

A general framework for the flatness of parabolic linear systems with boundary controls

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The flatness method

The flatness method has been introduced by M. Fliess et al.¹²³.

Roughly speaking, it states that the system, with state $y \in \mathbb{R}^n$ and control $u \in \mathbb{R}^m$,

$$\dot{y} = f(y, u) \quad (\star)$$

is flat if for every regular enough (y, u) we have,

there exists $z : \mathbb{R}_+ \rightarrow \mathbb{R}^p$ (which can be freely chosen), $\Psi_s : (\mathbb{R}^p)^{\kappa_s} \rightarrow \mathbb{R}^n$ and $\Psi_c : (\mathbb{R}^p)^{\kappa_c} \rightarrow \mathbb{R}^m$ such that

$$y = \Psi_s \left(z, z', \dots, z^{(\kappa_s)} \right) \quad \text{and} \quad u = \Psi_c \left(z, z', \dots, z^{(\kappa_c)} \right)$$

is solution of $(\star)$. Furthermore, z is a system output, i.e.,

$$z = \Phi \left(y, u, u', \dots, u^{(\kappa_o)} \right).$$

¹M. Fliess. "Automatique et corps différentiels. (Robotics and differential fields)". *Forum Math.* 1.3 (1989)

²M. Fliess. "Some basic structural properties of generalized linear systems". *Syst. Control Lett.* 15.5 (1990)

³M. Fliess, J. Lévine, P. Martin, and P. Rouchon. "Sur les systèmes non linéaires différentiellement plats. (On differentially flat nonlinear systems)". *C. R. Acad. Sci., Paris, Sér. I* 315.5 (1992)

Controllability and flatness

Let us recall that $(\star)$ is said controllable in time $T > 0$ if

for every y^0 and every y^1 there exists $u : [0, T] \rightarrow \mathbb{R}^m$ such that the solution of $(\star)$ with initial condition $y(0) = y^0$ satisfies $y(T) = y^1$.

For (finite dimensional) linear control system, flatness and controllability are equivalent³.

To build a control, one just have to find $z : [0, T] \rightarrow \mathbb{R}^p$ such that

$$y^0 = \Psi_s \left(z(0), \dots, z^{(\kappa_s)}(0) \right) \quad \text{and} \quad y^1 = \Psi_s \left(z(T), \dots, z^{(\kappa_s)}(T) \right)$$

and the control is given by

$$u(t) = \Psi_c \left(z(t), \dots, z^{(\kappa_s)}(t) \right) \quad (t \in [0, T]).$$

³M. Fliess, J. Lévine, P. Martin, and P. Rouchon. "Sur les systèmes non linéaires différentiellement plats. (On differentially flat nonlinear systems)". *C. R. Acad. Sci., Paris, Sér. I* 315.5 (1992)

Flatness for partial differential equations

Flatness has been proved for parabolic type equations:

- P. Martin, L. Rosier, and P. Rouchon. “Null controllability of the heat equation using flatness”. *Automatica* 50.12 (2014)
- P. Martin, L. Rosier, and P. Rouchon. “Null controllability of one-dimensional parabolic equations by the flatness approach”. *SIAM J. Control Optim.* 54.1 (2016)
- P. Martin, L. Rosier, and P. Rouchon. “On the reachable states for the boundary control of the heat equation”. *AMRX, Appl. Math. Res. Express* 2016.2 (2016)
- C. Laurent and L. Rosier. “Exact controllability of semilinear heat equations in spaces of analytic functions”. *Ann. Inst. Henri Poincaré, Anal. Non Linéaire* 37.4 (2020)
- B. Colle, J. Lohéac, and T. Takahashi. “Controllability of the Stefan problem by the flatness approach”. *Syst. Control Lett.* 174 (2023)

and also for linear Schrödinger⁴, Korteweg-De Vries⁵, beam⁶ equations. . .

Systems of heat equations have only been considered recently:

- B. Colle, J. Lohéac, and T. Takahashi. “Flatness approach for the boundary controllability of a system of heat equations”. *SIAM Journal on Control and Optimization* 62.3 (2024)

⁴P. Martin, L. Rosier, and P. Rouchon. “Controllability of the 1D Schrödinger equation using flatness”. *Automatica* 91 (2018)

⁵P. Martin, I. Rivas, L. Rosier, and P. Rouchon. “Exact controllability of a linear Korteweg-de Vries equation by the flatness approach”. *SIAM J. Control Optim.* 57.4 (2019)

⁶M. Fliess, H. Mounier, P. Rouchon, and J. Rudolph. “Linear systems over Mikusiński operators and control of a flexible beam”. *ESAIM, Proc.* 2 (1997)

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1 Case of a single heat equation

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2 General framework

3 Applications

4 Examples

5 Concluding remarks and open questions

Introductive words

We consider the following 1D heat equation:

$$\left\{ \begin{array}{ll} \dot{y}(t, x) = \partial_x^2 y(t, x) & (t > 0, x \in (0, 1)), \\ y(t, 0) = 0 & (t > 0), \\ y(t, 1) = u(t) & (t > 0), \\ y(0, x) = y^0(x) & (x \in (0, 1)). \end{array} \right. \quad (\text{H1})$$

It is well-known that **(H1) is null controllable in any time $T > 0$** , i.e.,

for every $y^0 \in L^2(0, 1)$, there exists $u \in L^2(0, T)$ such that the solution of (H1) satisfies $y(T, x) = 0$.

This has been proved using bi-orthogonal families⁷, Carleman estimates⁸, or Lebeau-Robbiano strategy⁹.

