of the observer (with $\bar{\lambda}_N < \bar{\lambda}_{N+1} < ...$)

- Finite-dimensional observer [Selivanov & EF, TAC'18]:
$\hat{w}(x,t) = \sum_{n=1}^{N} \hat{w}_n(t)\phi_n(x)$

$$\dot{\hat{w}}_n(t) = \bar{\lambda}_n \hat{w}_n(t) - \frac{\alpha\beta}{\lambda_n+\gamma} b_n \mathbf{u}(t) + b_n \mathbf{v}(t) - l_n[\hat{y}(t) - y(t)], \ t > 0, \qquad \hat{w}_n(0) = 0,$$

$$y(t) = \begin{bmatrix} w(x_1^*,t) + \psi(x_1^*)\mathbf{u}(t) \\ w(x_2^*,t) + \psi(x_2^*)\mathbf{u}(t) \end{bmatrix}, \qquad \hat{y}(t) = \begin{bmatrix} \hat{w}(x_1^*,t) + \psi(x_1^*)\mathbf{u}(t) \\ \hat{w}(x_2^*,t) + \psi(x_2^*)\mathbf{u}(t) \end{bmatrix}$$

where $l_n \in \mathbb{R}^{1\times 2}$, $n = 1, \ldots, N$, are observer gains to be determined.

- Finite-dimensional controller:

$$\mathbf{v}(t) = -K_0 \hat{w}^{N_0}(t), \qquad \hat{w}^{N_0}(t) = \text{col}\{\mathbf{u}(t), \hat{w}_1(t), \ldots, \hat{w}_{N_0}(t)\}$$

where controller gain K_0 is obtained from resulting LMIs.

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  where controller gain K_0 is obtained from resulting LMIs.

Given $z_0 \in H^1(0,1)$, w-system with the control input v has a unique classical solution $w \in C([0,\infty), L^2(0,1)) \cap C^1([0,\infty), L^2(0,1))$, and $w(\cdot,t) \in \mathcal{D}(\mathcal{A})$ for all $t > 0$

Controller & Observer

$$c_n = [\phi_n(x_1^*), \phi_n(x_2^*)]^{\mathrm{T}}, \quad C_0 = [c_1, \ldots, c_{N_0}] \in \mathbb{R}^{2\times N_0}, \quad A_0 = \mathrm{diag}\{\bar{\lambda}_n\}_{n=1}^{N_0}$$
$$\tilde{A}_0 = \begin{bmatrix} -\Xi_0 & 0 \\ -\Lambda_0 & A_0 \end{bmatrix}, \ \Lambda_0 = [\tfrac{\alpha\beta}{\lambda_1+\gamma} b_1^{\mathrm{T}}, \ldots, \tfrac{\alpha\beta}{\lambda_{N_0}+\gamma} b_{N_0}^{\mathrm{T}}]^{\mathrm{T}}, \quad \Xi_0 = \begin{bmatrix} \mu+\rho & 0 \\ 0 & \mu+\rho \end{bmatrix}$$
$$\tilde{B}_0 = \begin{bmatrix} I_2 \\ B_0 \end{bmatrix}, \quad B_0 = [b_1^{\mathrm{T}}, \ldots, b_{N_0}^{\mathrm{T}}]^{\mathrm{T}}$$

- Bilateral control $\Longrightarrow (\tilde{A}_0, \tilde{B}_0)$ is always controllable.
- Choose $x_1^*, x_2^* \in (0, 1]$ such that

  $$\mathrm{rank}[c_{n_1}, c_{n_2}] = 2, \ \text{if } \bar{\lambda}_{n_1} = \bar{\lambda}_{n_2}, 1 \le n_1 < n_2 \le N_0.$$

  [Wang & Fridman, Aut'24] $\Longrightarrow (A_0, C_0)$ is observable.
- Choose $L_0 = [l_1^{\mathrm{T}}, \ldots, l_{N_0}^{\mathrm{T}}]^{\mathrm{T}}$ satisfying

  $$P_c(A_0 - L_0C_0) + (A_0 - L_0C_0)^{\mathrm{T}} P_c < -2\delta P_c, \quad 0 < P_c \in \mathbb{R}^{N_0\times N_0}.$$

  Choose $l_n = 0$ for $n = N_0 + 1, \ldots, N$ ([Sakawa, SICON'83]).

Stability

Let $e_n(t) = w_n(t) - \hat{w}_n(t)$, $e^{N_0}(t) = \mathrm{col}\{e_n(t)\}_{n=1}^{N_0}$, $e^{N-N_0}(t) = \mathrm{col}\{e_n(t)\}_{n=N_0+1}^{N}$.

