

Control setting & separation

$$\dot{y}(t) + Ay(t) = Bu(t), \quad y(0) = y_0, \quad u(t) \in \mathcal{U} \subset U.$$

- X, U real Hilbert spaces; $-A$ generates $S_t = e^{-tA}$; $B \in \mathcal{L}(U, X)$.
- $\mathcal{U}$: nonempty, closed, bounded; $\mathcal{R}_{T,\mathcal{U}}(y_0)$ closed and convex.
- $\mathcal{Y}_f \subset X$: nonempty, closed, convex.

$$E_{\mathcal{U}} = \{u \in L^2(0, T; U) : u(t) \in \mathcal{U} \text{ a.e.}\},$$

$$L_T u = \int_0^T S_{T-t} B u(t) dt, \quad \mathcal{R}_{T,\mathcal{U}}(y_0) = S_T y_0 + L_T E_{\mathcal{U}}.$$

Separation equivalence

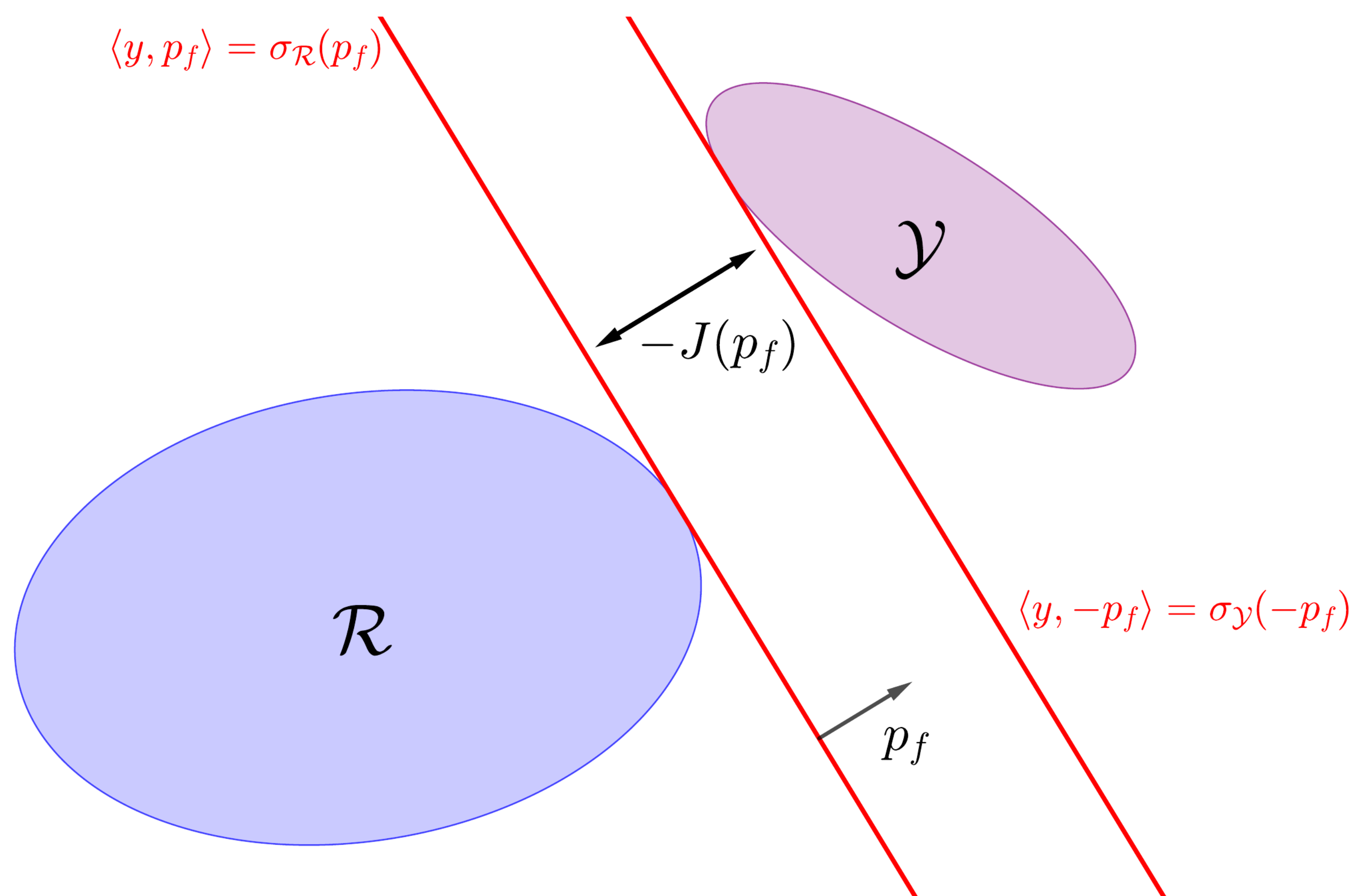
Introducing the separation functional

$$J(p_f; \mathcal{Y}_f) = \sigma_{\mathcal{R}_{T,\mathcal{U}}(y_0)}(p_f) + \sigma_{\mathcal{Y}_f}(-p_f), \quad \sigma_C(p) = \sup_{z \in C} \langle p, z \rangle,$$