⁷H. O. Fattorini and D. L. Russell. "Exact controllability theorems for linear parabolic equations in one space dimension". *Arch. Ration. Mech. Anal.* 43 (1971)

⁸A. V. Fursikov and O. Y. Imanuvilov. *Controllability of evolution equations*. Vol. 34. Lect. Notes Ser., Seoul. 1996

⁹G. Lebeau and L. Robbiano. "Exact control of the heat equation". *Commun. Partial Differ. Equations* 20.1-2 (1995)

Flatness strategy¹⁰

Power series

- ❶ Expand y in power series:
$$y(t, x) = \sum_{k=0}^{\infty} b_k(t) \frac{x^{2k}}{(2k)!} + \sum_{i=0}^{\infty} a_i(t) \frac{x^{2i+1}}{(2i+1)!}.$$
- ❷ $\dot{y} = \partial_x^2 y \Rightarrow a_{k+1} = \dot{a}_k \quad \text{and} \quad b_{k+1} = \dot{b}_k \quad (k \in \mathbb{N}).$
- ❸ $y(t, 0) = 0 \Rightarrow b_0 = 0.$

Hence, if y is a power series solution of (H1) then,

$$y(t, x) = \sum_{k=0}^{\infty} a^{(k)}(t) \frac{x^{2k+1}}{(2k+1)!} \quad (t > 0, x \in (0, 1)).$$

Note that $a(t) = a_0(t) = \partial_x y(t, 0)$ is a candidate to be a flat output.

¹⁰P. Martin, L. Rosier, and P. Rouchon. "Null controllability of the heat equation using flatness". *Automatica* 50.12 (2014)

Flatness strategy

Free solutions

Given $y^0 \in L^2(0, 1)$, the free solution $\bar{y}$ of (H1):

$$\begin{cases} \dot{\bar{y}}(t, x) = \partial_x^2 \bar{y}(t, x) & (t > 0, x \in (0, 1)), \\ \bar{y}(t, 0) = \bar{y}(t, 1) = 0 & (t > 0), \\ \bar{y}(0, x) = y^0(x) & (x \in (0, 1)). \end{cases}$$

is given by

$$\begin{aligned} \bar{y}(t, x) &= \sum_{n=1}^{\infty} y_n^0 e^{-(n\pi)^2 t} \sin(n\pi x) = \sum_{n=1}^{\infty} y_n^0 e^{-(n\pi)^2 t} \sum_{k=0}^{\infty} (-1)^k \frac{(n\pi x)^{2k+1}}{(2k+1)!} \\ &= \sum_{k=0}^{\infty} \left(\sum_{n=1}^{\infty} y_n^0 (-1)^k (n\pi)^{2k+1} e^{-(n\pi)^2 t} \right) \frac{x^{2k+1}}{(2k+1)!} \end{aligned}$$

with $y_n^0 = 2 \int_0^1 y^0(x) \sin(n\pi x) dx$, we have $(y_n^0)_{n \in \mathbb{N}^*} \in \ell^2(\mathbb{N}^*)$.

Set $\bar{a}(t) = \sum_{n=1}^{\infty} n\pi y_n^0 e^{-(n\pi)^2 t}$, we have $\bar{a} \in C^\omega(\mathbb{R}_+^*)$ and

$$\bar{y}(t, x) = \sum_{k=0}^{\infty} \bar{a}^{(k)}(t) \frac{x^{2k+1}}{(2k+1)!} \quad (t > 0, x \in (0, 1)).$$

Flatness strategy

Choosing a

Controlling (H1) from y^0 to 0 is equivalent to controlling (H1) from 0 to $-\bar{y}(T, \cdot)$.
Recall that

$$y(t, x) = \sum_{k=0}^{\infty} a^{(k)}(t) \frac{x^{2k+1}}{(2k+1)!} \quad \text{and} \quad \bar{y}(t, x) = \sum_{k=0}^{\infty} \bar{a}^{(k)}(t) \frac{x^{2k+1}}{(2k+1)!}.$$

The problem is thus to find $a \in C^\infty(0, T)$ such that
the sum converges, $y(0, x) = 0$ and $y(T, x) = -\bar{y}(T, x)$.

Set

$$a(t) = -\chi \left(\frac{t - \tau}{T - \tau} \right) \bar{a}(t),$$

with $0 < \tau < T$ and $\chi \in C^\infty(\mathbb{R})$ such that $\chi(t) = 0$ for $t < 0$ and $\chi(t) = 1$ for $t > 1$, so that $a \in C^\infty(0, T)$,

$$a^{(k)}(0) = 0 \quad \text{and} \quad a^{(k)}(T) = -\bar{a}^{(k)}(T).$$

Gevrey Functions¹¹

Definition and Example

Given $s > 0$, we say that $f \in C^\infty(\mathbb{R})$ is of class **Gevrey of order s** ($f \in G^s(\mathbb{R})$) if for every bounded interval $K \subset \mathbb{R}$, there exists $M, R > 0$ such that.

$$\left| f^{(i)}(t) \right| \leq M \frac{(i!)^s}{R^i} \quad (t \in K, i \in \mathbb{N}).$$

We have:

$$C^\omega(\mathbb{R}) = G^1(\mathbb{R}) \subset G^s(\mathbb{R}) \subset C^\infty(\mathbb{R}) \quad (s \geq 1),$$

$$f \in G^{s_1}(\mathbb{R}) \text{ and } g \in G^{s_2}(\mathbb{R}) \Rightarrow fg \in G^{\max(s_1, s_2)}(\mathbb{R}).$$

¹¹J.-P. Ramis. *Devissage Gevrey*. Astérisque 59-60, 173-204 (1978). 1978

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Given $s > 1$, the function defined by

$$\chi(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ \frac{\exp\left(-t^{-\frac{1}{s-1}}\right)}{\exp\left(-t^{-\frac{1}{s-1}}\right) + \exp\left(-(1-t)^{-\frac{1}{s-1}}\right)} & \text{if } 0 < t < 1, \\ 1 & \text{if } t \geq 1, \end{cases}$$

is Gevrey of order s .