$$\text{Reduced}: \begin{cases} \dot{\hat{w}}^{N_0}(t) = (\tilde{A}_0 - \tilde{B}_0 K_0)\hat{w}^{N_0}(t) + \tilde{L}_0 C_0 e^{N_0}(t) + \tilde{L}_0 \zeta(t) + \tilde{L}_0 C_1 e^{N-N_0}(t), \\ \dot{e}^{N_0}(t) = (A_0 - L_0 C_0) e^{N_0}(t) - L_0[\zeta(t) + C_1 e^{N-N_0}(t)], \\ \dot{w}_n(t) = \bar{\lambda}_n w_n(t) - \left[\frac{\alpha\beta b_n \mathbf{1}_0}{\lambda_n + \gamma} + b_n K_0\right] \hat{w}^{N_0}(t), \quad n > N, \end{cases}$$

$$\dot{e}^{N-N_0}(t) = A_1 e^{N-N_0}(t), \qquad A_1 = \mathrm{diag}\{\bar{\lambda}_n\}_{n=N_0+1}^{N} \text{ is stable}$$
$$\dot{\hat{w}}^{N-N_0}(t) = A_1 \hat{w}^{N-N_0}(t) - B_1 K_0 \hat{w}^{N_0}(t)$$

where $\zeta(t) = \sum_{n=N+1}^{\infty} c_n w_n(t)$ satisfying

$$|\zeta(t)|^2 \le \kappa_N \sum_{n=N+1}^{\infty} \lambda_n w_n^2(t), \quad \kappa_N = \sum_{n=N+1}^{\infty} \frac{|c_n|^2}{(n-\frac{1}{2})^2\pi^2}$$

Stability

- For reduced closed-loop system, consider (as in [Katz & EF, Aut'22] $p_e \to \infty$)

$$V(t) = |\hat{w}^{N_0}(t)|^2_{P_w} + |e^{N_0}(t)|^2_{P_e} + p_e|e^{N-N_0}(t)|^2 + \sum_{n=N+1}^{\infty} \lambda_n w_n^2(t)$$

- $\dot{V}(t) + 2\delta V(t) < 0 \Longrightarrow$ LMIs always feasible for large N

$$\begin{aligned} \Longrightarrow \quad & |\mathbf{u}(t)|^2 + \|w(\cdot,t)\|^2_{H^1} + \|w(\cdot,t) - \hat{w}(\cdot,t)\|^2_{H^1} \leq M\mathrm{e}^{-\delta t}\|z_0\|^2_{H^1}, \ t \geq 0 \\ \Longrightarrow \quad & \|z(\cdot,t)\|^2_{H^1} + \|v(\cdot,t)\|^2_{H^1} \leq \bar{M}\mathrm{e}^{-\delta t}\|z_0\|_{H^1}, \ t \geq 0 \end{aligned}$$

- We do not require $\gamma > 0$ ([Alalabi & Morris, Aut'25])

Example 1: single input & output

Example in [Alalabi & Morris, Aut'25]

$$
\begin{aligned}
z_t(x,t) =& z_{xx}(x,t) - \rho z(x,t) + 0.5v(x,t),\\
0 =& v_{xx}(x,t) - v(x,t) + z(x,t),\\
v_x(0,t) =& v_x(1,t) = z_x(0,t) = 0,\ \ z_x(1,t) = \mathbf{u}(t),\\
y(t) =& z(1,t),
\end{aligned}
$$

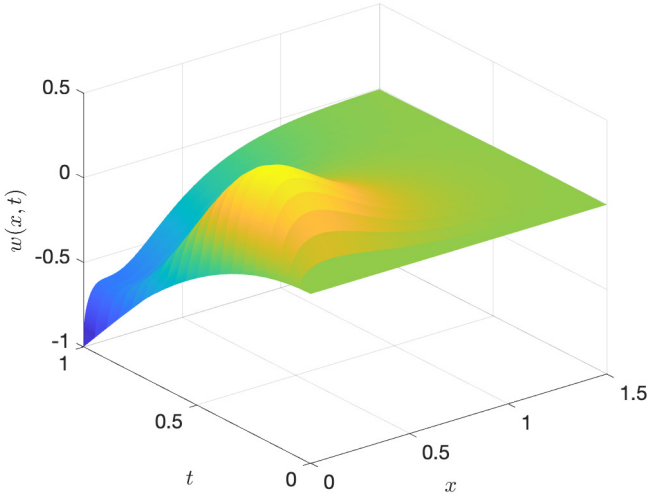
ρ	-20	-15	-8	-2	≥ 0
N_0	2	2	1	1	1
min N	10	6	3	2	1

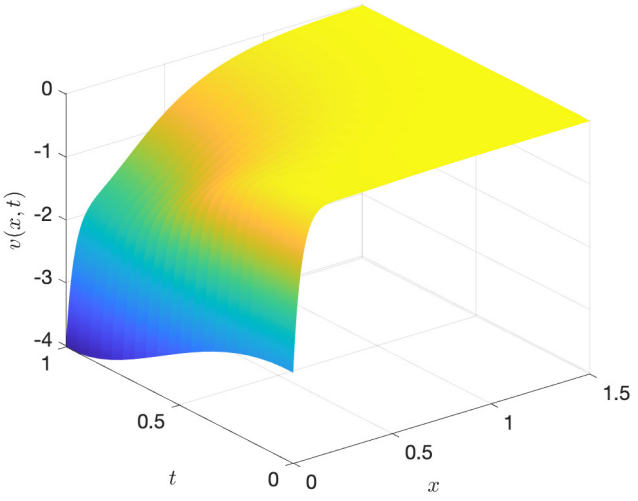
Theorem 10 in [Alalabi & Morris, Aut'25] (state-feedback) requires $\rho > 0.0615$.

