Stabilization of a class of delayed non-homogeneous semilinear systems via multiplicative feedback control

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SCINDIS
5TH WORKSHOP ON STABILITY AND CONTROL OF INFINITE-DIMENSIONAL SYSTEMS

Problem Statement & Infinite-Dimensional Setting

We consider a non-homogeneous semilinear distributed-delay system in a Hilbert space $\mathcal{H}$ endowed with the inner product $\langle\cdot,\cdot\rangle$ and associated norm $\|\cdot\|$, given by:

$$\begin{cases} \dfrac{dz}{dt}(t) = Az(t) + \varrho(t)(\mathcal{B}(z_t) + d), & t > 0, \\ z(t) = \phi(t), & t \in [-\tau, 0]. \end{cases} \tag{1}$$

- $\mathscr{C} := C([-\tau, 0], \mathcal{H})$ the space of continuous functions, equipped with the supremum norm $\|\phi\|_{\mathscr{C}} = \sup_{-\tau \leqslant s \leqslant 0} \|\phi(s)\|$.
- $(\mathscr{H}_1)$: The unbounded linear operator $A : D(A) \subset \mathcal{H} \longrightarrow \mathcal{H}$ generates a contraction semigroup $\mathcal{S}^t$ on $\mathcal{H}$.
- $(\mathscr{H}_2)$: $\mathcal{B} : \mathscr{C} \longrightarrow \mathcal{H}$ is a nonlinear locally Lipschitz operator satisfying $\mathcal{B}(0) = 0$.
- The history function $z_t \in \mathscr{C}$ of the state trajectory $z(t) \in \mathcal{H}$ is defined by

$$z_t(s) := z(t + s), \quad \forall s \in [-\tau, 0].$$

- $\varrho(t) \in \mathbb{R}$ is a scalar multiplicative feedback control, $d \in \mathcal{H}$.

Feedback Design, Well-Posedness & Fundamental Estimate

- Energy Dissipative Feedback: To guarantee that the energy $t \longmapsto E(t) = \|z(t)\|^2$ is non-increasing along trajectories, we introduce the feedback control law:

$$\varrho(t) = -\mu \frac{\langle \mathcal{B}(z_t) + d, z(t)\rangle}{f_\epsilon(z_t)} \mathbf{1}_{[\tau,+\infty)}(t), \quad \forall t \geqslant 0, \tag{2}$$

where $f_\epsilon(z_t) := (\|\mathcal{B}(z_t)\| + \|d\| + \epsilon)^2$ and $\mathbf{1}_{[\tau,+\infty)}$ stands for the characteristic function of the interval $[\tau, +\infty)$, $\mu, \epsilon > 0$, which ensures the following dissipation inequality:

$$\|z(t)\|^2 - \|z(s)\|^2 \leqslant -2\mu \int_s^t \frac{|\langle \mathcal{B}(z_\sigma) + d, z(\sigma)\rangle|^2}{f_\epsilon(z_\sigma)} d\sigma, \quad \forall\, 0 \leq s \leq t < +\infty. \tag{3}$$

Theorem 1

The system (1) under the feedback control (2) is globally well posed in $C([-\tau, +\infty), \mathcal{H})$. Moreover, the following estimate holds:

$$\int_\tau^T |\langle \mathcal{B}(\mathcal{S}^{\sigma+\cdot}\, z(t)) + d, \mathcal{S}^\sigma z(t)\rangle| d\sigma \leqslant C_* \left(\int_t^{t+T} \frac{|\langle \mathcal{B}(z_\sigma) + d, z(\sigma)\rangle|^2}{f_\epsilon(z_\sigma)} d\sigma \right)^{\frac{1}{2}}, \tag{4}$$

for any $T > \tau$, where $C_* := (1 + 2\mu T)\sqrt{T}\left(L_\eta \|\phi\|_{\mathscr{C}} + \|d\| + \epsilon\right)$.

Main Result

- Observability Assumption: There exist $T > \tau$ and $\delta > 0$ such that for all $\psi \in \mathcal{H}$:

$$\int_\tau^T |\langle \mathcal{B}(\mathcal{S}^{t+\cdot}\psi) + d, \mathcal{S}^t\psi\rangle|\, dt \geq \delta \|\mathcal{S}^T\psi\|^\beta, \tag{5}$$

where $\beta \geq 1$.