¹¹J.-P. Ramis. *Devissage Gevrey*. Astérisque 59-60, 173-204 (1978). 1978

Gevrey Functions

Sum of derivatives and conclusion

Lemma

Let $s < 2$, if $f \in G^s(\mathbb{R})$ then $\sum_{i=0}^{\infty} \frac{f^{(i)}}{(2i+1)!} \in G^s(\mathbb{R})$.

In summary given $s \in (1, 2)$,

- $a(t) = -\chi \left(\frac{t-\tau}{T-\tau} \right) \bar{a}(t) \in G^s(\mathbb{R})$;
- $y(t, x) = \sum_{k=0}^{\infty} a^{(k)}(t) \frac{x^{2k+1}}{(2k+1)!}$ is Gevrey of order 1 in space and s in time;
- $y(0, x) = 0$ and $y(T, x) = -\bar{y}(T, x)$

and y is solution of (H1) with control $u = y(\cdot, 1) \in G^s([0, T])$ given by

$$u(t) = \sum_{k=0}^{\infty} \frac{a^{(k)}(t)}{(2k+1)!} \quad (t \in (0, T)).$$

Ideas for generalizations

Let us define

$$\varphi(\lambda, x) = \sum_{k=0}^{\infty} \lambda^k \frac{x^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} \lambda^k \Phi_k(x) \quad (\lambda \in \mathbb{C}).$$

We have

$$\varphi(\lambda, 0) = 0 \quad \text{and} \quad \partial_x^2 \varphi(\lambda, x) = \lambda \varphi(\lambda, x) \quad (\lambda \in \mathbb{C}).$$

Given $f \in G^s(\mathbb{R})$, we define

$$\varphi[f](t, x) = \sum_{k=0}^{\infty} f^{(k)}(t) \Phi_k(x).$$

Then $\varphi[f](t, x)$ is solution of (H1) with control

$$u(t) = \Xi[f](t)$$

with $\Xi(\lambda) = \varphi(\lambda, 1)$.

The difficult part is to show the existence of $\tau \geq 0$ and $\bar{f} \in G^s(\tau, \infty)$ such that the free solution $\bar{y}$ of (H1) is given by $\bar{y}(t) = \varphi[\bar{f}](t)$ for every $t \geq \tau$.

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Entire function of order σ

Definition (Entire function of order σ)

Assume $\sigma \in [0, \infty]$ and that $\mathcal{X}$ is a Hilbert space. The entire function

$\varphi : \lambda \in \mathbb{C} \mapsto \sum_{k \in \mathbb{N}} \lambda^k \Phi_k \in \mathcal{X}$ is of order σ if $\limsup_{k \rightarrow \infty} \frac{k \ln k}{-\ln \|\Phi_k\|_{\mathcal{X}}} = \sigma$.

We denote by $\mathcal{E}^{\sigma}(\mathcal{X})$ the space of entire functions of order non-greater than σ .

In other words, $\lambda \in \mathbb{C} \mapsto \sum_{k \in \mathbb{N}} \lambda^k \|\Phi_k\|_{\mathcal{X}}$ is a (classical) entire function of order σ .

Using Stirling formula, for any $\tilde{\sigma} > \sigma$, there exists $C > 0$ such that

$$\|\Phi_k\|_{\mathcal{X}} \leq C \left(\frac{1}{k!} \right)^{1/\tilde{\sigma}} \quad (k \in \mathbb{N}).$$

If φ is of order σ , then all its derivatives are of order σ .

Action of entire functions on Gevrey function

Assume $\varphi \in \mathcal{E}^\sigma(\mathcal{X})$ for some Hilbert space $\mathcal{X}$: $\varphi(\lambda) = \sum_{k \in \mathbb{N}} \lambda^k \Phi_k$.

We define for $f \in \mathcal{G}^s(\mathbb{R})$:

$$\varphi[f](t) = \sum_{k \in \mathbb{N}} f^{(k)}(t) \Phi_k \quad (t \in I).$$

Proposition

Assume $\sigma \in (0, 1)$ and $s \in [1, 1/\sigma)$. Then for any $f \in \mathcal{G}^s(\mathbb{R})$, $\varphi[f] \in \mathcal{G}^s(\mathbb{R}; \mathcal{X})$. Moreover, we have the following relations

- $\frac{d}{dt}(\varphi[f]) = (\lambda\varphi)[f];$
- $\varphi[fg] = \sum_{k=0}^{\infty} \frac{f^{(k)}}{k!} \varphi^{(k)}[g] \quad (f, g \in \mathcal{G}^s(I));$
- $\varphi[e_\mu] = e_\mu \varphi(\mu), \quad \varphi[e_{\mu,k}] = \sum_{\ell=0}^k \frac{\varphi^{(\ell)}(\mu)}{\ell!} e_{\mu,k-\ell}. \quad (\mu \in \mathbb{R}, k \in \mathbb{N}),$

with $e_{\mu,k}(t) = \frac{t^k}{k!} e^{\mu t}$ and $e_\mu(t) = e_{\mu,0}(t) = e^{\mu t}$.

Abstract boundary control system

Consider abstract boundary control systems:

$$z' = Lz, \quad Gz = u \quad (\text{Sys})$$

with the following classical hypotheses¹²:

$\mathcal{Z}, \mathcal{X}$ Hilbert spaces with $\mathcal{Z} \subset \mathcal{X}$, $L \in \mathcal{L}(\mathcal{Z}, \mathcal{X})$, $G \in \mathcal{L}(\mathcal{Z}, \mathbb{C})$.

$\mathcal{X}_1 = \text{Ker } G$ dense in $\mathcal{X}$, G onto,

and the operator $(\mathcal{D}(A), A)$ defined by

$$\mathcal{D}(A) = \mathcal{X}_1, \quad A = L|_{\mathcal{X}_1}$$

has a nonempty resolvent set.

