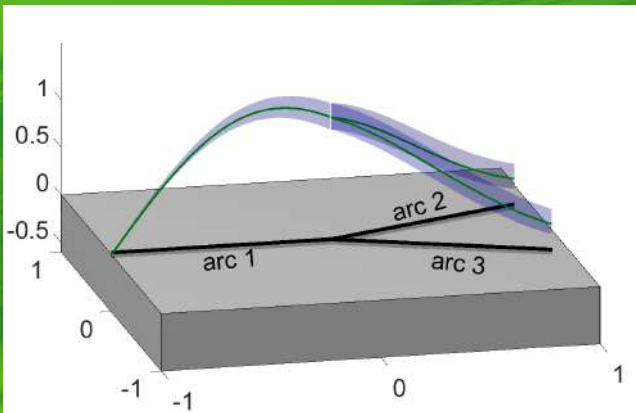


On the stabilizability of hyperbolic systems by boundary control: How to deal with cyclic networks

Martin Gugat, FAU, Department Mathematik, DCN (AvH-Professorship)

5th Workshop on Stability and Control of Infinite-Dimensional Systems
SCINDIS 2026



1. Motivation by application: Gas transportation through pipelines

2. The Example by Bastin and Coron

2.1 A sufficient condition for non-stabilizability

3. Networked systems

3.1 A network of strings with a stabilizing and a destabilizing source term

3.2 A sufficient condition for stabilizability

3.3 The quasilinear case: A cyclic network (pipeline and channel networks)

4. Time-delay

Application: Gas transport through pipelines

The system dynamics in a pipe is described by

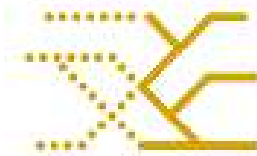
the isothermal Euler equations

$$\begin{cases} \rho_t + q_x = 0 \\ q_t + \left(p + \frac{q^2}{\rho}\right)_x = -\frac{f_g}{2\delta} \frac{q|q|}{\rho} - \rho g \sin(\alpha) \end{cases}$$

or a similar (linearized) model, see *A. Osiadacz, M. Chazykowski, Comparison of isothermal and non-isothermal pipeline gas flow models, 2001.*



See the results of DFG CRC 154:



**TRR
154**

Mathematical Modelling,
Simulation and Optimization Using
the Example of Gas Networks

(12 years!)

Application: Gas transport through pipelines

The system dynamics in a pipe is described by

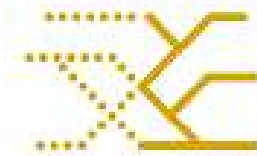
the isothermal Euler equations

$$\begin{cases} \rho_t + q_x = 0 \\ q_t + \left(p + \frac{q^2}{\rho}\right)_x = -\frac{f_g}{2\delta} \frac{q|q|}{\rho} - \rho g \sin(\alpha) \end{cases}$$

or a similar (linearized) model, see *A. Osiadacz, M. Chazykowski, Comparison of isothermal and non-isothermal pipeline gas flow models, 2001.*



See the results of DFG CRC 154:



**TRR
154**

Mathematical Modelling,
Simulation and Optimization Using
the Example of Gas Networks

(12 years!)

Ideal gas

In **ideal gas**, we have

$$p = c^2 \rho.$$

The sound speed c is constant!

Application: Gas transport through pipelines

The system dynamics in a pipe is described by

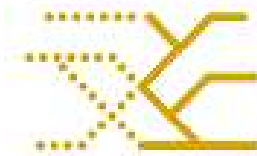
the isothermal Euler equations

$$\begin{cases} \rho_t + q_x = 0 \\ q_t + \left(p + \frac{q^2}{\rho}\right)_x = -\frac{f_g}{2\delta} \frac{q|q|}{\rho} - \rho g \sin(\alpha) \end{cases}$$

or a similar (linearized) model, see *A. Osiadacz, M. Chazykowski*, Comparison of isothermal and non-isothermal pipeline gas flow models, 2001.



See the results of DFG CRC 154:



**TRR
154**

Mathematical Modelling,
Simulation and Optimization Using
the Example of Gas Networks

(12 years!)

Ideal gas

In **ideal gas**, we have

$$p = c^2 \rho.$$

The sound speed c is constant!

In this case, the isothermal Euler equations ($\alpha = 0$)

are
$$\begin{cases} \rho_t + q_x = 0 \\ q_t + c^2 \rho_x + \left(\frac{q^2}{\rho}\right)_x = -\frac{f_g}{2\delta} \frac{q|q|}{\rho} \end{cases}$$

Application: Gas transport through pipelines

The system dynamics in a pipe is described by

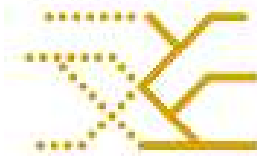
the isothermal Euler equations

$$\begin{cases} \rho_t + q_x = 0 \\ q_t + \left(p + \frac{q^2}{\rho}\right)_x = -\frac{f_g}{2\delta} \frac{q|q|}{\rho} - \rho g \sin(\alpha) \end{cases}$$

or a similar (linearized) model, see *A. Osiadacz, M. Chazykowski, Comparison of isothermal and non-isothermal pipeline gas flow models, 2001.*



See the results of DFG CRC 154:



**TRR
154**

Mathematical Modelling,
Simulation and Optimization Using
the Example of Gas Networks

(12 years!)

Ideal gas

In **ideal gas**, we have

$$p = c^2 \rho.$$

The sound speed c is constant!

In this case, the isothermal Euler equations ($\alpha = 0$)

are
$$\begin{cases} \rho_t + q_x = 0 \\ q_t + c^2 \rho_x + \left(\frac{q^2}{\rho}\right)_x = -\frac{f_g}{2\delta} \frac{q|q|}{\rho} \end{cases}$$

The Hamiltonian form: With $v = \frac{q}{\rho}$ and

$$H(\rho, v) = \frac{1}{2} \rho v^2 + c^2 [\rho \ln(\rho) - \rho]$$

is

$$\begin{cases} \rho_t + \partial_x H_v(\rho, v) = 0 \\ v_t + \partial_x H_\rho(\rho, v) = -\frac{f_g}{2\delta} v |v|, \quad (\text{see CRC1701}). \end{cases}$$

INVARIANTS

For ideal gas, we have the *Riemann invariants*

$$R_{\pm}(\rho, q) = c \ln(\rho) \pm \frac{q}{\rho}.$$

INVARIANTS

For ideal gas, we have the *Riemann invariants*

$$R_{\pm}(\rho, q) = c \ln(\rho) \pm \frac{q}{\rho}.$$

The *pressure* is given by $\exp\left(\frac{1}{2c} (r_+ + r_- + \bar{R}_+ + \bar{R}_-)\right) > 0$ and the gas *velocity* is $\frac{r_+ - r_- + \bar{R}_+ - \bar{R}_-}{2}$.

For ideal gas, we have the *Riemann invariants*

$$R_{\pm}(\rho, q) = c \ln(\rho) \pm \frac{q}{\rho}.$$

The *pressure* is given by $\exp\left(\frac{1}{2c} (r_+ + r_- + \bar{R}_+ + \bar{R}_-)\right) > 0$ and the gas *velocity* is $\frac{r_+ - r_- + \bar{R}_+ - \bar{R}_-}{2}$.

Travelling waves for the isothermal Euler equations. Let $\theta = \frac{f_g}{\delta}$.

Traveling waves solutions are (see *The isothermal Euler equations for ideal gas with source term: Product solutions, flow reversal and no blow up*; M GUGAT, S ULBRICH) (with a real parameter λ)

$$\begin{pmatrix} q(t, x) \\ \rho(t, x) \end{pmatrix} = \alpha(\lambda t - x) \begin{pmatrix} \lambda \\ 1 \end{pmatrix} \quad \text{with} \quad \alpha(z) = C \exp\left(\frac{\lambda |\lambda| \theta + 2g \sin(\alpha)}{2c^2} z\right).$$

Here the *Riemann invariants* are $R_{\pm}(t, x) = \frac{\lambda |\lambda| \theta + 2g \sin(\alpha)}{2c} (\lambda t - x) \pm \lambda$.

- The velocity $v = \frac{q}{\rho} = \lambda$ is constant.

Another solution is $\rho = \text{const}, v = \frac{1}{\text{const} + \frac{1}{2}\theta t}$.

In terms of the RIEMANN invariants the hyperbolic system has the *quasilinear* form

In terms of the RIEMANN invariants the hyperbolic system has the *quasilinear* form

$$R_t + D(R) R_x = F(R), \quad D(R) = \begin{pmatrix} \lambda_+(R) & 0 \\ 0 & \lambda_-(R) \end{pmatrix}$$

with a *diagonal matrix* $D(R)$ that contains the *eigenvalues*

$$\lambda_+ = c + \frac{q}{\rho}, \quad \lambda_- = -c + \frac{q}{\rho}.$$

In terms of the RIEMANN invariants the hyperbolic system has the *quasilinear* form

$$R_t + D(R) R_x = F(R), \quad D(R) = \begin{pmatrix} \lambda_+(R) & 0 \\ 0 & \lambda_-(R) \end{pmatrix}$$

with a *diagonal matrix* $D(R)$ that contains the *eigenvalues*

$$\lambda_+ = c + \frac{q}{\rho}, \quad \lambda_- = -c + \frac{q}{\rho}.$$

Let a stationary state $\bar{R}$ with $D \bar{R}_x = F(\bar{R})$ be given. We have $\bar{R}_x = D^{-1}(\bar{R}) F(\bar{R})$.