Example 2: multiple eigenvalues & 2 inputs/outputs

$$
\begin{aligned}
z_t(x,t) =& z_{xx}(x,t) + 3\pi^2 z(x,t) - \frac{\pi^2}{2} v(x,t) \\
0 =& v_{xx}(x,t) - \frac{\pi^2}{2} v(x,t) + \frac{5\pi^2}{2} z(x,t) \\
z(0,t) =& \mathbf{u}_1(t), \;\; z_x(1,t) = \mathbf{u}_2(t), \quad v(0,t) = v_x(1,t) = 0 \\
y(t) =& [z(\frac{1}{2},t), z(1,t)]^{\mathrm{T}}
\end{aligned}
$$

- $\bar{\lambda}_1 = \bar{\lambda}_2 > 0$ and $\bar{\lambda}_n < 0$ for $n \geq 3 \Longrightarrow N_0 = 2$.
- Take the decay rate $\delta = 1$, $N = 3$, and initial condition $z_0(x) = x^2 - 2x$.





Example 3: ONE input/output vs TWO

$$
\begin{aligned}
z_t(x,t) =& z_{xx}(x,t) + 3\pi^2 z(x,t) - \frac{\pi^2}{2} v(x,t), \\
0 =& v_{xx}(x,t) - (\frac{\pi^2}{2} + 0.1) v(x,t) + \frac{5\pi^2}{2} z(x,t), \\
v(0,t) =& v_x(1,t) = 0
\end{aligned}
$$

$$
\begin{array}{ll}
z(0,t) = \mathbf{u}_1(t), z_x(1,t) = \mathbf{u}_2(t), & N = 3 \\
z(0,t) = \mathbf{u}_1(t), z_x(1,t) = 0, & N = 26 \\
z(0,t) = 0, z_x(1,t) = \mathbf{u}_2(t), & N = 7
\end{array}
\quad \Longrightarrow \quad
\begin{array}{l}
\text{Bilateral control reduces} \\
\text{the observer dimension}
\end{array}
$$

Plan

1. Intro: delay-robust control of PDEs - spatial & modal decomposition
2. Finite-dimensional output-feedback for Parabolic-Elliptic systems
3. Stabilization of Non-Standard Singularly Perturbed PDE systems
4. Extensions: 2D parabolic, L_2 separation,...
5. Conclusions

Singularly perturbed infinite-dimensional systems

Stability of

- Coupled ODE with hyperbolic eq [Tang & Mazanti, Aut'17], with wave eq [Cerpa & Prieur, TAC'20], with KdV eq [Marx & Cerpa, TAC'24]
- Coupled hyperbolic systems [Tang, Prieur, and Girard, TAC'15]
- Coupled heat-wave systems [Arias, Cerpa, & Marx, JMAA'25]

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Singularly perturbed reaction-diffusion systems

- **Applications**:
  - Interaction between vegetation and water [Doelman, Nonlinearity'22]
  - Woodland ecosystem model [Ambrosio & Alaoui, Actabiotheoretica'16]
  - Thermal fluid convection [Vazquez & Krstic, SCL'06; TCST'09]
- **Boundary stabilization of SP heat PDEs via Backstrepping** [Arias & Parada, SICON'25], [Vazquez & Krstic, SCL'06; TCST'09]

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Shortcomings:

- Stability is achieved for small enough $\varepsilon > 0$ - no quantitative bounds
- Standard SP systems

Control of non-standard singularly perturbed systems

Consider

$$E^{\varepsilon}\dot{x}(t) = Ax(t) + Bu(t), \tag{1}$$

where $x(t) = col\{x_1(t), x_2(t)\} \in \mathcal{R}^{n_1} \times \mathcal{R}^{n_2}$, $\varepsilon > 0$ is a small parameter,

$$E^{\varepsilon} = \begin{bmatrix} I_{n_1} & 0 \\ 0 & \varepsilon I_{n_2} \end{bmatrix}, \quad A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

Non-standard problem [Khalil TAC'89]: A_4 can be singular $\Rightarrow$

No solution to

$$A_3x_1 + A_4x_2 + B_2u = 0$$

w.r.t. the fast variable x_2

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Slow system is **descriptor**, i.e. (1) with $\varepsilon = 0$.
Controllability assumptions and design for slow descriptor and fast ODE

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$$\dot{x}_2 = A_4x_2 + B_2u.$$

- Initiated in [Khalil TAC'89] (for small enough $\varepsilon > 0$)
- Constructive with quantitative bound on ε and sampled-data control in [Fridman TAC'06]

Control of non-standard singularly perturbed reaction-diffusion systems [Wang & EF, CDC'25]

$$
\begin{aligned}
z_t(x,t) &= z_{xx}(x,t) - \rho z(x,t) + \alpha v(x,t) \\
\varepsilon v_t(x,t) &= v_{xx}(x,t) - \gamma v(x,t) + \beta z(x,t) \\
z(x,0) &= z_0(x), \quad v(x,0) = v_0(x), \ x \in (0,1)
\end{aligned}
$$

- $\rho, \gamma, \alpha, \beta \in \mathbb{R}$, $\alpha\beta \neq 0$
- $\varepsilon > 0$: the singular perturbation parameter
- Neumann actuation:

$$
z_x(0,t) = 0, \quad z_x(1,t) = u_1(t), \qquad v_x(0,t) = 0, \quad v_x(1,t) = u_2(t)
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$$