We assume that

- $(\mathcal{D}(A), A)$ is the infinitesimal generator of an analytic semigroup;
- the spectrum of A consists of isolated simple eigenvalues $(\lambda_n)_{n \in \mathbb{N}}$ with finite algebraic multiplicity $m_n \in \mathbb{N}^*$. In addition, we assume $\sup_{n \in \mathbb{N}} m_n =: m_\infty < \infty$.

¹²M. Tucsnak and G. Weiss. *Observation and control for operator semigroups*. Birkhäuser Adv. Texts, Basel-Lehrbüch. 2009.

Additional assumptions

Spectral/semi-group assumptions

Let us denote by $\mathcal{P}_n$ the Riesz projection operator associated with λ_n :

$$\mathcal{P}_n = \frac{1}{2i\pi} \oint_{|\lambda - \lambda_n| = \delta} (\lambda I - A)^{-1} d\lambda \quad (\text{with } \delta > 0 \text{ small enough}),$$

Recall¹³ that $\text{Ran}(\mathcal{P}_n)$ is the subspace of generalized eigenvectors of A associated with λ_n .

We assume that A is complete:

- $\sum_{n \geq 0} \text{Ran}(\mathcal{P}_n)$ is dense in $\mathcal{X}$.

There exist $C_0, \tau_0, \tau_1 > 0$ such that:

- $\sum_{n \in \mathbb{N}} e^{\text{Re}(\lambda_n) \tau_0} < \infty,$
- $e^{\text{Re}(\lambda_n) \tau_1} \sum_{k=0}^{m_n-1} \frac{t^k}{k!} \left\| (A - \lambda_n I)^k \mathcal{P}_n \right\|_{\mathcal{L}(\mathcal{X})} \leq C_0 e^{C_0 t} \quad (n \in \mathbb{N}, t \geq 0),$

¹³K.-J. Engel and R. Nagel. *One-parameter semigroups for linear evolution equations*. Vol. 194. Graduate Texts in Mathematics. 2000

Additional assumptions

Existence of φ

Assume there exists an entire function $\lambda \in \mathbb{C} \mapsto \varphi(\lambda) \in \mathcal{Z} \setminus \{0\}$ of order $\sigma < 1$ such that

- for every $\lambda \in \mathbb{C}$, $L\varphi(\lambda) = \lambda\varphi(\lambda)$;
- $\text{Ker}(L - \lambda_n I) = \text{Span } \varphi(\lambda_n)$ ($n \in \mathbb{N}$).

In particular, we have $G\varphi(\lambda_n) = 0$ and $\text{Ker}(A - \lambda_n I) = \text{Span } \varphi(\lambda_n)$;

Finally, we assume that there exists $\tau_2 \geq 0$ such that

- $\limsup_{n \rightarrow \infty} \frac{\ln \|\varphi(\lambda_n)\|_{\mathcal{X}}}{\text{Re}(\lambda_n)} \leq \tau_2$.

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- $\limsup_{n \rightarrow \infty} \frac{\ln \|\varphi(\lambda_n)\|_{\mathcal{X}}}{\text{Re}(\lambda_n)} \leq \tau_2$.

The above assumptions ensure the following result.

Proposition

Assume $\sigma \in (0, 1)$ and $s \in [1, 1/\sigma]$. Consider $L \in \mathcal{L}(\mathcal{Z}, \mathcal{X})$ and $\varphi \in \mathcal{E}^\sigma(\mathcal{Z})$ such that $\varphi(\lambda) \in \text{Ker}(L - \lambda I)$ for every $\lambda \in \mathbb{C}$.

Then, for any $f \in \mathcal{G}^s(\mathbb{R})$, $z = \varphi[f]$ is a solution of (Sys), with control $u = Gz = \Xi[f]$.

Proof. We have $L(\varphi[f]) = (L\varphi)[f] = (\lambda\varphi)[f] = \frac{d}{dt}(\varphi[f])$.

Then, it is sufficient to take $u = (G\varphi)[f] = \Xi[f]$. □

Main results

Theorem

With the previous assumptions, for any $s \in (1, 1/\sigma)$, $z^0 \in \mathcal{X}$ and $T > \tau_0 + \tau_1 + m_\infty \tau_2$, there exists $u \in \mathcal{G}^s(\mathbb{R})$ such that the corresponding solution z of (Sys) with $z(0) = z^0$ satisfies $z(T) = 0$.

Moreover, we have $z \in \mathcal{G}^s((\tau, T]; \mathcal{Z})$.

Theorem

With the previous assumptions, let us consider $\mu_1, \dots, \mu_J \in \mathbb{C}$, with $J \in \mathbb{N}^*$ and $N \in \mathbb{N}^*$. For any

$$z^0, z^1 \in \text{Span} \left\{ \varphi^{(\ell)}(\mu_j) : \ell \in \{1, \dots, N\}, j \in \{1, \dots, J\} \right\},$$

for any $s \in (1, 1/\sigma)$, and $T > 0$, there exists $u \in \mathcal{G}^s(\mathbb{R})$ such that the corresponding solution z of (Sys) with $z(0) = z^0$ satisfies $z(T) = z^1$.

Particular cases are:

- z^0 and z^1 are (finite) linear combinations of generalized eigenfunctions of A ;
- $z^0, z^1 \in \text{Span } \varphi(0)$, i.e., z^0 and z^1 are steady states.

Key lemma

Lemma

For every $n \in \mathbb{N}$, λ_n is a zero of $\Xi = G\varphi$ of order m_n . Moreover, we have

$$\text{Ran } \mathcal{P}_n = \text{Span} \left\{ \varphi^{[k]}(\lambda_n), k \in \{0, \dots, m_n - 1\} \right\},$$

with $\varphi^{[k]} = \varphi^{(k)} / k!$.