In terms of the RIEMANN invariants the hyperbolic system has the *quasilinear* form

$$R_t + D(R) R_x = F(R), \quad D(R) = \begin{pmatrix} \lambda_+(R) & 0 \\ 0 & \lambda_-(R) \end{pmatrix}$$

with a *diagonal matrix* $D(R)$ that contains the *eigenvalues*

$$\lambda_+ = c + \frac{q}{\rho}, \quad \lambda_- = -c + \frac{q}{\rho}.$$

Let a stationary state $\bar{R}$ with $D \bar{R}_x = F(\bar{R})$ be given. We have $\bar{R}_x = D^{-1}(\bar{R}) F(\bar{R})$.

Linearizing around $\bar{R}$ with $r = R - \bar{R}$ yields

$$r_t + D(\bar{R}) r_x = F'(\bar{R}) r - [D'(\bar{R})(r)](\bar{R})_x$$

In terms of the RIEMANN invariants the hyperbolic system has the *quasilinear* form

$$R_t + D(R) R_x = F(R), \quad D(R) = \begin{pmatrix} \lambda_+(R) & 0 \\ 0 & \lambda_-(R) \end{pmatrix}$$

with a *diagonal matrix* $D(R)$ that contains the *eigenvalues*

$$\lambda_+ = c + \frac{q}{\rho}, \quad \lambda_- = -c + \frac{q}{\rho}.$$

Let a stationary state $\bar{R}$ with $D \bar{R}_x = F(\bar{R})$ be given. We have $\bar{R}_x = D^{-1}(\bar{R}) F(\bar{R})$.

Linearizing around $\bar{R}$ with $r = R - \bar{R}$ yields

$$r_t + D(\bar{R}) r_x = F'(\bar{R}) r - [D'(\bar{R})(r)](\bar{R})_x$$

For the ideal gas we get

$$F'(\bar{R}) = -2 \frac{f_g}{\delta} |\bar{R}_+(x) - \bar{R}_-(x)| \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad [D'(\bar{R})(r)] D^{-1}(\bar{R}) = \frac{1}{2} \begin{pmatrix} \frac{r_+ - r_-}{\bar{\lambda}_+} & 0 \\ 0 & \frac{r_+ - r_-}{\bar{\lambda}_-} \end{pmatrix}.$$

The matrix $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ is *positive semi-definite*.

We are interested in *boundary control* of the system!

For this purpose, we introduce a control $u(t)$ in the boundary condition.

We are interested in *boundary control* of the system!

For this purpose, we introduce a control $u(t)$ in the boundary condition.

For example DIRICHLET boundary control

$$r_+(t, 0) = u_+(t), \quad r_-(t, L) = u_-(t).$$

We are interested in *boundary control* of the system!

For this purpose, we introduce a control $u(t)$ in the boundary condition.

For example DIRICHLET boundary control

$$r_+(t, 0) = u_+(t), \quad r_-(t, L) = u_-(t).$$

What happens, if in the pde

$$r_t + D r_x = -M r$$

the matrix is *not* positive definite?

Can this cause difficulties?

We are interested in *boundary control of the system!*

For this purpose, we introduce a control $u(t)$ in the boundary condition.

For example DIRICHLET boundary control

$$r_+(t, 0) = u_+(t), \quad r_-(t, L) = u_-(t).$$

What happens, if in the pde

$$r_t + D r_x = -M r$$

the matrix is *not* positive definite?

Can this cause difficulties?

Yes!

1. Motivation by application: Gas transportation through pipelines

2. The Example by Bastin and Coron

2.1 A sufficient condition for non-stabilizability

3. Networked systems

3.1 A network of strings with a stabilizing and a destabilizing source term

3.2 A sufficient condition for stabilizability

3.3 The quasilinear case: A cyclic network (pipeline and channel networks)

4. Time-delay

The example by Bastin & Coron (interval-case)

In *Stability and Boundary Stabilization of 1-d Hyperbolic Systems*, Bastin & Coron consider

$$\begin{pmatrix} U \\ V \end{pmatrix}_t + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}_x + \begin{pmatrix} 0 & \beta \\ \beta & 0 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with a real parameter $\beta > 0$.

The example by Bastin & Coron (interval-case)

In *Stability and Boundary Stabilization of 1-d Hyperbolic Systems*, Bastin & Coron consider

$$\begin{pmatrix} U \\ V \end{pmatrix}_t + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}_x + \begin{pmatrix} 0 & \beta \\ \beta & 0 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with a real parameter $\beta > 0$.

Together with the boundary conditions $U(t, 0) = k V(t, 0)$, $V(t, L) = U(t, L)$

where $k \in (-1, 1)$ is a feedback parameter (and initial data $U(0, \cdot)$, $V(0, \cdot)$)) we have a closed loop-system.

The example by Bastin & Coron (interval-case)

In *Stability and Boundary Stabilization of 1-d Hyperbolic Systems*, Bastin & Coron consider

$$\begin{pmatrix} U \\ V \end{pmatrix}_t + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}_x + \begin{pmatrix} 0 & \beta \\ \beta & 0 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with a real parameter $\beta > 0$.

Together with the boundary conditions $U(t, 0) = k V(t, 0)$, $V(t, L) = U(t, L)$

where $k \in (-1, 1)$ is a feedback parameter (and initial data $U(0, \cdot)$, $V(0, \cdot)$)) we have a closed loop-system.

Is it stable?

The example by Bastin & Coron (interval-case)

In *Stability and Boundary Stabilization of 1-d Hyperbolic Systems*, Bastin & Coron consider

$$\begin{pmatrix} U \\ V \end{pmatrix}_t + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}_x + \begin{pmatrix} 0 & \beta \\ \beta & 0 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with a real parameter $\beta > 0$.

Together with the boundary conditions $U(t, 0) = k V(t, 0)$, $V(t, L) = U(t, L)$

where $k \in (-1, 1)$ is a feedback parameter (and initial data $U(0, \cdot)$, $V(0, \cdot)$)) we have a **closed loop-system**.

Is it stable?

BASTIN & CORON look at *product solutions* of the form

$$\begin{pmatrix} U(t, x) \\ V(t, x) \end{pmatrix} = \exp(\sigma t) \begin{pmatrix} f(x) \\ g(x) \end{pmatrix}.$$

If such a solution can be found with $\sigma > 0$,
the system is *exponentially unstable* and cannot be stabilized.

The example by BASTIN and CORON

Some TECHNICAL DETAILS

We consider the separation ansatz

$$U(t, x) = \exp(\sigma t) f(x), \quad V(t, x) = \exp(\sigma t) g(x) \text{ for the pde } \begin{pmatrix} U \\ V \end{pmatrix}_t + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}_x + \begin{pmatrix} 0 & \beta \\ \beta & 0 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

For $\sigma \in (0, \beta)$ define $\omega = \sqrt{\beta^2 - \sigma^2} > 0$. The pde and $f(0) = k g(0)$ imply

$$\begin{aligned} f(x) &= (\beta + k \sigma) \sin(\omega x) - k \omega \cos(\omega x), \\ g(x) &= -(\sigma + k \beta) \sin(\omega x) - \omega \cos(\omega x). \end{aligned}$$

For $k \neq -1$ we have $f(L) = g(L)$ if $\sigma \in (0, \beta)$ is such that

$$(\sigma + \beta) \frac{\tan(\omega L)}{\omega} = \frac{k-1}{k+1}.$$

For $k = -1$ we have $f(L) = g(L)$ if $\sigma \in (0, \beta)$ is such that $0 = \cos(\sqrt{\beta^2 - \sigma^2} L)$.

This is possible if βL is sufficiently large in the sense that

$$\beta L > \pi/2.$$

The example by BASTIN and CORON

More TECHNICAL DETAILS

For $k \neq -1$, in terms of ωL , the condition for non-stabilizability is

$$\left(\beta L + \sqrt{(\beta L)^2 - (\omega L)^2} \right) \frac{\tan(\omega L)}{\omega L} = \frac{k - 1}{k + 1}. \quad (1)$$

So we analyze the *range* of the function on the left-hand side of the equation!

The example by BASTIN and CORON

More TECHNICAL DETAILS

For $k \neq -1$, in terms of ωL , the condition for non-stabilizability is

$$\left(\beta L + \sqrt{(\beta L)^2 - (\omega L)^2} \right) \frac{\tan(\omega L)}{\omega L} = \frac{k-1}{k+1}. \quad (1)$$

So we analyze the *range* of the function on the left-hand side of the equation!

- We consider

$$G(\beta L) = \inf_{s \in (0, \beta L)} \left[\beta L + \sqrt{(\beta L)^2 - s^2} \right] \frac{\tan(s)}{s}.$$

Note that

$$\frac{\tan(s)}{s}$$

is continuous on $(\frac{\pi}{2}, \pi)$ with range from $-\infty$ to zero. (On $(0, \pi/2)$ only positive values are attained.)

The example by BASTIN and CORON

More TECHNICAL DETAILS

For $k \neq -1$, in terms of ωL , the condition for non-stabilizability is

$$\left(\beta L + \sqrt{(\beta L)^2 - (\omega L)^2} \right) \frac{\tan(\omega L)}{\omega L} = \frac{k-1}{k+1}. \quad (1)$$

So we analyze the *range* of the function on the left-hand side of the equation!

- We consider

$$G(\beta L) = \inf_{s \in (0, \beta L)} \left[\beta L + \sqrt{(\beta L)^2 - s^2} \right] \frac{\tan(s)}{s}.$$

Note that

$$\frac{\tan(s)}{s}$$

is continuous on $(\frac{\pi}{2}, \pi)$ with range from $-\infty$ to zero. (On $(0, \pi/2)$ only positive values are attained.)

We substitute

$y = \beta L$. For $y > 0$, define $G(y) = \inf_{s \in (0, y)} \left[y + \sqrt{y^2 - s^2} \right] \frac{\tan(s)}{s}$.

If $y > \frac{\pi}{2}$, then $G(y) \leq \lim_{s \rightarrow \frac{\pi}{2}^+} \left[y + \sqrt{y^2 - s^2} \right] \frac{\tan(s)}{s} = -\infty$.