Objective: - Design **finite-dim** stabilizing $[u_1(t), u_2(t)]^{\mathrm{T}} = \mathcal{K}[z^N(t), v^N(t)]$

$z^N(t) = \mathrm{col}\{\langle z(\cdot,t), \phi_n \rangle\}_{n=1}^N$, $v^N(t) = \mathrm{col}\{\langle v(\cdot,t), \phi_n \rangle\}_{n=1}^N$

- Provide **constructive** conditions for minimal N and maximal ε

Two boundary inputs for non-standard SP system

- [Arias & Parada, SICON'25]:
  $z_x(0,t) = 0,\ z_x(1,t) = u_1(t),\quad v_x(0,t) = 0,\ v_x(1,t) = 0$
  - $\gamma > -\lambda_1$: Requires fast system to be stable
  - $\gamma \neq -\lambda_n$: Cannot handle non-standard case [1]

- **This paper**: $z_x(0,t) = 0,\ z_x(1,t) = u_1(t),\quad v_x(0,t) = 0,\ v_x(1,t) = u_2(t)$
  - $\gamma \leq -\lambda_1$ ✔ Handle unstable fast system
  - $\gamma = -\lambda_n$ ✔ Address non-standard singular perturbation
  - Manage multiple eigenvalue case
  - Potentially reduce controller gain

[1] The algebraic eq for $\varepsilon = 0$ is not solvable w.r.t. fast variable.

Modal decomposition

- Eigenvalues and eigenfunctions of $\mathcal{A}h = -h''$, $h'(0) = h'(1) = 0$:

$$\lambda_n = (n-1)^2\pi^2,\ n = 1, 2, \dots$$
$$\phi_1(x) \equiv 1,\ \phi_n(x) = \sqrt{2}\cos(\sqrt{\lambda_n}x), n \geq 2$$

- $\{\phi_n\}_{n=1}^{\infty}$ is a basis of $L^2(0,1)$:

$$z(\cdot,t) \overset{L^2}{=} \sum_{n=1}^{\infty} z_n(t)\phi_n,\ \ z_n(t) = \langle z(\cdot,t), \phi_n \rangle$$
$$v(\cdot,t) \overset{L^2}{=} \sum_{n=1}^{\infty} v_n(t)\phi_n,\ \ v_n(t) = \langle v(\cdot,t), \phi_n \rangle$$

- $z_n(t)$ and $v_n(t)$ satisfy:

$$\dot{z}_n(t) = (-\lambda_n - \rho)z_n(t) + \alpha v_n(t) + b_n u_1(t)$$
$$\varepsilon\dot{v}_n(t) = (-\lambda_n - \gamma)v_n(t) + \beta z_n(t) + b_n u_2(t)$$

where $b_n = \phi_n(1)$.

Slow (descriptor) system & fast system

- $\varepsilon = 0$ (slow system):

$$\begin{aligned}\dot{\bar{z}}_n(t) &= (-\lambda_n - \rho)\bar{z}_n(t) + \alpha\bar{v}_n(t) + b_n\bar{u}_1(t)\\ 0 &= (-\lambda_n - \gamma)\bar{v}_n(t) + \beta\bar{z}_n(t) + b_n\bar{u}_2(t), \quad n \in \mathbb{N}\end{aligned}$$

  - Corresponds to coupled parabolic-elliptic PDE
    [Alalabi & Morris, Aut'25; Wang & Fridman, L-CSS'24]
  - Controller is designed for the descriptor system

- fast system:

$$\dot{\bar{v}}_n(t) = (-\lambda_n - \gamma)\bar{v}_n(t) + b_n\bar{u}_2(t), \ n \in \mathbb{N}$$

Modes for controller

Let $\delta > 0$ be the decay rate.
$\lambda_n = (n-1)^2\pi^2 \to \infty \quad \Rightarrow \quad \exists N \in \mathbb{N}$:

$$\lambda_{f,n} + \delta < 0, \ \lambda_{f,n} := -\lambda_n - \gamma, \ n > N$$
$$\lambda_{s,n} + \delta < 0, \ \lambda_{s,n} := -\lambda_n - \rho + \frac{\alpha\beta}{\lambda_n + \gamma}, \ n > N$$

- $\lambda_{f,n}$, characteristic roots of the fast system with $n > N$
  $\lambda_{s,n}$, characteristic roots of the slow system with $n > N$

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- $\lambda_{f,n}$, characteristic roots of the fast system with $n > N$
  $\lambda_{s,n}$, characteristic roots of the slow system with $n > N$
- N determines the number of unstable modes

Controller design: Stabilization of slow descriptor system ($\varepsilon = 0$)

$$\dot{\bar{z}}_n(t) = (-\lambda_n - \rho)\bar{z}_n(t) + \alpha\bar{v}_n(t) + b_n\bar{u}_1(t)$$
$$0 = (-\lambda_n - \gamma)\bar{v}_n(t) + \beta\bar{z}_n(t) + b_n\bar{u}_2(t),\ n \in \mathbb{N}$$

$$\bar{X} = [\bar{z}_1, \ldots, \bar{z}_N, \bar{v}_1, \ldots, \bar{v}_N]^{\mathrm{T}}, \quad \bar{X}_n = [\bar{z}_n, \bar{v}_n]^{\mathrm{T}}, \quad \bar{\mathbf{u}}(t) = [\bar{u}_1(t), \bar{u}_2(t)]^{\mathrm{T}}$$

$$E_N\dot{\bar{X}}(t) = A\bar{X}(t) + B\bar{\mathbf{u}}(t), \qquad E_N = \mathrm{diag}\{I_N, 0_N\}$$
$$E_1\dot{\bar{X}}_n(t) = A_n\bar{X}_n(t) + B_n\bar{\mathbf{u}}(t),\ n > N, \quad E_1 = \mathrm{diag}\{1, 0\}$$

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(E_N, A, B) **is controllable**.