Theorem 2

Assuming hypotheses $(\mathscr{H}_1 - \mathscr{H}_2)$ and the observability condition (5) hold, the decay rate of system (1) depends on the parameter $\beta \geq 1$ and the control gain $\mu > 0$. More precisely:
If $\beta = 1$, system (1) is exponentially stable under the feedback control law (2), which means there exist constants $M \geq 1$ and $\sigma > 0$ such that:

$$\|z(t)\| \leq Me^{-\sigma t}\|\phi\|_{\mathscr{C}}, \quad \forall t \geq 0$$

for $\mu \in \left(0, \frac{1}{2^{3/4} T \|\mathcal{B}\|^2}\right)$.
If $\beta > 1$, system (1) is polynomially stable with the decay rate:

$$\|z(t)\| = \mathcal{O}\left(t^{-\frac{1}{2(\beta-1)}}\right) \quad \text{as } t \to +\infty, \tag{6}$$

for any $0 < \mu < \mu_0 := 2^{\frac{1-2\beta}{2\beta}} T^{-1}$.

Stabilization Results via the Decomposition Approach

Assume that the following hypotheses are satisfied:

- $(\mathscr{G}_1)$: $\mathcal{H} = \mathcal{H}_u \overset{\perp}{\oplus} \mathcal{H}_s$, $\mathcal{S}^t$-invariant subspaces, where $A^u = A|_{\mathcal{H}^u}$ and $A^s = A|_{\mathcal{H}^s}$ generate strongly continuous semigroups $\mathcal{S}_u^t$ on $\mathcal{H}^u$ and S_s^t on $\mathcal{H}^s$, respectively.
- $(\mathscr{G}_2)$: $\mathcal{B} \in \mathcal{L}(\mathscr{C}, \mathcal{H})$ satisfies $\mathcal{B}(\mathscr{C}^u) \subset \mathcal{H}^u$ and $\mathcal{B}(\mathscr{C}^s) \subset \mathcal{H}^s$, where $\mathscr{C}^u := C([-\tau, 0]; \mathcal{H}^u)$ and $\mathscr{C}^s := C([-\tau, 0]; \mathcal{H}^s)$.
- $(\mathscr{G}_3)$: $\mathcal{S}_u^t$ is a contraction semigroup and $\mathcal{S}_s^t$ is an exponentially stable semigroup. (i.e. $\|\mathcal{S}_s^t\| \leq Ne^{-\gamma t}$, $N \geq 1, \gamma > 0$).
- $(\mathscr{G}_4)$: **Observability Inequality:** There exist $T > \tau$ and $\gamma > 0$ such that, for all $\phi \in \mathcal{H}_u$, we have

$$\int_\tau^T \left| \left\langle \mathcal{B}^u(\mathcal{S}_u^{t+\cdot}\phi) + d_u, \mathcal{S}_u^t\phi \right\rangle \right| dt \geqslant \gamma \left\| \mathcal{S}_u^T\phi \right\|^\beta, \tag{7}$$

where $\beta \in [1, 2]$. Under the hypotheses $(\mathscr{G}_1) - (\mathscr{G}_2)$, system (1) admits the following decomposition into two subsystems:

$$\begin{cases} \dfrac{dz^u}{dt}(t) = A_u z^u(t) + \varrho_u(t)(\mathcal{B}^u(z_t^u) + d_u), & t > 0, \\ z^u(t) = \phi_u(t), & t \in [-\tau, 0], \end{cases} \tag{8}$$

and

$$\begin{cases} \dfrac{dz^s}{dt}(t) = A_s z^s(t) + \varrho_u(t)(\mathcal{B}^s(z_t^s) + d_s), & t > 0, \\ z^s(t) = \phi_s(t), & t \in [-\tau, 0], \end{cases} \tag{9}$$

with $z = z^u + z^s$, the feedback control law $\varrho_u(t)$ is given by:

$$\varrho_u(t) = -\mu \frac{\langle \mathcal{B}^u(z_t^u) + d_u, z^u(t)\rangle}{f_\epsilon(z_t^u)} \mathbf{1}_{[\tau,+\infty)}(t), \quad \forall t \geqslant 0, \tag{10}$$

where $f_\epsilon(z_t^u) := \left(\|\mathcal{B}^u(z_t^u)\| + \|d_u\| + \epsilon\right)^2$ and $\mu, \epsilon > 0, d = d_u + d_s$.