If $y \geq \pi$, then $F(y) = \sup_{s \in (0, y)} \left[y + \sqrt{y^2 - s^2} \right] \frac{\tan(s)}{s} \geq \lim_{s \rightarrow \pi^-} \left[\pi + \sqrt{\pi^2 - s^2} \right] \frac{\tan(s)}{s} = 0$.

The *range* of a function

Final TECHNICAL DETAILS

For $y > 0$ we consider

$$F(y) = \sup_{s \in (0, y)} \left[y + \sqrt{y^2 - s^2} \right] \frac{\tan(s)}{s}.$$

$$\text{If } y \geq \pi, \text{ then we have } F(y) \geq \sup_{s \in (\pi/2, \pi]} \left[\pi + \sqrt{\pi^2 - s^2} \right] \frac{\tan(s)}{s} \geq \lim_{s \rightarrow \pi^-} \left[\pi + \sqrt{\pi^2 - s^2} \right] \frac{\tan(s)}{s} = 0.$$

For $y > 0$, define

$$G(y) = \inf_{s \in (0, y)} \left[y + \sqrt{y^2 - s^2} \right] \frac{\tan(s)}{s}.$$

$$\text{If } y > \frac{\pi}{2}, \text{ then } G(y) \leq \lim_{s \rightarrow \frac{\pi}{2}^+} \left[y + \sqrt{y^2 - s^2} \right] \frac{\tan(s)}{s} = -\infty.$$

Thus for $y = \beta L \geq \pi$, the function $\left[y + \sqrt{y^2 - s^2} \right] \frac{\tan(s)}{s}$ takes **all values** in $(-\infty, 0]$ for $s \in (\frac{\pi}{2}, \pi]$!

For all $k \in (-1, 1]$, this implies the instability of the system

because our equation has a solution $s = \omega L \in (\pi/2, \pi)$, where $\omega = \sqrt{\beta^2 - \sigma^2}$ correspond to $\sigma = \sqrt{\beta^2 - \omega^2} \in (0, \beta)$.

In fact, the following proposition is already proved by BASTIN and CORON:

Proposition: If

$$\boxed{\beta L \geq \pi,}$$

there is *no* real value of k such that the closed loop system with the pde

$$\begin{pmatrix} U \\ V \end{pmatrix}_t + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}_x + \begin{pmatrix} 0 & \beta \\ \beta & 0 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and the boundary conditions $U(t, 0) = k V(t, 0)$, $V(t, L) = U(t, L)$
is exponentially stable. The bound is sharp!

In fact, the following proposition is already proved by BASTIN and CORON:

Proposition: If

$$\boxed{\beta L \geq \pi,}$$

there is *no* real value of k such that the closed loop system with the pde

$$\begin{pmatrix} U \\ V \end{pmatrix}_t + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}_x + \begin{pmatrix} 0 & \beta \\ \beta & 0 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and the boundary conditions $U(t, 0) = k V(t, 0)$, $V(t, L) = U(t, L)$
is exponentially stable. The bound is sharp!

Boundary stabilization becomes impossible if the length or the source term is too large!

- The problem is caused by the source term!
- In many applications, the source term is an essential part of the dynamics!

In fact, the following proposition is already proved by BASTIN and CORON:

Proposition: If

$$\boxed{\beta L \geq \pi,}$$

there is *no* real value of k such that the closed loop system with the pde

$$\begin{pmatrix} U \\ V \end{pmatrix}_t + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}_x + \begin{pmatrix} 0 & \beta \\ \beta & 0 \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and the boundary conditions $U(t, 0) = k V(t, 0)$, $V(t, L) = U(t, L)$
is exponentially stable. The bound is sharp!

Boundary stabilization becomes impossible if the length or the source term is too large!

- The problem is caused by the source term!
- In many applications, the source term is an essential part of the dynamics!

How is the situation on *networks*?

1. Motivation by application: Gas transportation through pipelines

2. The Example by Bastin and Coron

2.1 A sufficient condition for non-stabilizability

3. Networked systems

3.1 A network of strings with a stabilizing and a destabilizing source term

3.2 A sufficient condition for stabilizability

3.3 The quasilinear case: A cyclic network (pipeline and channel networks)

4. Time-delay

Now we consider a networks of strings.

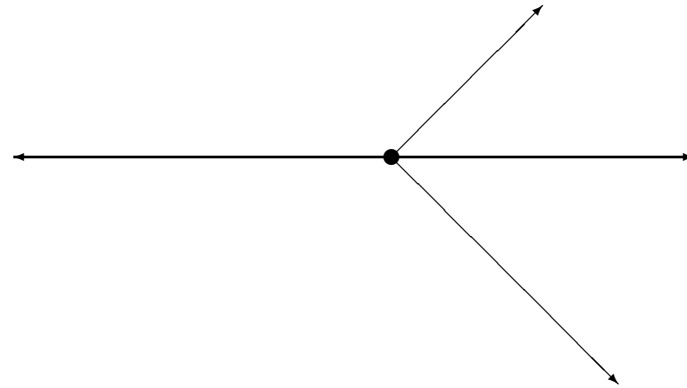


Figure: A star-shaped network with $N = 4$ edges.

Now we consider a networks of strings.

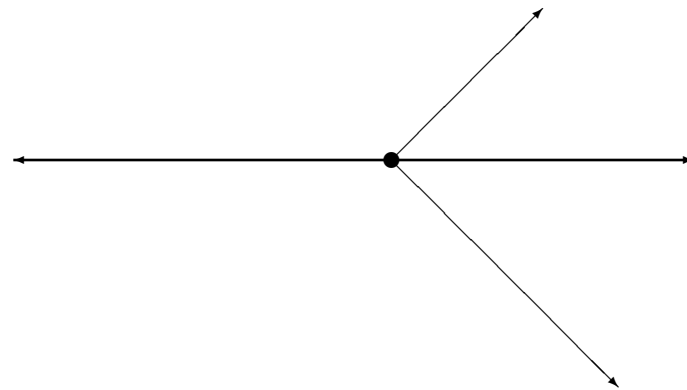


Figure: A star-shaped network with $N = 4$ edges.

- We consider feedback control at all boundary nodes except one homogeneous Dirichlet node.
- Let $N \in \{2, 3, 4, \dots\}$ denote the number of strings.

Now we consider a networks of strings.

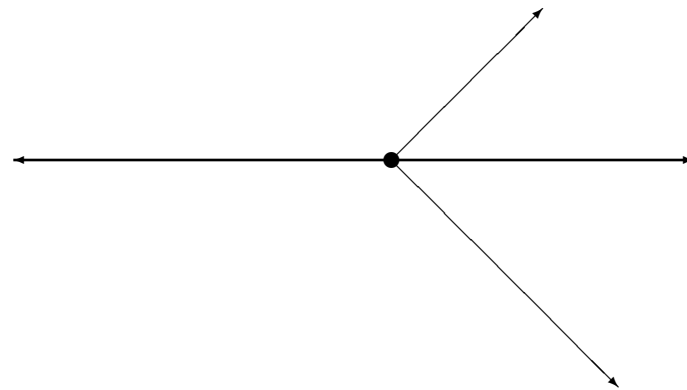


Figure: A star-shaped network with $N = 4$ edges.

- We consider feedback control at all boundary nodes except one homogeneous Dirichlet node.
- Let $N \in \{2, 3, 4, \dots\}$ denote the number of strings.
- For $i \in \{1, 2, \dots, N\}$ let $L_i > 0$, $\beta_i > 0$ and $\varepsilon_i \geq 0$ be given and on $[0, L_i]$ consider the wave equation

$$U_{tt}^i = U_{xx}^i - 2\varepsilon_i U_t^i - (\varepsilon_i^2 - \beta_i^2) U^i = 0.$$

Now we consider a networks of strings.

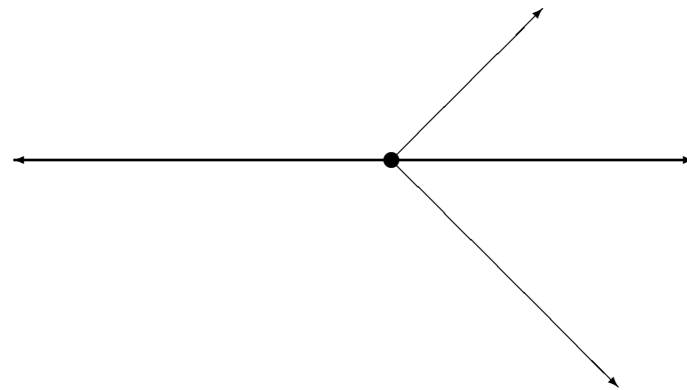


Figure: A star-shaped network with $N = 4$ edges.

- We consider feedback control at all boundary nodes except one homogeneous Dirichlet node.
- Let $N \in \{2, 3, 4, \dots\}$ denote the number of strings.
- For $i \in \{1, 2, \dots, N\}$ let $L_i > 0$, $\beta_i > 0$ and $\varepsilon_i \geq 0$ be given and on $[0, L_i]$ consider the wave equation

$$U_{tt}^i = U_{xx}^i - 2\varepsilon_i U_t^i - (\varepsilon_i^2 - \beta_i^2) U^i = 0.$$

- The edges are coupled at $x = 0$ by node conditions, see the next slide.