State-feedback control: $\bar{\mathbf{u}}(t) = -K\bar{X}(t),\ K = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \in \mathbb{R}^{2\times 2N}$

and $\bar{u}_2(t) = -K_{22}\bar{v}^N(t)$ stabilizes the fast ODE system

Stabilization of the full-order system with a bound on ε

- Closed-loop system:

$$E_N^\varepsilon \dot{X}(t) = (A - BK)X(t), \qquad E_N^\varepsilon = \mathrm{diag}\{I_N, \varepsilon I_N\} \tag{2a}$$

$$E_1^\varepsilon \dot{X}_n(t) = A_n X_n(t) - b_n K X(t),\ n > N, \quad E_1^\varepsilon = \mathrm{diag}\{1, \varepsilon 1\} \tag{2b}$$

- For stability of (2a), $V(t) = X^{\mathrm{T}}(t) E_N^\varepsilon P_\varepsilon X(t)$ ([Xu & Mizukami, Aut'97])

$$P_\varepsilon = \begin{bmatrix} P_1 & \varepsilon P_2^{\mathrm{T}} \\ P_2 & P_3 \end{bmatrix}, \quad P_1, P_3 > 0, \quad E_N^\varepsilon P_\varepsilon > 0, \quad \varepsilon > 0$$

- $P_\varepsilon^{\mathrm{T}}(A - BK) + (A - BK)^{\mathrm{T}} P_\varepsilon + 2\delta E_N^\varepsilon P_\varepsilon < 0 \Rightarrow \dot{V}(t) + 2\delta V(t) \le 0$
  $\Rightarrow \boxed{|X(t)|^2 \le M_1(\varepsilon) \mathrm{e}^{-2\delta t} |X(0)|^2, \quad t \ge 0}$

Stability of infinite-dimensional tail

- For stability of the residual part we choose

$$V_\infty(t) = \sum_{n=N+1}^{\infty} X_n^{\mathrm{T}}(t) E_1^\varepsilon Q_{\varepsilon,n} X_n(t)$$

with

$$Q_{\varepsilon,n} = \begin{bmatrix} q_{n1} & \varepsilon q_{n2} \\ q_{n2} & q_{n3} \end{bmatrix}, \; q_{n1}, q_{n3} > 0, \; E_1^\varepsilon Q_{\varepsilon,n} > 0, \; N < n \le \tilde{N}$$

$$Q_{\varepsilon,n} = I_2, \quad n > \tilde{N}, \quad A_{\tilde{N}} + A_{\tilde{N}}{}^{\mathrm{T}} + 2\delta E_1^\varepsilon < 0$$

Here $Q_{\varepsilon,n} = I_2$ for $n > \tilde{N}$ guarantees for some ε-independent $q_*, q^* > 0$

$$\varepsilon q_* \sum_{n=N+1}^{\infty} |X_n(t)|^2 \le \bar{V}_\infty(t) \le q^* \sum_{n=N+1}^{\infty} |X_n(t)|^2, \tag{3}$$

Stability of the residual part

Matrix inequalities

$$Q_{\varepsilon,n}^{\mathrm{T}} A_n + A_n^{\mathrm{T}} Q_{\varepsilon,n} + 2\delta E_1^{\varepsilon} Q_{\varepsilon,n} < 0 \quad (N < n \leq \tilde{N})$$

$\Rightarrow \dot{V}_\infty(t) + 2\delta V_\infty(t) - \chi |KX(t)|^2 \leq 0$ for large enough $\chi > 0$

$\Rightarrow \sum_{n=N+1}^{\infty} |X_n(t)|^2 \leq M_2(\varepsilon) \mathrm{e}^{-2\delta t} [\|z_0(\cdot)\|_{L^2}^2 + \|v_0(\cdot)\|_{L^2}^2],\ t \geq 0$

$\Rightarrow \|z(\cdot,t)\|_{L^2}^2 + \|v(\cdot,t)\|_{L^2}^2 \leq M_3(\varepsilon) \mathrm{e}^{-2\delta t} [\|z_0(\cdot)\|_{L^2}^2 + \|v_0(\cdot)\|_{L^2}^2],\ t \geq 0$