Proof arguments. It is clear that the eigenvalues $\{\lambda_n\}_{n \in \mathbb{N}}$ of A are the zeros of Ξ . To prove the rest of the result, we use the relation:

$$L\varphi^{[k+1]}(\lambda) = \lambda\varphi^{[k+1]}(\lambda) + \varphi^{[k]}(\lambda) \quad (k \in \mathbb{N}, \lambda \in \mathbb{C}),$$

together with the assumption $\dim \text{Ker}(A - \lambda_n I) = 1$.



Free solutions I

Proposition

Let $\tau = \tau_0 + \tau_1 + m_\infty \tau_2$. For any $z^0 \in \mathcal{X}$, there exists $\bar{f} \in \mathcal{G}^1(\tau, \infty)$ such that

$$e^{tA} z^0 = \varphi[\bar{f}](t) \quad (t > \tau).$$

Proof. From the key lemma, we have: $\mathcal{P}_n z^0 = \sum_{\ell=0}^{m_n-1} a_{n,\ell} \varphi^{[\ell]}(\lambda_n)$.

The semigroup associated with A can be written as follows¹⁴:

$$e^{tA} = \sum_{n \in \mathbb{N}} e^{\lambda_n t} \sum_{k=0}^{m_n-1} \frac{t^k}{k!} (\lambda_n I - A)^k \mathcal{P}_n = \sum_{n \in \mathbb{N}} \sum_{k=0}^{m_n-1} e_{\lambda_n, k}(t) (\lambda_n I - A)^k \mathcal{P}_n \quad (t \geq \tau_0 + \tau_1).$$

Using $(\lambda_n I - A)^k \varphi^{[\ell]}(\lambda_n) = \varphi^{[\ell-k]}(\lambda_n)$ with $\varphi^{[\ell-k]} \equiv 0$ for $k > \ell$, we have for $t > \tau_0 + \tau_1$,

$$e^{tA} z^0 = \sum_{n \in \mathbb{N}} \sum_{k=0}^{m_n-1} e_{\lambda_n, k}(t) \sum_{\ell=0}^{m_n-1} a_{n,\ell} \varphi^{[\ell-k]}(\lambda_n) = \sum_{n \in \mathbb{N}} \varphi \left[\sum_{\ell=0}^{m_n-1} a_{n,\ell} e_{\lambda_n, \ell} \right] (t).$$

¹⁴G. Q. Xu and S. P. Yung. "Properties of a class of C_0 semigroups on Banach spaces and their applications". *J. Math. Anal. Appl.* 328.1 (2007)

Free solutions II

Set $\bar{f} = \sum_{n \in \mathbb{N}} \sum_{\ell=0}^{m_n-1} a_{n,\ell} e_{\lambda_n, \ell}$, the aim is to show that $e^{tA} z_0 = \varphi[\bar{f}](t)$ for $t > \tau$. To this end, we will show that $\bar{f}$ is analytic in (τ, ∞) .

We need to estimate the coefficients $(a_{n,\ell})$.

Using again $(\lambda_n I - A)^k \varphi^{[\ell]}(\lambda_n) = \varphi^{[\ell-k]}(\lambda_n)$, we deduce that $(a_{n,m_n-1}, \dots, a_{n,0})$ satisfies the “triangular” linear system

$$\begin{cases} a_{n,m_n-1} \varphi^{[0]}(\lambda_n) = (A - \lambda_n I)^{m_n-1} \mathcal{P}_n z^0, \\ a_{n,m_n-1} \varphi^{[1]}(\lambda_n) + a_{n,m_n-2} \varphi^{[0]}(\lambda_n) = (A - \lambda_n I)^{m_n-2} \mathcal{P}_n z^0, \\ \vdots \\ a_{n,m_n-1} \varphi^{[m_n-1]}(\lambda_n) + \dots + a_{n,0} \varphi^{[0]}(\lambda_n) = (A - \lambda_n I)^0 \mathcal{P}_n z^0 \end{cases}$$

The discrete Grönwall lemma¹⁵, the fact that $\varphi \in \mathcal{E}^\sigma(\mathcal{X})$ and $m_n \leq m_\infty < \infty$, lead to

$$|a_{n,m_n-\ell}| \leq \frac{1}{\|\varphi(\lambda_n)\|_{\mathcal{X}}} \left\| \mathcal{P}_{n,m_n-\ell} z^0 \right\|_{\mathcal{X}} + \frac{C e^{|\lambda_n| \tilde{\sigma}}}{\|\varphi(\lambda_n)\|_{\mathcal{X}}^2} \sum_{j=1}^{\ell-1} \left(1 + \frac{C e^{|\lambda_n| \tilde{\sigma}}}{\|\varphi(\lambda_n)\|_{\mathcal{X}}} \right)^{\ell-j-1} \left\| \mathcal{P}_{n,m_n-j} z^0 \right\|_{\mathcal{X}}$$

with $\mathcal{P}_{n,k} = (A - \lambda_n I)^k \mathcal{P}_n$ and for C and $\tilde{\sigma} \in (\sigma, 1)$ independent of n and ℓ .

¹⁵D. Willett and J. S. W. Wong. “On the discrete analogues of some generalizations of Gronwall’s inequality”. *Monatsh. Math.* 69 (1965)

Free solutions III

Using the assumption $\limsup_{n \rightarrow \infty} \frac{\ln \|\varphi(\lambda_n)\|_{\mathcal{X}}}{\operatorname{Re}(\lambda_n)} \leq \tau_2$ together with the fact that A is the generator of an analytic semi-group, we get:

$$\frac{e^{|\lambda_n|^{\tilde{\sigma}}}}{\|\varphi(\lambda_n)\|_{\mathcal{X}}} \leq C e^{-\tilde{\tau}_2 \operatorname{Re}(\lambda_n)} \quad (n \in \mathbb{N}).$$

for any $\tilde{\tau}_2 \geq \tau_2$ and some constant C independent of n . We then deduce that

$$|a_{n, m_n - \ell}| \leq C e^{-m_\infty \tilde{\tau}_2 \operatorname{Re}(\lambda_n)} \sum_{j=1}^{\ell} \left\| \mathcal{P}_{n, m_n - j} z^0 \right\|_{\mathcal{X}}.$$