we have the equivalence:

$$\mathcal{R}_{T,\mathcal{U}}(y_0) \cap \mathcal{Y}_f = \emptyset \iff \exists p_f \in X : J(p_f; \mathcal{Y}_f) < 0.$$

- p_f : dual certificate of non-reachability;
- $-J(p_f) > 0$ for $\|p_f\| = 1$: strict separation gap.



Computation of the separation functional

$$\begin{aligned} \sigma_{\mathcal{R}_{T,\mathcal{U}}(y_0)}(p_f) &= \langle S_T y_0, p_f \rangle + \sigma_{E_{\mathcal{U}}}(L_T^* p_f) \\ &= \langle S_T y_0, p_f \rangle + \int_0^T \sigma_{B\mathcal{U}}(S_t^* p_f) dt. \end{aligned}$$

Explicit discretisation error

- Gelfand triple $V \hookrightarrow X \hookrightarrow V'$; $A : V \rightarrow V'$ continuous and coercive.
 - Any finite-dimensional family $V_h \subset V$ satisfying
- $$\forall f \in X : \inf_{v_h \in V_h} \|A^{-1}f - v_h\|_V + \inf_{v_h \in V_h} \|(A^*)^{-1}f - v_h\|_{V'} \leq C_0 h \|f\|. \quad (V_1)$$

Discretisation error theorem. For $p_f \in D(A^*)$, $p_{f,h} \in V_h$ and bounded $\mathcal{Y}_f$, and J discretised over V_h using implicit Euler in time:

$$\begin{aligned} |J(p_f; \mathcal{Y}_f) - J_{\Delta t,h}(p_{f,h}; \mathcal{Y}_f)| &\leq C(T, \mathcal{U}, \|y_0\|, \|\mathcal{Y}_f\|) \\ &\quad \times (h^2 + \Delta t)(\|A^* p_f\| + \|p_f - p_{f,h}\|), \end{aligned}$$

where C is **explicit** and depends at most **linear** with respect to each of its parameters.

References

Six steps methodology

A. Candidate search

1. Discretise

V_h^1 satisfying (V_1)
coarse $J_{\Delta t^1, h^1}$

2. Search

"minimise" $J_{\Delta t^1, h^1}$ or $J_{\Delta t^1, h^1}^\lambda$
in floating point arithmetic