LMI Conditions for ε^*

Verify LMIs at $\varepsilon_0 = 0$ and $\varepsilon_1 = \varepsilon^*$:

$$E_N^{\varepsilon_1} P_{\varepsilon_1} > 0, \quad P_\varepsilon = \begin{bmatrix} P_1 & \varepsilon P_2^{\mathrm{T}} \\ P_2 & P_3 \end{bmatrix},$$

$$P_{\varepsilon_i}{}^{\mathrm{T}}(A - BK) + (A - BK)^{\mathrm{T}} P_{\varepsilon_i} + 2\delta E_N^{\varepsilon_i} P_{\varepsilon_i} < 0, \quad i = 0, 1$$

$$E_1^{\varepsilon_1} Q_{\varepsilon_1, n} > 0, \quad Q_{\varepsilon, n} = \begin{bmatrix} q_{n1} & \varepsilon q_{n2} \\ q_{n2} & q_{n3} \end{bmatrix}, N < n \leq \tilde{N}$$

$$Q_{\varepsilon_i, n}{}^{\mathrm{T}} A_n + A_n^{\mathrm{T}} Q_{\varepsilon_i, n} + 2\delta E_1^{\varepsilon_i} Q_{\varepsilon_i, n} < 0, \quad i = 0, 1$$

- Exponential stability guaranteed for any $\varepsilon \in [0, \varepsilon^*]$
- Quantitative bound ε^*
- Always feasible for small enough ε^*

Numerical Examples

1. **Non-standard case**:

$$\alpha = 3\pi^2,\ \beta = -\pi^2,\ \rho = -10\pi,\ \gamma = -\pi^2 \tag{a}$$

   $\gamma = -\lambda_2$ cannot be handled by [Arias & Parada, SICON'25].

2. **Multiple eigenvalues**:

$$\alpha = \pi^2,\ \beta = \pi - 1,\ \rho = -3\pi,\ \gamma = -\pi \tag{b}$$

3. **Coefficients from** [Arias & Parada, SICON'25]:

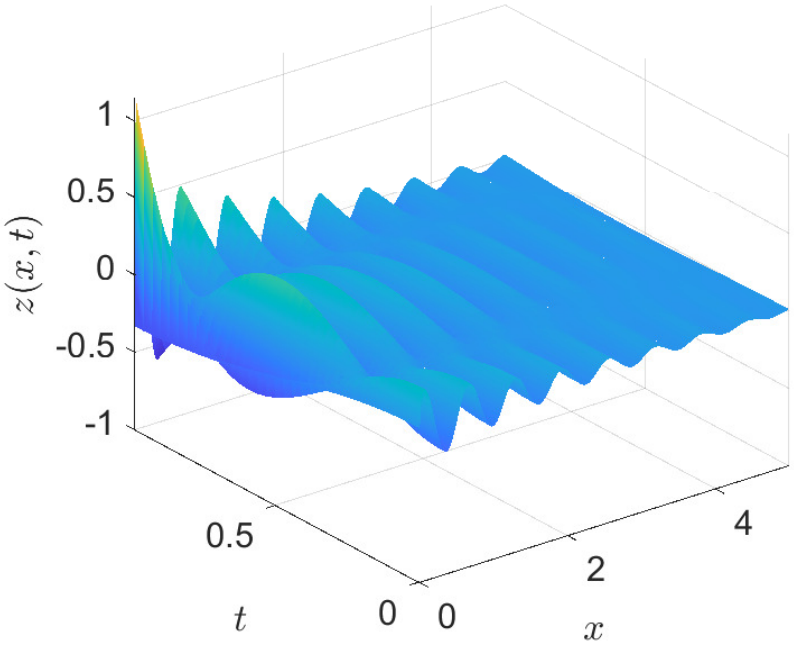
$$\alpha = 5,\ \beta = 6,\ \rho = -10,\ \gamma = 0.1 \tag{c}$$