A network of strings: Node conditions for coupling

At the central node: Continuity and Kirchhoff

For $i, j \in \{1, \dots, N\}$

$$\begin{aligned} U^i(t, 0) - U^j(t, 0) &= 0, \\ \sum_{k=1}^N U_x^k(t, 0) &= 0. \end{aligned}$$

At the boundary node of edge number 1 at $x = L_1$ we have a homogeneous DIRICHLET condition

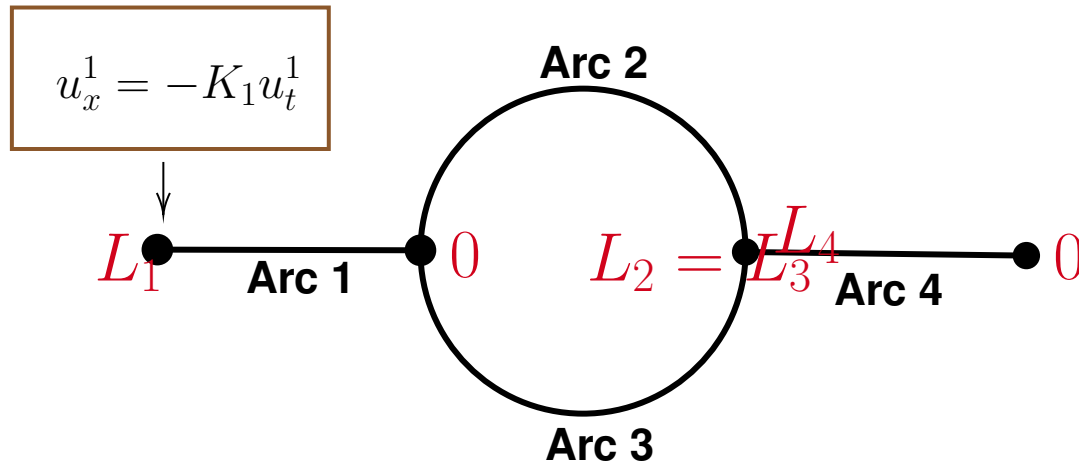
$$U_1(t, L_1) = 0$$

and **at the other boundary nodes** for $j \in \{2, \dots, N\}$ at $x = L_j$ we have a NEUMANN *velocity feedback*

$$U_x^j(t, L_j) = K_j U_t^j(t, L_j)$$

with a real *feedback* parameter K_j .

A system with cyclic graph



A cyclic network with 4 edges: Boundary Feedback control at L_1

- We consider a network with a circle and two additional edges.
- At one boundary node, feedback control takes place.
- At the other boundary node, a homogeneous Dirichlet condition is prescribed.

For $k \in \{1, 2, 3, 4\}$, let $\beta_k > 0, \varepsilon_k \geq 0$ be given.

For $I = \{1, 2, 3, 4\}$, consider the system:

$$\left\{ \begin{array}{l} U_{tt}^k = U_{xx}^k - 2\varepsilon_k U_t^k - (\varepsilon_k^2 - \beta_k^2) U^k, \quad t > 0, x \in (0, L_k), k \in I \\ U^1(t, 0) = U^2(t, 0) = U^3(t, 0), \\ U^2(t, L_2) = U^3(t, L_3) = U^4(t, L_4), \\ \sum_{k=1,2,3} U_x^k(t, 0) = 0, \\ \sum_{k=2,3,4} U_x^k(t, L_k) = 0, \\ U^4(t, 0) = 0, \\ U_x^1(t, L_1) = -K_1 U_t^1(t, L_1). \end{array} \right.$$

(2)

The following result is from GUGAT, HUANG, WANG: *Limits of stabilization of a networked hyperbolic system with a circle*, Control and Cybernetics 52 (2023):

Theorem

Assume that $\beta_k = \beta > 0$, $\varepsilon_k = \varepsilon > 0$, $L_k = L$, that is the length of the arcs in the network and the parameters are the same for all arcs. Assume that for the initial state the C^0 compatibility conditions are satisfied and $\varepsilon \in (0, \beta)$.

- If $L < L_{min} = \frac{\arctan \sqrt{\frac{2}{7}}}{\sqrt{\beta^2 - \varepsilon^2}}$, system (2) is stabilizable (with $|K_1|$ sufficiently small).
- If $L > L_{max} = \frac{\pi}{2\sqrt{\beta^2 - \varepsilon^2}}$, the system is **not** stabilizable.

- If ε is sufficiently close to $\beta > 0$, the value of L_{min} can become arbitrarily **large**.
If β is sufficiently large, the value of L_{max} can become arbitrarily **small**.
- There is still a *gap* between L_{min} and L_{max} . (Open problem)

Now we state a result for the special case where $\varepsilon = 0$. In this case L_{max} is minimal as a function of ε .

Proposition Under the previous assumptions, for $\beta > 0$ and $\varepsilon = 0$, if

$$L > L_{max} = \frac{\pi}{2\beta},$$

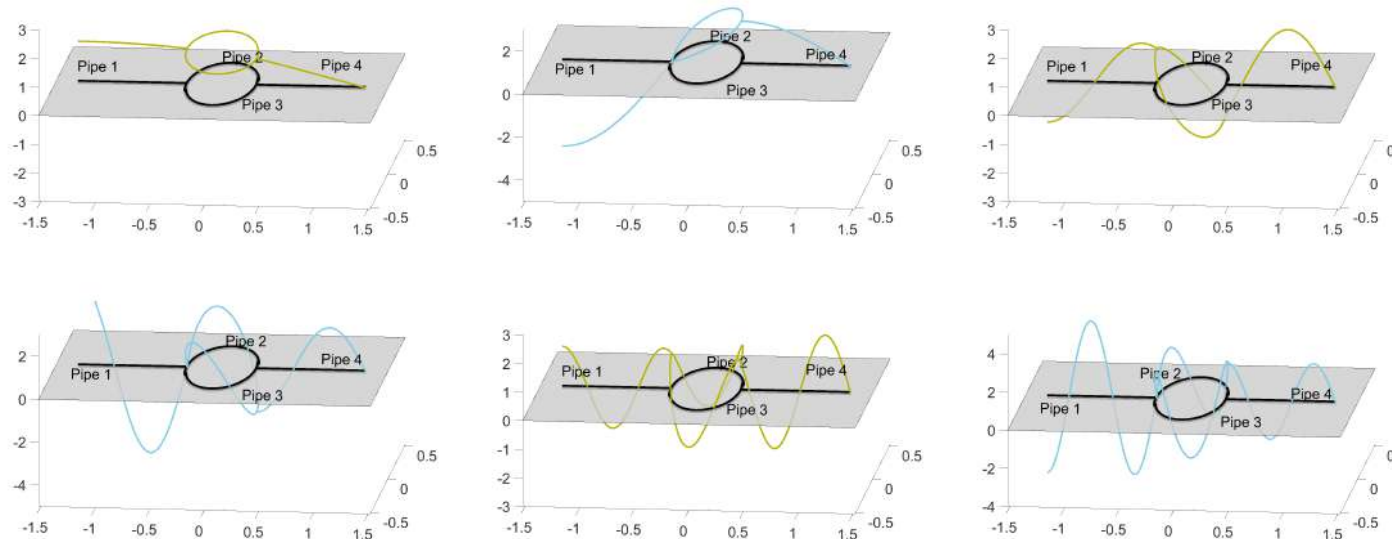
the system is **not** stabilizable.

A system with cyclic graph

Illustrations of eigenfuncions

In this subsection, we present the figures of some of the eigenfunction in the system. We take $\varepsilon = \pi, c = \sqrt{1.01}\varepsilon, L = 1$. We obtain the eigenvalues and the corresponding eigenfuctions. The following figures shows the eigenfunction corresponding to the eigenvalues

$$\lambda_{0,1}^- = -\pi + \sqrt{1.01\pi^2 - \arctan^2 \sqrt{\left(\frac{2}{7}\right)}}, \lambda_{0,2}^- = (-1 + \sqrt{0.76})\pi, \lambda_{1,1}^- = -\pi + \sqrt{-1.01\pi^2 + (\arctan \sqrt{\frac{2}{7}} + \pi)^2}i,$$
$$\lambda_{1,2}^- = -\pi + \sqrt{1.24}\pi i, \lambda_{2,1}^- = -\pi + \sqrt{-1.01\pi^2 + (\arctan \sqrt{\frac{2}{7}} + 2\pi)^2}i, \lambda_{2,2}^- = -\pi + \sqrt{5.24}\pi i.$$



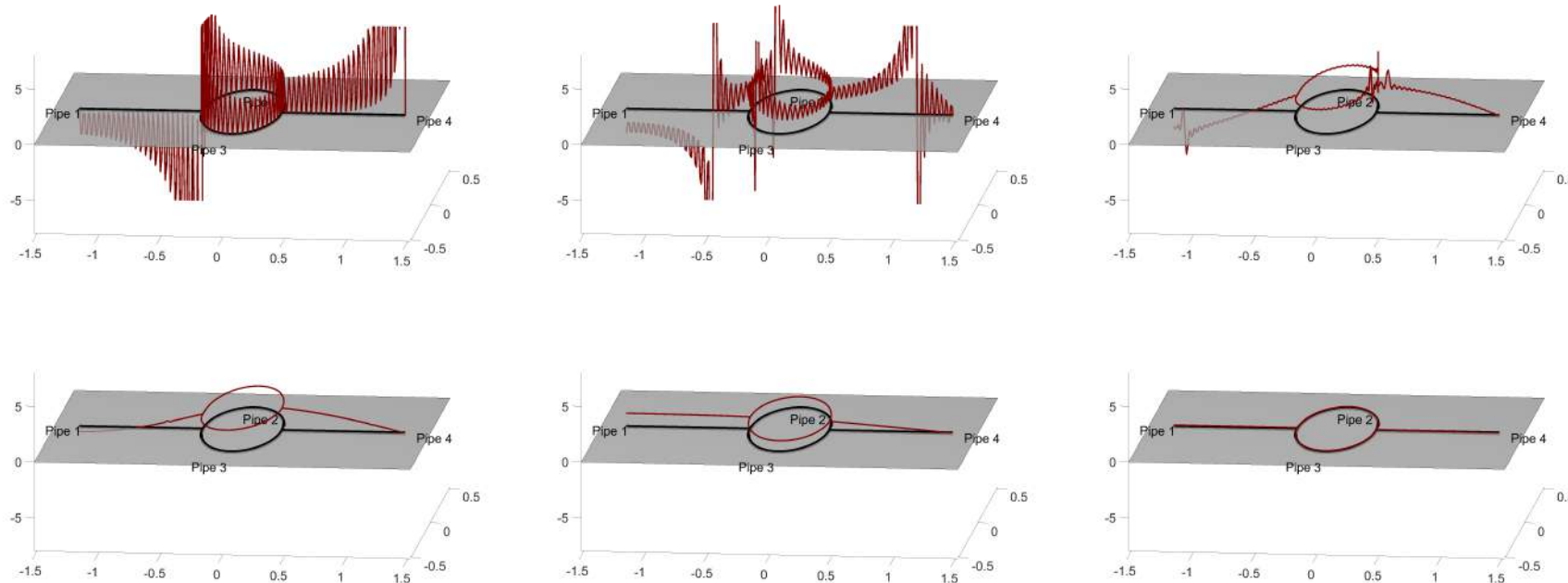
A system with cyclic graph

Illustrations of the evolution

We take $\varepsilon = \pi$, $\beta = \sqrt{1.01}\varepsilon$, Then $L_{\min} = \frac{\arctan \sqrt{\frac{2}{7}}}{\sqrt{\beta^2 - \varepsilon^2}} \approx 1.5625$.