Combining this estimate together with the assumption

$$e^{\operatorname{Re}(\lambda_n) \tau_1} \sum_{k=0}^{m_n-1} \frac{t^k}{k!} \|\mathcal{P}_{n, k}\|_{\mathcal{L}(\mathcal{X})} \leq C_0 e^{C_0 t} \quad (n \in \mathbb{N}, t > 0),$$

leads to the analyticity of $\bar{f}$ on $(\tau_0 + \tau_1 + m_\infty \tilde{\tau}_2, \infty)$. □

Proof of the results for $z^0 \in \mathcal{X}$

From the previous proposition, for every $z^0 \in \mathcal{X}$, there exists $\bar{f} \in \mathcal{G}^1(\tau, \infty)$ such that $e^{tA}z^0 = \varphi[\bar{f}](t)$ holds for every $t > \tau$.

By assumption, $T > \tau$, and there exists $\varepsilon > 0$ such that $T - \varepsilon > \tau + \varepsilon$. Let us consider $s \in (1, 1/\sigma)$ and $\eta_s \in \mathcal{G}^s(\mathbb{R})$ such that

$$\eta_s \equiv 0 \quad \text{in } (-\infty, \tau + \varepsilon), \quad \eta_s \equiv 1 \quad \text{in } (T - \varepsilon, \infty).$$

We define

$$\zeta = \varphi[\eta_s \bar{f}] \in \mathcal{G}^s(\mathbb{R}; \mathcal{Z}).$$

We then have

$$\zeta'(t) = L\zeta(t), \quad G\zeta(t) = v(t) \quad (t \in \mathbb{R}),$$

with $v = (G\varphi)[\eta_s \bar{f}] \in \mathcal{G}^s(\mathbb{R})$ (in particular, $v \equiv 0$ on $(0, \tau + \varepsilon)$). Then setting

$$z(t) = e^{tA}z^0 - \zeta(t) \quad (t \geq 0), \quad u = -v,$$

we have that z is solution of (Sys) associated with u , $z(0) = z^0$ and $z(T) = 0$. □

- 1 Case of a single heat equation
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- 3 **Applications**
 - Simultaneous controllability
 - Controllability of cascade systems
- 4 Examples
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Abstract parallel system

Assume $(L^\sharp, G^\sharp, \mathcal{Z}^\sharp, \mathcal{X}^\sharp)$ and $(L^b, G^b, \mathcal{Z}^b, \mathcal{X}^b)$ are two abstract boundary control systems (satisfying all the previous assumptions).

Then, we consider the simultaneous control problem:

$$\begin{cases} (z^\sharp)' = L^\sharp z^\sharp, \\ G^\sharp z^\sharp = \textcolor{red}{u}, \end{cases} \quad \begin{cases} (z^b)' = L^b z^b, \\ G^b z^b = \textcolor{red}{u}. \end{cases}$$

We can write this system as (Sys) by setting

$$\mathcal{X} = \mathcal{X}^\sharp \times \mathcal{X}^b, \quad \mathcal{Z} = \left\{ z \in \mathcal{Z}^\sharp \times \mathcal{Z}^b : G^\sharp z^\sharp = G^b z^b \right\},$$

$$Gz = \frac{1}{2} \left(G^\sharp z^\sharp + G^b z^b \right), \quad L = \begin{pmatrix} L^\sharp & 0 \\ 0 & L^b \end{pmatrix}.$$

We define $\Xi^\sharp = G^\sharp \varphi^\sharp$ and $\Xi^b = G^b \varphi^b$.

Simultaneous controllability of flat systems

Theorem

Assume $\sigma(A^\sharp) \cap \sigma(A^b) = \emptyset$.

Then $(L, G, \mathcal{Z}, \mathcal{X})$ is an abstract boundary control system that satisfies all the flatness assumptions with

$$\tau_0 = \max(\tau_0^\sharp, \tau_0^b), \quad \tau_1 = \max(\tau_1^\sharp, \tau_1^b) \quad \text{and} \quad \tau_2 = \max(\tilde{\tau}_2^\sharp, \tilde{\tau}_2^b),$$

$$\text{with } \tilde{\tau}_2^\sharp = \tau_2^\sharp + \limsup_{n \rightarrow \infty} \frac{\ln |\Xi^b(\lambda_n^\sharp)|}{\operatorname{Re}(\lambda_n^\sharp)} \quad \text{and} \quad \tilde{\tau}_2^b = \tau_2^b + \limsup_{n \rightarrow \infty} \frac{\ln |\Xi^\sharp(\lambda_n^b)|}{\operatorname{Re}(\lambda_n^b)}.$$

Proof idea. Consider $\varphi = \begin{pmatrix} \Xi^b \varphi^\sharp \\ \Xi^\sharp \varphi^b \end{pmatrix} \in \mathcal{E}^{\max(\sigma^\sharp, \sigma^b)}(\mathcal{Z})$. □

Remark

If $\sigma(A^\sharp)$ and $\sigma(A^b)$ are not disjoint, then there is no simultaneous controllability.

An abstract cascade system

Assume $(L, G, \mathcal{Z}, \mathcal{X})$ is an abstract boundary control system (satisfying all the previous assumptions). We consider the cascade system for $d \geq 2$:

$$\begin{aligned} z_1' &= Lz_1, & Gz_1 &= b_1 u, \\ z_i' &= Lz_i - z_{i-1}, & Gz_i &= b_i u \end{aligned} \quad (i = 2, \dots, d),$$

where $b = (b_1, \dots, b_d) \in \mathbb{C}^d$, $|b| = 1$.