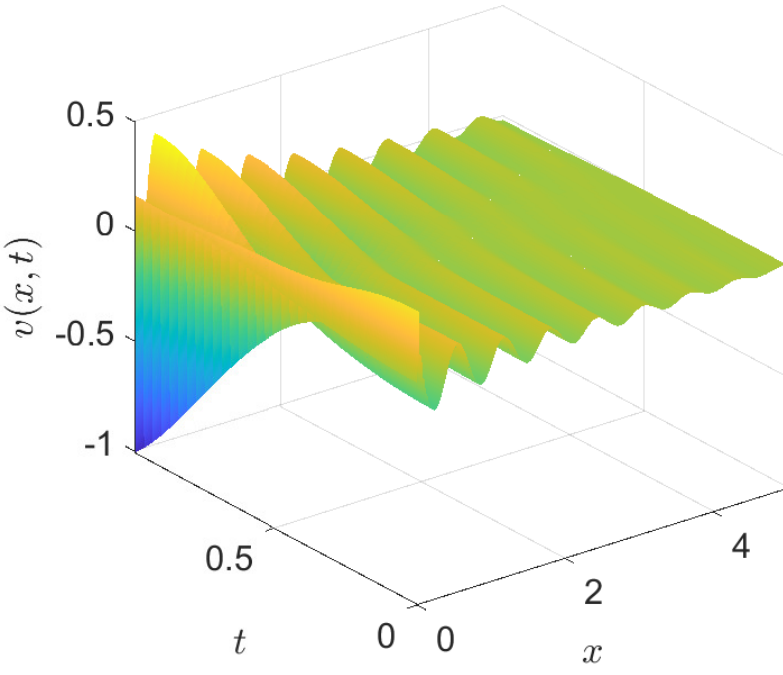
Simulations

- All cases: $N = 2$ unstable modes
- Quantitative bounds via LMIs

Coefficients	Controller gain K from slow system	ε^*
(a)	$\begin{bmatrix} 22.98 & 8.66 & -96.63 & -37.70 \\ 5.04 & -6.34 & 39.26 & 11.83 \end{bmatrix}$	0.169
(b)	$\begin{bmatrix} 329.72 & -468.34 & -45.44 & -75.77 \\ -1.26 & 6.78 & 3.82 & 4.09 \end{bmatrix}$	0.136
(c)	$\begin{bmatrix} 5.911 & -2.371 & 3.739 & 1.221 \\ 7.705 & -3.684 & 2.858 & 7.933 \end{bmatrix}$	0.558
(c) $u_2 = 0$	$\begin{bmatrix} 7875.92 & 1141.02 & 29.869 & 333.42 \end{bmatrix}$	0.113

Simulations: non-standard case





- $\alpha = 3\pi^2,\ \beta = -\pi^2,\ \rho = -10\pi,\ \gamma = -\pi^2$
- Initial conditions: $z_0(x) = \frac{2}{3}x^3 - x^2,\ v_0(x) = -\frac{1}{3}x^3 + \frac{1}{2}x^2$
- From simulations $\varepsilon = 0.19 > \varepsilon^* = 0.169$

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Finite-dim control of 2D parabolic PDEs

[Wang & EF, Aut'24; SCL'26]

$$\begin{aligned} \mathrm{d}z(x,t) &= [\Delta z(x,t) + qz(x,t) + f(z(x,t))]\mathrm{d}t \\ &\quad + g(z(x,t))\mathrm{d}\mathcal{W}(t), \quad x \in \mathcal{O}, \\ z(x,0) &= z_0(x), \quad \mathbb{E}\|z_0\|^2_{L^2(\mathcal{O})} < \infty. \end{aligned}$$

- $\mathcal{O} \subset \mathbb{R}^2$: bounded open connected set ($\partial\mathcal{O} = \Gamma_1 \cup \Gamma_2$ is of class C^2 or $\mathcal{O}$ is a rectangular domain).
- $\mathcal{W}(t)$ is a 1-dimensional standard Brownian motion.
- Random parameter variation of $f(z)\mathrm{d}t$ leads to $g(z)\mathrm{d}\mathcal{W}(t)$.
- Nonlinear functions $f, g : \mathbb{R} \to \mathbb{R}$

$$\begin{aligned} &f(0) = 0, \ |f(z_1) - f(z_2)| \le \sigma_f(z_1 - z_2), \\ &g(0) = 0, \ |g(z_1) - g(z_2)| \le \sigma_g(z_1 - z_2), \ \forall z_1, z_2 \in \mathbb{R}, \end{aligned}$$

- Dirichlet actuation:

$$z(x,t) = 0, \ x \in \Gamma_1, \quad z(x,t) = u(x,t),$$

$u(x,t)$ is the control input to be determined.

Challenges in 2D case are due to multiple eigenvalues and slower convergence

- For 1D case
  - Stability analysis employs:

$$\lambda_n = O(n^2) \Longrightarrow \sum_{n=1}^{\infty} \frac{1}{\lambda_n} < \infty \quad \text{and} \quad \sum_{n=N+1}^{\infty} \frac{1}{\lambda_n} = O(\frac{1}{N})$$

  - [Karafyllis & Krstic, Aut'18; Katz & EF, Aut'20; Lhachemi & Prieur, Aut'22] (controllability and observability):

$$\lambda_1 < \lambda_2 < \cdots < \lambda_k < \ldots.$$

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  - $\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \ldots$ – controllability and observability ☹

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  - $\lambda_1 < \lambda_2 \leq \cdots \leq \lambda_k \leq \ldots$ – controllability and observability ☹
- [Feng, Lang & Liu, Aut'12] – semilinear case ☹
- [Munteanu, IJC'17; Lhachemi, Munteanu & Prieur, Aut'25] – how to find N ☹

Our design gives efficient bound on N

L_2-separation of residue for modal decomposition

Anton Selivanov (Sheffield University, UK)

[Selivanov & Wang & EF, L-CSS'24]

Semilinear heat equation with boundary input

$$z_t = z_{xx} + qz + f(\cdot, t, z(\cdot, t)) \qquad z\colon [0,\pi] \times [0,\infty) \to \mathbb{R}$$
$$z_x(0,t) = 0, \quad z_x(\pi,t) = u(t) \qquad u\colon [0,\infty) \to \mathbb{R}$$

Assumption: $f(x,t,0) \equiv 0$ and $|f(x,t,z_1) - f(x,t,z_2)| \le \sigma |z_1 - z_2|$

Objective: Find a stabilizing $u(t) = \mathcal{K} z(\cdot, t)$.