We take $L_1 = 1 < L_{\min}$. The time evolution of the network can be shown.

The initial data contains highly oscillatory parts that vanish rather quickly with time.



Our wave equation can be transformed to a 2×2 system:

Our wave equation can be transformed to a 2×2 system:

For $i \in \{1, \dots, N\}$ define $V^i = -\frac{1}{\beta_i} (U_t^i + U_x^i + \varepsilon_i U^i)$.

Our wave equation can be transformed to a 2×2 system:

For $i \in \{1, \dots, N\}$ define $V^i = -\frac{1}{\beta_i} (U_t^i + U_x^i + \varepsilon_i U^i)$.

Then due to the definition of V^i and (14), the function (U^i, V^i) solve

$$\begin{pmatrix} U^i \\ V^i \end{pmatrix}_t + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} U^i \\ V^i \end{pmatrix}_x + \begin{pmatrix} \varepsilon_i & \beta_i \\ \beta_i & \varepsilon_i \end{pmatrix} \begin{pmatrix} U^i \\ V^i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Relation to first order system

Our wave equation can be transformed to a 2×2 system:

For $i \in \{1, \dots, N\}$ define $V^i = -\frac{1}{\beta_i} (U_t^i + U_x^i + \varepsilon_i U^i)$.

Then due to the definition of V^i and (14), the function (U^i, V^i) solve

$$\begin{pmatrix} U^i \\ V^i \end{pmatrix}_t + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} U^i \\ V^i \end{pmatrix}_x + \begin{pmatrix} \varepsilon_i & \beta_i \\ \beta_i & \varepsilon_i \end{pmatrix} \begin{pmatrix} U^i \\ V^i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Also the node conditions and boundary conditions can be transformed similarly:

Relation to first order system

Our wave equation can be transformed to a 2×2 system:

For $i \in \{1, \dots, N\}$ define $V^i = -\frac{1}{\beta_i} (U_t^i + U_x^i + \varepsilon_i U^i)$.

Then due to the definition of V^i and (14), the function (U^i, V^i) solve

$$\begin{pmatrix} U^i \\ V^i \end{pmatrix}_t + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} U^i \\ V^i \end{pmatrix}_x + \begin{pmatrix} \varepsilon_i & \beta_i \\ \beta_i & \varepsilon_i \end{pmatrix} \begin{pmatrix} U^i \\ V^i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Also the node conditions and boundary conditions can be transformed similarly:

For example, the homogenous DIRICHLET conditions yield

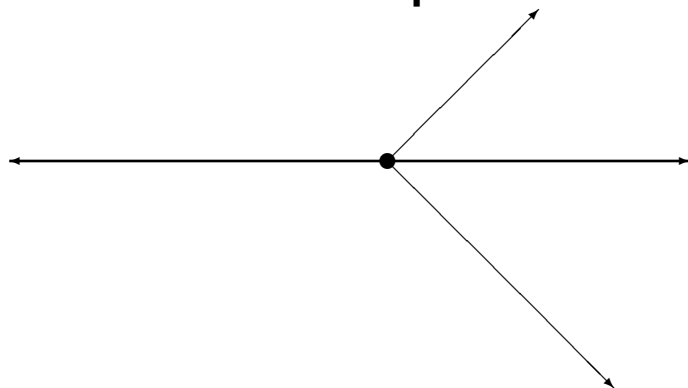
$$V^1(t, L_1) = -\frac{1}{\beta_1} (U_x^1 + U_t^1).$$

The **feedback law** becomes

$$V^i(t, L_i) = -\frac{1}{\beta_i} (\varepsilon_i U^i + (K_i + 1) U_t^i).$$

-
- Also for our **cyclic graph**, *boundary feedback stabilization is not always possible!*

- Also for our **cyclic graph**, *boundary feedback stabilization is not always possible!*
- Now we consider a **star-shaped** networks of strings.



- We consider feedback control at all boundary nodes except one homogeneous Dirichlet node.

A star of strings

Gugat Gerster Syst. Control Lett. 131(2019)

Limits of stabilizability: Assume that for all $i \in \{1, \dots, N\}$ we have $\beta_i > \varepsilon_i$

A star of strings

Gugat Gerster Syst. Control Lett. 131(2019)

Limits of stabilizability: Assume that for all $i \in \{1, \dots, N\}$ we have $\beta_i > \varepsilon_i$

- **Instability if ONE edge is sufficiently long:**

If

$$\beta_1^2 > \varepsilon_1^2 + \frac{9}{4} \frac{\pi^2}{L_1^2} \quad \text{and} \quad \beta_1 - \varepsilon_1 \leq \min_{i \in \{2, \dots, N\}} \{\beta_i - \varepsilon_i\}, \quad (3)$$

there are **no values** of $K_2, \dots, K_N \in (-\infty, 0]$ such that the closed loop system is **asymptotically stable**.

Limits of stabilizability: Assume that for all $i \in \{1, \dots, N\}$ we have $\beta_i > \varepsilon_i$

- **Instability if ONE edge is sufficiently long:**

If

$$\beta_1^2 > \varepsilon_1^2 + \frac{9}{4} \frac{\pi^2}{L_1^2} \quad \text{and} \quad \beta_1 - \varepsilon_1 \leq \min_{i \in \{2, \dots, N\}} \{\beta_i - \varepsilon_i\}, \quad (3)$$

there are **no values** of $K_2, \dots, K_N \in (-\infty, 0]$ such that the closed loop system is **asymptotically stable**.

- **Instability for a large number of short edges:**

If

$$\beta_i = \beta_1, \quad \varepsilon_i = \varepsilon_1, \quad L_i = L_1, \quad K_i = K_2 \quad \text{and} \quad \sin^2(\sqrt{\beta_1^2 - \varepsilon_1^2} L_1) = \frac{1}{N},$$

there are **no values** of $K_2 \in (-\infty, \infty)$ ($j \in \{2, \dots, N\}$) such that the closed loop system is asymptotically stable. This implies

$$L_1^2 (\beta_1^2 - \varepsilon_1^2) = \left(\arcsin\left(\frac{1}{\sqrt{N}}\right) \right)^2.$$

Since we have $\lim_{N \rightarrow \infty} \arcsin\left(\frac{1}{\sqrt{N}}\right) = 0$, for N *sufficiently large* we obtain *arbitrarily small values of the lengths* $L_i > 0$, for which the system is not exponentially stable!

So here the *total length* of the strings $N L_1^2$ must be sufficiently large!

Assume that $\beta_i = \beta_1$, $\varepsilon_i = \varepsilon_1$, $L_i = L_1$, $K_i = K_2$. If $\varepsilon_1 > 0$ and

$$\varepsilon_1 \geq \beta_1 \geq 0$$

with $K_2 = 0$ the closed loop system is exponentially stable.

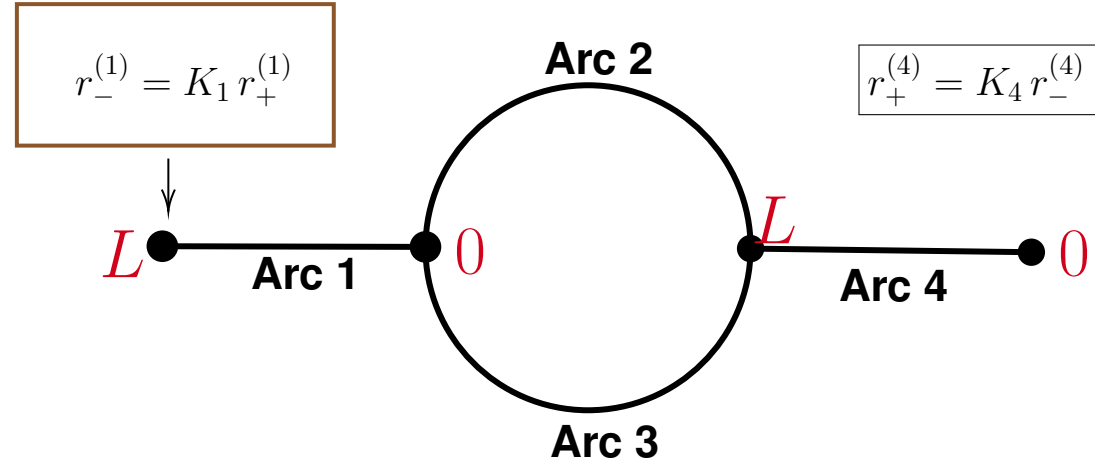
Assume that $\beta_i = \beta_1$, $\varepsilon_i = \varepsilon_1$, $L_i = L_1$, $K_i = K_2$. If $\varepsilon_1 > 0$ and

$$\varepsilon_1 \geq \beta_1 \geq 0$$

with $K_2 = 0$ the closed loop system is exponentially stable.