We set

$$\mathcal{Z}_d = \left\{ z \in \mathcal{Z}^d : Gz \in \text{Span } b \right\}, \quad L_d = L - M,$$

with $Gz = (Gz_1, \dots, Gz_d) \in \mathbb{C}^d$, $Lz = (Lz_1, \dots, Lz_d) \in \mathcal{X}^d$, $Mz = (0, z_1, \dots, z_{d-1}) \in \mathcal{X}^d$, and we set

$$G_d z = \langle Gz, b \rangle_{\mathbb{C}^d}.$$

We have $L_d \in \mathcal{L}(\mathcal{Z}_d, \mathcal{X}^d)$, $G_d \in \mathcal{L}(\mathcal{Z}_d, \mathbb{C})$, and we can write the system under the form

$$z' = L_d z, \quad G_d z = u.$$

Controllability of the cascade of flat systems

Theorem

Assume $m_\infty = 1$ and $b_1 \neq 0$.

Then $(L_d, G_d, \mathcal{Z}_d, \mathcal{X}^d)$ is an abstract boundary control system that satisfies all the flatness assumptions with

$$\tau_{0,d} = \tau_0, \quad \tau_{1,d} = \tau_1 \quad \text{and} \quad \tau_{2,d} = \tau_2 + (d-1) \limsup_{n \rightarrow \infty} \frac{\ln |\Xi'(\lambda_n)|}{\operatorname{Re}(\lambda_n)}.$$

Proof idea. By solving $L_d \varphi_d(\lambda) = \lambda \varphi_d(\lambda)$, we observe that

$$[\varphi_d(\lambda)]_i = \sum_{k=1}^i [\operatorname{Cof}(\mathbb{M}(\lambda))^\top b]_k \varphi^{[i-k]}(\lambda), \text{ with } \mathbb{M} = \begin{pmatrix} \Xi & 0 & \dots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ \Xi^{[d-1]} & \dots & \dots & \Xi \end{pmatrix}.$$

In particular, we have $\|\varphi_d(\lambda_n)\|_{\mathcal{X}^d} = |\Xi'(\lambda_n)|^{d-1} \|b_1\| \|\varphi(\lambda_n)\|_{\mathcal{X}}$. □

Remark

If $b_1 = 0$, then the system is not controllable.

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 - Examples based on the 1D heat equation
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A cascade system I

We consider the system

$$\left\{ \begin{array}{ll} \partial_t z_1 = \frac{1}{\delta_1} \partial_{xx} z_1 & \text{in } (0, T) \times (0, 1), \\ \partial_t z_2 = \frac{1}{\delta_2} \partial_{xx} z_2 + z_1 & \text{in } (0, T) \times (0, 1) \\ \partial_t z_3 = \frac{1}{\delta_3} \partial_{xx} z_3 + z_2 & \text{in } (0, T) \times (0, 1) \\ z_1(t, 0) = z_2(t, 0) = z_3(t, 0) = 0 & t \in (0, T), \\ z_1(t, 1) = u(t), \quad z_2(t, 1) = z_3(t, 1) = 0 & t \in (0, T), \end{array} \right.$$

with $\delta_1, \delta_2, \delta_3 > 0$, $\sqrt{\frac{\delta_j}{\delta_k}} \notin \mathbb{Q}$ for $j \neq k$.

This corresponds to (Sys) with

$$\mathcal{X} = L^2(0, 1)^3, \quad \mathcal{Z} = \left\{ z \in H^2(0, 1)^3 : z(0) = 0, \quad z_2(1) = z_3(1) = 0 \right\},$$

$$Lz = D \partial_{xx} z + Mz, \quad Gz = z_1(1).$$

$$\text{where } D = \text{diag} \left(\frac{1}{\delta_1}, \frac{1}{\delta_2}, \frac{1}{\delta_3} \right), \quad M = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

A cascade system II

We can check that the flatness assumptions hold with any $\tau_0 > 0$ and $\tau_1 = 0$ and the eigenvalues are

$$\lambda_{k,n} = -\frac{n^2\pi^2}{\delta_k}, \quad (n \in \mathbb{N}^*, k \in \{1, 2, 3\}).$$

We can also show that

$$\varphi(\lambda)(x) = \sum_{j=0}^{\infty} \frac{x^{2j+1}}{(2j+1)!} \mathcal{M}_{\lambda}^j \text{Cof}(\mathcal{N}_{\lambda})^{\top} b \in \mathcal{E}^{1/2}(\mathcal{Z}),$$

with $\mathcal{M}_{\lambda} = D^{-1}(\lambda I - M) = \begin{pmatrix} \delta_1\lambda & 0 & 0 \\ -\delta_2 & \delta_2\lambda & 0 \\ 0 & -\delta_3 & \delta_3\lambda \end{pmatrix}$ and $\mathcal{N}_{\lambda} = \sum_{j=0}^{\infty} \frac{1}{(2j+1)} \mathcal{M}_{\lambda}^j$, is solution of $L\varphi(\lambda) = \lambda\varphi(\lambda)$. Finally, we can show that $\tau_2 = \max(\tau_{2,1}, \tau_{2,2}, \tau_{2,3})$ with

$$\tau_{2,1} = \limsup_{n \rightarrow \infty} \frac{\ln \|\varphi(\lambda_{1,n})\|_{\mathcal{X}}}{\text{Re } \lambda_{1,n}} = \limsup_{n \rightarrow \infty} \frac{-\delta_1 \ln \left| \sin \left(n\pi \sqrt{\frac{\delta_2}{\delta_1}} \right) \sin \left(n\pi \sqrt{\frac{\delta_3}{\delta_1}} \right) \right|}{n^2\pi^2} \dots$$

A cascade system III

Proposition

Assume $\tau_2 < \infty$, for any $s \in (1, 2)$, $z^0 \in L^2(0, 1)^3$ and $T > \tau_2$, there exists $u \in \mathcal{G}^s(\mathbb{R})$ such that the corresponding solution z with $z(0, \cdot) = z^0$ satisfies $z(T, \cdot) = 0$.