Theorem 3 ($\beta = 1$)

Let A generates a C_0-semigroup $\mathcal{S}^t$ and assume that $(\mathscr{G}_1) - (\mathscr{G}_4)$ hold. Then, system (1) is exponentially stabilized by the feedback control low (10) for all $\mu \in (0, \frac{1}{\sqrt{2T}})$. More precisely, there exist two constants $C > 0$ and $\sigma > 0$ such that

$$\|z(t)\| \leqslant Ce^{-\sigma t}\|\phi\|_{\mathscr{C}}, \quad \forall t \geqslant 0. \tag{11}$$

Theorem 4 ($\beta \in (1, 2]$)

Let A generates a C_0-semigroup $\mathcal{S}^t$ and assume that $(\mathscr{G}_1) - (\mathscr{G}_4)$ hold. Then, there exists $\mu_0 > 0$ such that for all $\mu \in (0, \mu_0)$, the feedback control low (10) polynomially stabilizes system (1). Moreover, we have

$$\|z(t)\| = \mathcal{O}\left(t^{-\frac{1}{2(\beta-1)}}\right) \text{ as } t \to +\infty.$$

1 D wave equation

- Consider a nonhomogeneous wave equation with delay on the spatial domain $\Gamma = (0, 1)$:

$$\ddot{y}(x,t) = y_{xx}(x,t) + \varrho(t)\left(\|\dot{y}(\cdot,t)\|_{L^2(\Gamma)}^\beta \dot{y}(x,t-2) + k\int_{-4}^0 y(x,t+s)\, ds + \theta(x)\right), \ (x,t) \in \Gamma \times (0,+\infty),$$

(with Dirichlet boundary conditions $z = 0$ on $\partial\Gamma \times (0, +\infty)$, and initial history $\Phi = (\Phi_1, \Phi_2)$ on $[-2, 0]$).

- First-Order Formulation:

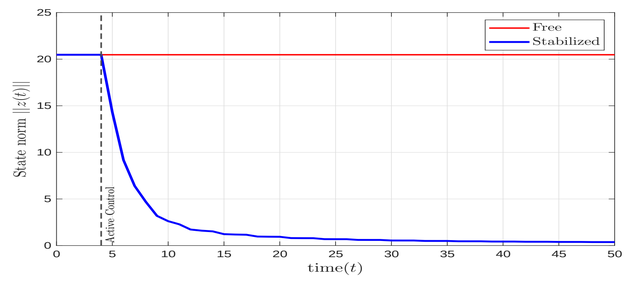
Defining the state $z = (y, \dot{y})^\top \in \mathcal{H} = H_0^1(\Gamma) \times L^2(\Gamma)$, the model is recast in the abstract form (1) with constant term $d = (0, \theta)^\top$, generator $A = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}$, and delay operator $\mathcal{B} : C([-4, 0]; \mathcal{H}) \to \mathcal{H}$ defined for $\phi = (\phi_1, \phi_2)^\top$ by:

$$\mathcal{B}(\phi) = \begin{pmatrix} 0 \\ \|\phi_2(0)\|_{L^2(\Gamma)}^\beta \phi_2(-2) + k\int_{-4}^0 \phi_1(s) ds \end{pmatrix},$$

where $\beta \geq 1$ and $k \geq 0$. The skew-adjoint operator A generates an isometry semigroup $S(t)$. For a state $z = (\sum \beta_j\varphi_j, \sum j\pi\beta_j\varphi_j)^\top$ with $(\beta_j, \beta_j) \in \mathbb{R}^2$ and eigenbasis $\varphi_j(x) = \sqrt{2}\sin(j\pi x)$, its action is:

$$S(t)z = \sum_{j=1}^\infty \begin{pmatrix} \beta_j \cos(j\pi t) + \beta_j \sin(j\pi t) \\ -\beta_j j\pi \sin(j\pi t) + \beta_j j\pi \cos(j\pi t) \end{pmatrix} \varphi_j(x), \quad \forall t \geq 0.$$

Figure 1. State norm evolution



- Assumptions $(\mathscr{H}_1)$ and $(\mathscr{H}_2)$ are satisfied and and inequality (5) with $T = 8$ holds for all $\beta \geq 3$. Therefore, the estimate (6) stated in Theorem 2 holds.

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