- This can be shown by the analysis of the *eigenfunctions* of the system.
- For $\varepsilon_1 \geq \beta_1 \geq 0$, the matrix of the source term is positive semi-definite.

Consider again the network



- For $\theta = 0$, **stabilization is impossible** for a *cyclic* network also if the length is small!

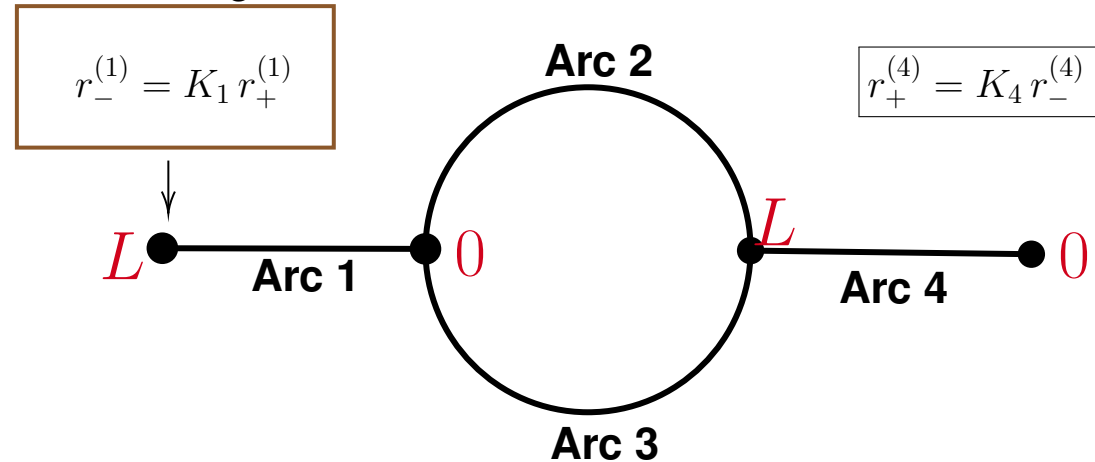
but with the *isothermal* EULER equations on the edges. At the interior nodes $\bar{x} \in \{0, L\}$, the coupling is

$$\begin{pmatrix} r_+^{(1)}(t, \bar{x}) \\ r_+^{(2)}(t, \bar{x}) \\ r_+^{(3)}(t, \bar{x}) \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} r_-^{(1)}(t, \bar{x}) \\ r_-^{(2)}(t, \bar{x}) \\ r_-^{(3)}(t, \bar{x}) \end{pmatrix}$$

Boundary Feedback control at $x_1 = L$, $x_4 = 0$.

The matrix in the coupling conditions (conservation of mass and continuity of pressure) in terms of Riemann invariants **is orthogonal!**

Consider again the network



but with the *isothermal* EULER equations on the edges. At the interior nodes $\bar{x} \in \{0, L\}$, the coupling is

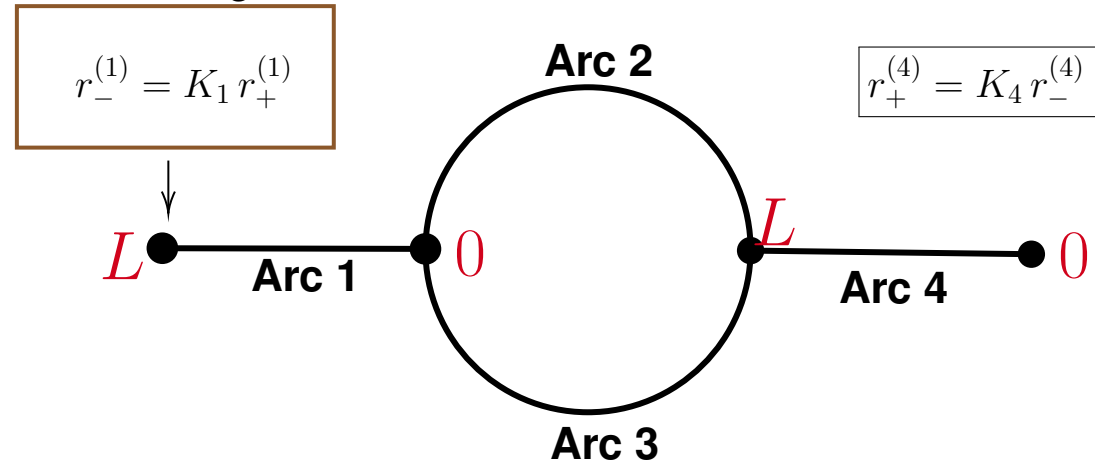
$$\begin{pmatrix} r_+^{(1)}(t, \bar{x}) \\ r_+^{(2)}(t, \bar{x}) \\ r_+^{(3)}(t, \bar{x}) \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} r_-^{(1)}(t, \bar{x}) \\ r_-^{(2)}(t, \bar{x}) \\ r_-^{(3)}(t, \bar{x}) \end{pmatrix}$$

Boundary Feedback control at $x_1 = L, x_4 = 0$.

The matrix in the coupling conditions (conservation of mass and continuity of pressure) in terms of Riemann invariants **is orthogonal!**

- For $\theta = 0$, **stabilization is impossible** for a *cyclic* network also **if the length is small!**
This can be seen as follows: Take a steady state with constant pressure, $v > 0$ in the cycle and $v = 0$ outside.

Consider again the network



but with the *isothermal* EULER equations on the edges. At the interior nodes $\bar{x} \in \{0, L\}$, the coupling is

$$\begin{pmatrix} r_+^{(1)}(t, \bar{x}) \\ r_+^{(2)}(t, \bar{x}) \\ r_+^{(3)}(t, \bar{x}) \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} r_-^{(1)}(t, \bar{x}) \\ r_-^{(2)}(t, \bar{x}) \\ r_-^{(3)}(t, \bar{x}) \end{pmatrix}$$

Boundary Feedback control at $x_1 = L, x_4 = 0$.

The matrix in the coupling conditions (conservation of mass and continuity of pressure) in terms of Riemann invariants is **orthogonal**!

- For $\theta = 0$, **stabilization is impossible** for a *cyclic* network also if the **length is small**! This can be seen as follows: Take a steady state with constant pressure, $v > 0$ in the cycle and $v = 0$ outside.
- A remedy is to include control action at an interior node, see *Stabilization of a cyclic network of strings by nodal control*, J. Evol. Equ. 2024, M. Gugat

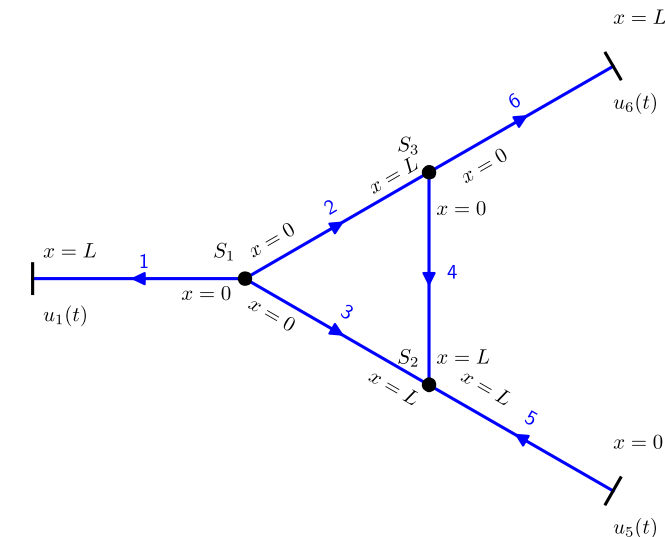


Figure: The graph of the string network

The quasi-linear case

Evolution of a Lyapunov function

We need an **additional damper** at a *cycle node*: Replace the coupling matrix Q at the **interior node** $x = L$ by μA !

Consider the Lyapunov function

$$\mathcal{E}^{(i)}(t) = \frac{1}{2} \int_{-L}^L w_+(x) |r_+^{(i)}(t, x)|^2 + w_-(x) |r_-^{(i)}(t, x)|^2 dx,$$

$$\mathcal{E}(t) = \mathcal{E}^{(1)}(t) + \mathcal{E}^{(2)}(t) + \mathcal{E}^{(3)}(t) + \mathcal{E}^{(4)}(t)$$

with **hyperbolic weights**

$$w_{\pm}(x) = \sqrt{v} \cosh(\psi(x + \eta)) \mp \sinh(\psi(x + \eta)),$$

$\psi > 0$, $v > \tanh^2(\psi(L + |\eta|))$ see

New Lyapunov functions for systems with source terms, Control Cybern. 2024, *BOUNDARY STABILIZATION OF QUASI-LINEAR HYPERBOLIC SYSTEMS WITH VARYING TIME-DELAY*, SICON 2025, M. Gugat.

- For $v = 1$, we obtain the exponential weights $w_+(x) = \exp(-\psi(x + \eta))$, $w_-(x) = \exp(\psi(x + \eta))$, (see CORON, BASTIN, D'ANDREA-NOVEL, IEEE Trans. Autom. Control. 2007).