Moreover, $z \in \mathcal{G}^s((\tau_2, T]; H^2(0, 1)^3)$.

In addition, τ_2 is the condensation index of the family $\{\lambda_{k,n}\}_{k \in \{1,2,3\}, n \in \mathbb{N}^*}$.

Therefore, we already know¹⁶ that the system is null-controllable in time $T > \tau_2$ and not null-controllable in time $T < \tau_2$.

¹⁶F. Ammar Khodja, A. Benabdallah, M. González-Burgos, and L. de Teresa. "Minimal time for the null controllability of parabolic systems: the effect of the condensation index of complex sequences". *J. Funct. Anal.* 267.7 (2014)

A degenerate heat equation I

Let us consider the system

$$\begin{cases} \rho \partial_t z = \partial_{xx} z & \text{in } (0, T) \times (0, 1), \\ [\alpha_0 z + \beta_0 \partial_x z](t, 0) = 0 & t \in (0, T), \\ [\alpha_1 z + \beta_1 \partial_x z](t, 1) = u(t) & t \in (0, T), \end{cases}$$

with

$$\rho \in L^1(0, 1), \quad \rho > 0 \quad \text{in } (0, 1), \quad (\alpha_0, \beta_0), (\alpha_1, \beta_1) \in \mathbb{R}^2 \setminus \{(0, 0)\}.$$

To write under the abstract form (Sys), we introduce the spaces

$$\begin{aligned} L_\rho^2(0, 1) &= \left\{ f : (0, 1) \rightarrow \mathbb{R} \text{ measurable s.t. } \int_0^1 |f|^2 \rho \, dx < \infty \right\}, \\ L_{1/\rho}^2(0, 1) &= \left\{ f : (0, 1) \rightarrow \mathbb{R} \text{ measurable s.t. } \int_0^1 \frac{|f|^2}{\rho} \, dx < \infty \right\}, \end{aligned}$$

and set

$$\mathcal{X} = L_\rho^2(0, 1), \quad \mathcal{Z} = \left\{ z \in H^1(0, 1) : z'' \in L_{1/\rho}^2(0, 1), [\alpha_0 z + \beta_0 \partial_x z](0) = 0 \right\},$$

$$Lz = \frac{1}{\rho} \partial_{xx} z, \quad Gz = [\alpha_1 z + \beta_1 \partial_x z](1).$$

A degenerate heat equation II

It is shown¹⁶ that

- the corresponding operator $(\mathcal{D}(A), A)$ is self-adjoint and is the generator of an analytic semigroup;
- the eigenvalues $(\lambda_n)_{n \in \mathbb{N}}$ of A are simple (geometrically and algebraically), (i.e., $m_\infty = 1$);
- there exist $c_1, c_2 > 0$ such that $\lambda_n \leq c_1 - c_2 n$ (i.e., $\tau_1 = 0$ and τ_0 is any positive number);
- there exists an orthonormal basis of $\mathcal{X}$ composed by eigenvectors of A (i.e., A is complete).

We also have¹⁷

- there exists a unique $\varphi(\lambda) \in W^{2,1}(0, 1)$ such that

$$\begin{cases} \rho \lambda \varphi(\lambda) = \partial_{xx} \varphi(\lambda) & \text{in } (0, 1), \\ [\alpha_0 \varphi(\lambda) + \beta_0 \partial_x \varphi(\lambda)](0) = 0, \\ [\beta_0 \varphi(\lambda) - \alpha_0 \partial_x \varphi(\lambda)](0) = 1. \end{cases}$$

In particular, $\varphi(\lambda) \in \mathcal{Z} \setminus \{0\}$ and $\text{Ker}(L - \lambda I) = \text{Span } \varphi(\lambda)$ and $\varphi \in \mathcal{E}^{1/2}(\mathcal{Z})$.

¹⁶P. Martin, L. Rosier, and P. Rouchon. "Null controllability of one-dimensional parabolic equations by the flatness approach". *SIAM J. Control Optim.* 54.1 (2016)

¹⁷A. Zettl. *Sturm-Liouville theory*. Vol. 121. Math. Surv. Monogr. 2005

A degenerate heat equation III

Finally¹⁶

- $\frac{1}{\|\varphi(\lambda_n)\|_{\mathcal{X}}} \lesssim 1 + |\lambda_n|^{3/2}$ (i.e., $\tau_2 = 0$).

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A degenerate heat equation III

Finally¹⁶

$$\bullet \frac{1}{\|\varphi(\lambda_n)\|_{\mathcal{X}}} \lesssim 1 + |\lambda_n|^{3/2} \text{ (i.e., } \tau_2 = 0\text{)}.$$

We have obtained the following result:

Proposition

For any $s \in (1, 2)$, $z^0 \in L^2_\rho(0, 1)$ and $T > 0$, there exists $u \in \mathcal{G}^s(\mathbb{R})$ such that the corresponding solution z with $z(0, \cdot) = z^0$ satisfies

$$z(T, \cdot) = 0.$$

Moreover, $z \in \mathcal{G}^s((0, T]; \mathcal{Z})$.

¹⁶P. Martin, L. Rosier, and P. Rouchon. "Null controllability of one-dimensional parabolic equations by the flatness approach". *SIAM J. Control Optim.* 54.1 (2016)

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Conclusion and comments

- What is the flat output?
- In regard to
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the connection between our minimal controllability time and the condensation index has to be analyzed.
- Connection with the moment method?
- What about
 - More general coupled systems (boundary coupling, different operators. . .)?
 - Nonlinear systems?
 - Systems in higher space dimension?

The presented results are taken from:

- J. Lohéac and T. Takahashi. “A general framework for the flatness of parabolic linear systems with boundary controls”. 2026

Conclusion and comments

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Thank you for your attention.