3. Lift

$p_{f,h}^1 \mapsto p_f \in D(A^*)$

B. Rigorous certification

4. Refine

fine mesh V_h^2
re-discretise lifted
 $p_f \mapsto p_{f,h}^2$

5. Enclose with INTLAB

bound discretisation and
round-off errors of
 $J_{\Delta t^2, h^2}(p_{f,h}^2)$

6. Decide

check whether
 $\text{Eval}(J_{\Delta t^2, h^2}(p_{f,h}^2))$
 $+e_r + e_d < 0$

Heat equation: a target with a corner

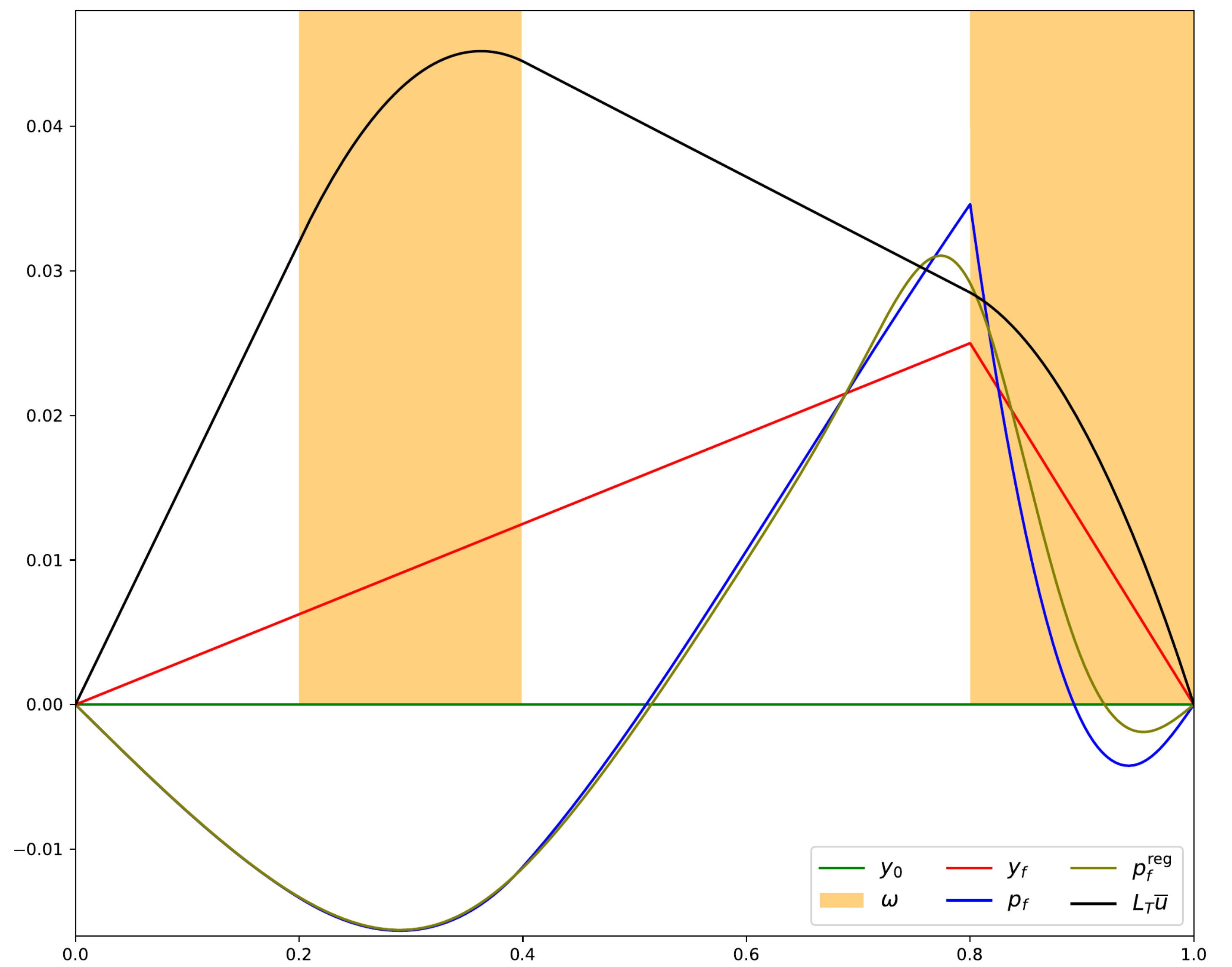
$$\begin{cases} \partial_t y - \partial_{xx} y = \chi_\omega u, \\ y(t, 0) = y(t, 1) = 0, & y(0, \cdot) = y_0 = 0, \\ 0 \leq u(t, x) \leq 1, \end{cases} \quad y_f(x) = \begin{cases} x/32, & 0 \leq x \leq 4/5, \\ (1-x)/8, & 4/5 \leq x \leq 1. \end{cases}$$

$$\omega = \left(\frac{1}{5}, \frac{2}{5}\right) \cup \left(\frac{4}{5}, 1\right), \quad T = 1.$$

- corner at $x = 4/5$;

- $y_f \in H_0^1(0, 1)$ only.

$$0 \leq y_f \leq L_T \bar{u}, \quad \bar{u} \equiv 1 \implies \text{comparison inconclusive.}$$



- corner in $y_f \implies$ rough optimal p_f , large $\|A^* p_f\|$, prohibitive error;
- $J^\lambda(p_f) = J(p_f) + \frac{\lambda}{2} \|A^* p_f\|^2$, $\lambda = 10^{-12} \implies$ similar gap, much smoother direction.

Unregularised

$$\begin{aligned} J(p_f) &\approx -1.4 \times 10^{-3} \\ \|A^* p_f\| &\approx 1350 \end{aligned}$$

Regularised

$$\begin{aligned} J(p_f^{\text{reg}}) &\approx -1.3 \times 10^{-3} \\ \|A^* p_f^{\text{reg}}\| &\approx 125 \end{aligned}$$

Rigorous enclosure

$$J(p_f^{\text{reg}}; \{y_f\}) \in [-1.7, -0.9] \times 10^{-3} < 0.$$

y_f **not reachable** from $y_0 = 0$ at $T = 1$ under $0 \leq u \leq 1$.

Discrete evaluation

$$\begin{aligned} N_x &= 3,250 \\ N_t &= 5,000,000 \end{aligned}$$

Errors

$$\begin{aligned} \text{discretisation} & e_{\text{disc}} \approx 1.8 \times 10^{-4} \\ \text{round-off} & e_{\text{round}} \approx 2.0 \times 10^{-4} \\ \text{comparable contributions} & \end{aligned}$$