- Let $\psi > 0$, η and $v > \tanh^2(\psi(L + |\eta|))$ be given.
For $x \in [-L, L]$, define

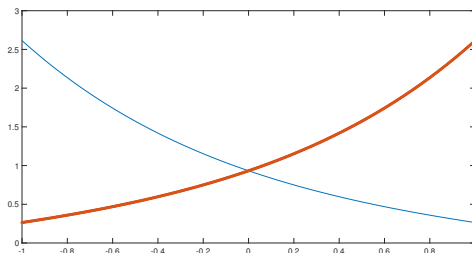
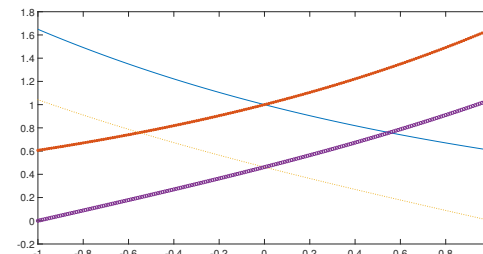
$$h_{\pm}(x) = \sqrt{v} \cosh(\psi x) \mp \sinh(\psi x),$$

- (a) The functions $h_{\pm}(\cdot + \eta)$ are *strictly positive* and *strictly convex* on $[-L, L]$ (since $h''_{\pm} = \psi^2 h_{\pm}$).
- (b) We have

$$h'_+(x + \eta) = -\psi \left(\frac{v+1}{2\sqrt{v}} \right) h_+(x + \eta) - \psi \left[\frac{1-v}{2\sqrt{v}} \right] h_-(x + \eta),$$

$$h'_-(x + \eta) = \psi \left(\frac{1-v}{2\sqrt{v}} \right) h_+(x + \eta) + \psi \left[\frac{v+1}{2\sqrt{v}} \right] h_-(x + \eta).$$

If $v \in (\tanh^2(\psi(L + |\eta|)), 1]$, $h_+(\cdot + \eta)$ is decreasing and $h_-(\cdot + \eta)$ is increasing.



The quasi-linear case

Evolution of the Lyapunov function

For a steady state $\bar{R}$, let $r = R - \bar{R}$ and ε_0 be such that

$$|\partial_x (\lambda_{\pm}(x, \bar{R} - \bar{R})| < \varepsilon_0 \quad (4)$$

Consider the system in diagonal form

$$\begin{pmatrix} r_+^{(i)} \\ r_-^{(i)} \end{pmatrix}_t + \begin{pmatrix} \lambda_+(x, r^{(i)}) & 0 \\ 0 & \lambda_-(x, r^{(i)}) \end{pmatrix} \begin{pmatrix} r_+^{(i)} \\ r_-^{(i)} \end{pmatrix}_x = \begin{pmatrix} G_+(x, r^{(i)}) \\ G_-(x, r^{(i)}) \end{pmatrix}. \quad (5)$$

The quasi-linear case

Evolution of the Lyapunov function

For a steady state $\bar{R}$, let $r = R - \bar{R}$ and ε_0 be such that

$$|\partial_x (\lambda_{\pm}(x, \bar{R} - \bar{R})| < \varepsilon_0 \quad (4)$$

Consider the system in diagonal form

$$\begin{pmatrix} r_+^{(i)} \\ r_-^{(i)} \end{pmatrix}_t + \begin{pmatrix} \lambda_+(x, r^{(i)}) & 0 \\ 0 & \lambda_-(x, r^{(i)}) \end{pmatrix} \begin{pmatrix} r_+^{(i)} \\ r_-^{(i)} \end{pmatrix}_x = \begin{pmatrix} G_+(x, r^{(i)}) \\ G_-(x, r^{(i)}) \end{pmatrix}. \quad (5)$$

Assume that $|G_{\pm}(x, r)| \leq \beta (|r_+| + |r_-|)$.

The quasi-linear case

Evolution of the Lyapunov function

For a steady state $\bar{R}$, let $r = R - \bar{R}$ and ε_0 be such that

$$|\partial_x (\lambda_{\pm}(x, \bar{R} - \bar{R})| < \varepsilon_0 \quad (4)$$

Consider the system in diagonal form

$$\begin{pmatrix} r_+^{(i)} \\ r_-^{(i)} \end{pmatrix}_t + \begin{pmatrix} \lambda_+(x, r^{(i)}) & 0 \\ 0 & \lambda_-(x, r^{(i)}) \end{pmatrix} \begin{pmatrix} r_+^{(i)} \\ r_-^{(i)} \end{pmatrix}_x = \begin{pmatrix} G_+(x, r^{(i)}) \\ G_-(x, r^{(i)}) \end{pmatrix}. \quad (5)$$

Assume that $|G_{\pm}(x, r)| \leq \beta (|r_+| + |r_-|)$.

Theory of semi-global solutions (see TATSIEN LI)

For given $\varepsilon_0 > 0$, $T > 0$, there exists $\varepsilon_T > 0$ such that for all initial states $r \in \mathcal{B}_{\varepsilon_T}^{C^1}(0)$ that are C^1 -compatible with the feedback laws and the node conditions the system has a classical solution on $[0, T]$ that satisfies

$|\partial_x (\lambda_{\pm}(x, \bar{R} - \bar{R})| < \varepsilon_0$ and for some $\delta \geq \gamma > 0$ we have

$$-\delta \leq \lambda_-(\cdot) \leq -\gamma, \quad \gamma \leq \lambda_+(\cdot) \leq \delta. \quad (6)$$

The quasi-linear case

Evolution of the Lyapunov function

For a steady state $\bar{R}$, let $r = R - \bar{R}$ and ε_0 be such that

$$|\partial_x (\lambda_{\pm}(x, \bar{R} - \bar{R})| < \varepsilon_0 \quad (4)$$

Consider the system in diagonal form

$$\begin{pmatrix} r_+^{(i)} \\ r_-^{(i)} \end{pmatrix}_t + \begin{pmatrix} \lambda_+(x, r^{(i)}) & 0 \\ 0 & \lambda_-(x, r^{(i)}) \end{pmatrix} \begin{pmatrix} r_+^{(i)} \\ r_-^{(i)} \end{pmatrix}_x = \begin{pmatrix} G_+(x, r^{(i)}) \\ G_-(x, r^{(i)}) \end{pmatrix}. \quad (5)$$

Assume that $|G_{\pm}(x, r)| \leq \beta (|r_+| + |r_-|)$.

Theory of semi-global solutions (see TATSIEN LI)

For given $\varepsilon_0 > 0$, $T > 0$, there exists $\varepsilon_T > 0$ such that for all initial states $r \in \mathcal{B}_{\varepsilon_T}^{C^1}(0)$ that are C^1 -compatible with the feedback laws and the node conditions the system has a classical solution on $[0, T]$ that satisfies

$|\partial_x (\lambda_{\pm}(x, \bar{R} - \bar{R})| < \varepsilon_0$ and for some $\delta \geq \gamma > 0$ we have

$$-\delta \leq \lambda_-(\cdot) \leq -\gamma, \quad \gamma \leq \lambda_+(\cdot) \leq \delta. \quad (6)$$

Stability-Theorem: Assume that the closed loop system has a classical solution on $[0, T]$

such that (4) and (6) hold for all $t \in [0, T]$, that we have $(2\varepsilon_0 + 6\beta)\delta L < \gamma^2$ and the feedback gains $|K_1|$, $|K_4|$ and $|\mu|$ are sufficiently small

Then the closed loop system (the L^2 -norm of r) **decays exponentially fast** on $[0, T]$.

The quasi-linear case

Decay of the Lyapunov function: Choice of ψ , η , v

Under the assumptions of the Stability-Theorem, there exist weight-parameters

$$\psi > 0, \eta > 0, v \in (\tanh^2(\psi(L + \eta)), 1]$$

such that

$$\sqrt{v} \leq \tanh(\psi\eta) \frac{1 + \frac{\gamma}{\delta}}{1 - \frac{\gamma}{\delta}}, \quad (7)$$

and

$$3\beta + \varepsilon_0 \leq \frac{\gamma}{2} \psi \frac{1 - v}{\sqrt{v}}. \quad (8)$$

For each such choice, $K_1 > 0$, $K_4 > 0$, $\mu \in (0, 1)$ can be chosen such that the **Lyapunov function** $\mathcal{E}(t)$ decays exponentially with the rate

$$\tilde{\psi} = \gamma \psi \sqrt{v}. \quad (9)$$

Our analysis also applies to networks of water-channels,

where the dynamics on each edge is governed by the

Saint-Venant equations.

See e.g. the work of AMAURY HAYAT and PEIPEI SHANG.



Less regular solutions

Although in many applications these states are not desirable, also solutions with shocks can be considered, see e.g. *Feedback stabilization for entropy solutions of a 2×2 hyperbolic system of conservation laws at a junction*, by GIUSEPPE MARIA COCLITE, NICOLA DE NITTI, MAURO GARAVELLO , FRANCESCA MARCELLINI.

The analysis is only for pipes **without friction** (all steady states are constant).

- The graph is *star-shaped* with feedback at the boundary nodes.



non-classical solutions

- For solutions in L^2 (Finite-Energy), the Lyapunov analysis could be adapted.
LIONS P.L., PERTHAME B., TADMOR E., *Kinetic formulation for the isentropic gas dynamics and p -system*, Commun. Math. Phys., 163 (1994), pp. 415-431

1. Motivation by application: Gas transportation through pipelines

2. The Example by Bastin and Coron

2.1 A sufficient condition for non-stabilizability

3. Networked systems

3.1 A network of strings with a stabilizing and a destabilizing source term

3.2 A sufficient condition for stabilizability

3.3 The quasilinear case: A cyclic network (pipeline and channel networks)

4. Time-delay

The example of Bastin and Coron:

The influence of constant time-delay

Let $\tau > 0$ be a given time delay. For $t \geq 2\tau$, we replace the feedback law by

$$\delta_+(t, 0) = k \delta_-(t - \tau, 0).$$

To complete the system, a **compatible starting phase** on the time interval $[0, 2\tau)$ has to be added.

The example of Bastin and Coron:

The influence of constant time-delay

Let $\tau > 0$ be a given time delay. For $t \geq 2\tau$, we replace the feedback law by

$$\delta_+(t, 0) = k \delta_-(t - \tau, 0).$$

To complete the system, a **compatible starting phase** on the time interval $[0, 2\tau)$ has to be added.