L_2-separation of residue for modal decomposition

Output-feedback via residual separation:

- Design of the full-order gains K and L based on N modes that enlarge σ;
- Proof that Lipschitz constant does not decrease for larger N [CDC'26]
- Linear wave eq with Kelvin Voigt damping [IFAC Congress'26]

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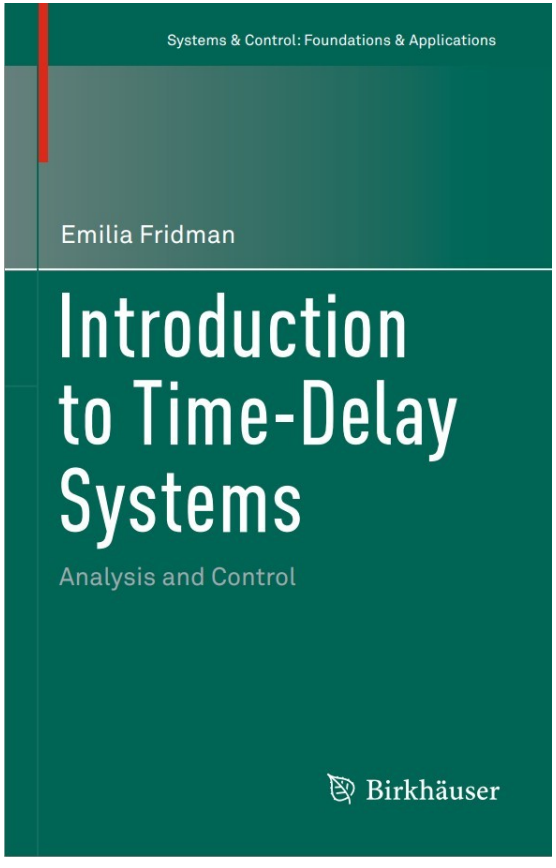
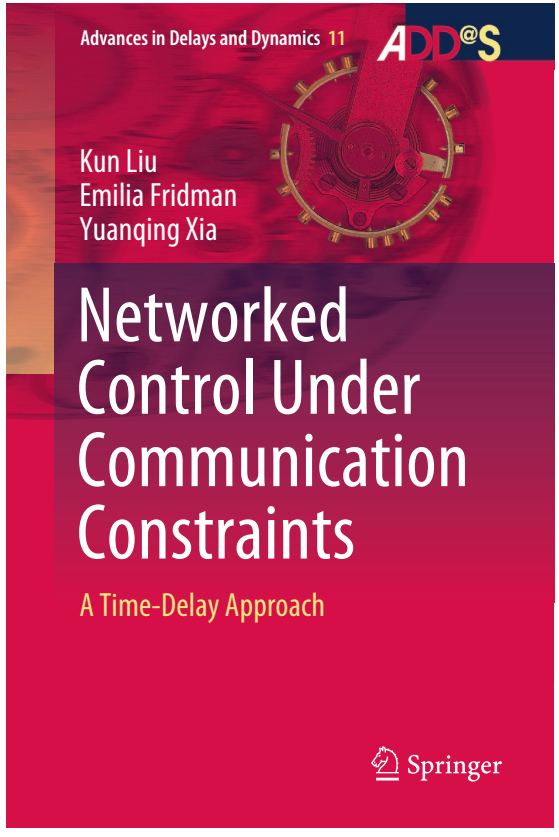
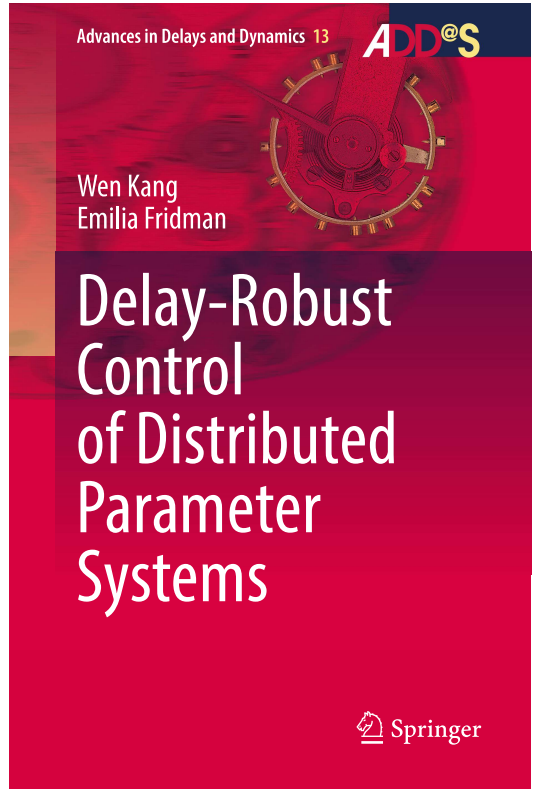
2014



2020



2025



Conclusions

Constructive LMI-based method was introduced for finite-dimensional observer-based control of parabolic PDEs via **modal decomposition** [Katz & EF, Aut'20]

Our recent results:

- → Finite-dim observer-based boundary control of damped 1D wave [Selivanov & EF, TAC'24]
- → Disturbance attenuation of 1D damped Euler-Bernoulli beam [Selivanov & EF, SCL'24; CDC'26]
- → Nonlinear Homogenous control of heat, Burgers with advantages over linear controller [Ayamou & Espitia & Polyakov & EF, EJC'25, IMA'25, Aut'26]
- → Delayed regional stabilization of KdVB eq.[W. Kang,& EF&E-X. Pang, EJC'25]
- → 2D KSE with disturbance rejection [J. Zhang, W. Kang, P. Wang & EF, CDC'26]

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- **Spatial decomposition** method is applicable to a wide class of PDEs including e.g. KdV and Wave eq with boundary damping, where modal decomposition seems to be not applicable