For $t \in [0, 2\tau)$ we define

$$\delta_+(t, 0) = k \delta_-(t - \zeta(t), 0).$$

We choose $\zeta(t)$ as a smooth function with $\zeta(0) = 0$, $\zeta(2\tau) = \tau$, $\zeta'(0) = \zeta'(2\tau) = 0$ and $\zeta(t) \leq t$ for all $t \in [0, 2\tau]$.

The example of Bastin and Coron:

The influence of constant time-delay

Let $\tau > 0$ be a given time delay. For $t \geq 2\tau$, we replace the feedback law by

$$\delta_+(t, 0) = k \delta_-(t - \tau, 0).$$

To complete the system, a **compatible starting phase** on the time interval $[0, 2\tau)$ has to be added.

For $t \in [0, 2\tau)$ we define

$$\delta_+(t, 0) = k \delta_-(t - \zeta(t), 0).$$

We choose $\zeta(t)$ as a smooth function with $\zeta(0) = 0$, $\zeta(2\tau) = \tau$, $\zeta'(0) = \zeta'(2\tau) = 0$ and $\zeta(t) \leq t$ for all $t \in [0, 2\tau]$.

Proposition, see *New Lyapunov functions for systems with source terms*, (Control and Cybernetics 2024)

Let $\hat{k} > 0$ be given. If

$$\mathcal{M}L \in \left(\frac{3}{4}\pi, \pi \right) \tag{10}$$

and $\tau > 0$ is sufficiently large, there is no value of $k \in (-\hat{k}, \hat{k})$ such that the closed loop system is exponentially stable.

The example of Bastin and Coron:

The influence of constant time-delay

Let $\tau > 0$ be a given time delay. For $t \geq 2\tau$, we replace the feedback law by

$$\delta_+(t, 0) = k \delta_-(t - \tau, 0).$$

To complete the system, a **compatible starting phase** on the time interval $[0, 2\tau)$ has to be added.

For $t \in [0, 2\tau)$ we define

$$\delta_+(t, 0) = k \delta_-(t - \zeta(t), 0).$$

We choose $\zeta(t)$ as a smooth function with $\zeta(0) = 0$, $\zeta(2\tau) = \tau$, $\zeta'(0) = \zeta'(2\tau) = 0$ and $\zeta(t) \leq t$ for all $t \in [0, 2\tau]$.

Proposition, see *New Lyapunov functions for systems with source terms*, (Control and Cybernetics 2024)

Let $\hat{k} > 0$ be given. If

$$\mathcal{M}L \in \left(\frac{3}{4}\pi, \pi \right) \tag{10}$$

and $\tau > 0$ is sufficiently large, there is no value of $k \in (-\hat{k}, \hat{k})$ such that the closed loop system is exponentially stable.

- Sufficiently large time-delay leads to a decrease of the critical length!
- Is this bound sharp? To be clarified (best in the workshop)!
- At this point, it is appropriate to mention DATKO's classical contributions to the study of time-delay, where it is shown that *arbitrarily small time-delay can destabilize a system that is otherwise stable*.

With MARIUS TUCSNAK:

Boundary feedback control of a vibrating string with piecewise constant (i.e. switching) time-delay, see

Gugat, Martin, and Marius Tucsnak.

An example for the switching delay feedback stabilization of an infinite dimensional system: The boundary stabilization of a string, Systems & Control Letters 60.4 (2011): 226-23.

- The Neumann boundary condition with delay is

$$y_x(t, L) = k y_t(t - \delta(t), L),$$

where $\delta(t)$ is a simple function with values in e.g. $\{4\frac{L}{c}, 8\frac{L}{c}\}$.

- Certain delays do not cause a problem!

The time-delay is described by a continuously differentiable function $\tau(t)$

with $\tau(t) \geq 0$ and $\tau'(t) < 1$. Thus we have $t - \tau(t) \geq -\tau(0)$. Assume that

$$\sup_t \tau(t) = \bar{\tau} < 1, \quad \sup_t |\tau'(t)| = \hat{\tau} < 1. \quad (11)$$

In order to **initialize the system with time-delay**, we proceed as follows:

Let $t_0 > \bar{\tau}$ be given such that $\tau(t_0) = \bar{\tau}$. For $t \in [0, t_0]$ define the function

$$\vartheta_0(t) = \left(1 - \frac{t}{t_0}\right)^2 \left(5 - 2\frac{t}{t_0}\right).$$

Then we have

$$\vartheta_0(t_0) = \vartheta'_0(t_0) = 0, \quad \vartheta_0(2t_0) = 1, \quad \vartheta'_0(2t_0) = 0. \quad (12)$$

Let $\sigma \in \{-1, 1\}$ and the initial state $y_+, y_- \in C^1([0, L])$ be given such that at $x = 0$ the C^1 -compatibility conditions

$$y_+(0) = \sigma y_-(0), \quad (13)$$

$$-\lambda_+(0, y_+(0), y_-(0)) y'_+(0) + G_+(0, y_+(0), y_-(0)) = \sigma \left[-\lambda_-(0, y_+(0), y_-(0)) y'_-(0) + G_-(0, y_+(0), y_-(0)) \right]$$

hold.

The influence of varying time-delay

Compatible starting phase

- We consider the initial condition

$$\delta_+(0, x) = y_+(x), \quad \delta_-(0, x) = y_-(x). \quad (14)$$

Compatible starting phase

- We consider the initial condition

$$\delta_+(0, x) = y_+(x), \quad \delta_-(0, x) = y_-(x). \quad (14)$$

- Let a function $f_0 \in C^1([0, 2t_0])$ be given that satisfies the C^1 compatibility conditions with (14) at $x = L$, that is $f_0(0) = y_-(L)$,

$$f'_0(0) = G_-(L, y_+(L), y_-(L)) - \lambda_-(L, y_+(L), y_-(L)) y'_-(L).$$

For $t \geq 0$ system (5) is completed with the boundary conditions

$$\delta_+(t, 0) = \sigma \delta_-(t, 0) \quad (15)$$

and (with a real number k_L that serves as the feedback gain)

$$\delta_-(t, L) = \begin{cases} f_0(t), & t \in [0, t_0) \\ (1 - \vartheta_0(t)) f_0(t) + \vartheta_0(t) k_L \delta_+(t - \tau(t), L), & t \in [t_0, 2t_0) \\ k_L \delta_+(t - \tau(t), L) & t \in (2t_0, T]. \end{cases} \quad (16)$$

Due to (12) the boundary condition is continuously differentiable with respect to t .

The influence of time-delay

The quasi-linear case interval $[0, L]$

Theorem StabiD: Assume that the closed loop system has a classical solution on $[0, T]$

such that (4) and (6) hold for all $t \in [0, T]$, that we have $(2\varepsilon_0 + 6\beta)\delta L < \gamma^2$ and the feedback gains $|K_0|, |K_L|$ are sufficiently small.

Then the closed loop system (the L^2 -norm of r) **decays exponentially fast** on $[0, T]$.

Theorem StabiD: Assume that the closed loop system has a classical solution on $[0, T]$

such that (4) and (6) hold for all $t \in [0, T]$, that we have $(2\varepsilon_0 + 6\beta)\delta L < \gamma^2$ and the feedback gains $|K_0|, |K_L|$ are sufficiently small.

Then the closed loop system (the L^2 -norm of r) **decays exponentially fast** on $[0, T]$.

- To be precise that is there exist constants $C_1 > 0$ and $\tilde{\psi} > 0$ that are independent of T such that for all $t \in [2t_0, T]$

$$\begin{aligned} & \int_0^L \delta_+^2(t, x) + \delta_-^2(t, x) dx \\ & \leq C_1 e^{-\tilde{\psi}(t-2t_0)} \left(\int_0^L \delta_+^2(2t_0, x) + \delta_-^2(2t_0, x) dx + \int_0^{\tau(2t_0)} e^{-\tilde{\psi}s} \delta_+^2(2t_0 - s, L) ds \right). \end{aligned} \quad (17)$$

Theorem StabiD: Assume that the closed loop system has a classical solution on $[0, T]$

such that (4) and (6) hold for all $t \in [0, T]$, that we have $(2\varepsilon_0 + 6\beta)\delta L < \gamma^2$ and the feedback gains $|K_0|, |K_L|$ are sufficiently small.

Then the closed loop system (the L^2 -norm of r) **decays exponentially fast** on $[0, T]$.

- To be precise that is there exist constants $C_1 > 0$ and $\tilde{\psi} > 0$ that are independent of T such that for all $t \in [2t_0, T]$

$$\begin{aligned} & \int_0^L \delta_+^2(t, x) + \delta_-^2(t, x) dx \\ & \leq C_1 e^{-\tilde{\psi}(t-2t_0)} \left(\int_0^L \delta_+^2(2t_0, x) + \delta_-^2(2t_0, x) dx + \int_0^{\tau(2t_0)} e^{-\tilde{\psi}s} \delta_+^2(2t_0 - s, L) ds \right). \end{aligned} \quad (17)$$

- In the proof, in the Lyapunov function a **delay term** $\mathcal{D}(t)$ is added to $\mathcal{E}(t)$:

$$\mathcal{D}(t) := \frac{K_L}{2} \int_0^{\tau(t)} \exp(-\tilde{\psi}s) |\delta_+(t-s, L)|^2 ds. \quad (18)$$

See *BOUNDARY STABILIZATION OF QUASI-LINEAR HYPERBOLIC SYSTEMS WITH VARYING TIME-DELAY*, SICON 2025, M. Gugat.



- Thank you for your attention!



- **Thank you for your attention!**

Acknowledgements This work was supported by Deutsche Forschungsgemeinschaft (DFG) in the Collaborative Research Centre CRC/Transregio 154, Mathematical Modelling, Simulation and Optimization Using the Example of Gas Networks, Projects C03 and C05. This work was also supported by DFG in the Project *Regelung interagierender Partikelsysteme und ihre probabilistischen und fluid-dynamischen Beschreibungen*, Projektnummer: 560288187.