



Long- and short-time decay of solutions of evolution equations

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Three classes of evolution equations

- ▶ Classical linear **evolution equation** $\dot{x} = -Bx$, $x(0) = x_0$, $B \in \mathcal{B}(\mathcal{H}, \mathcal{H}^*)$, in (finite or infinite dim.) Hilbert space $\mathcal{H}$.
- ▶ Linear **operator differential-algebraic equation (DAE)** in (finite or infinite dim.) Hilbert spaces, $\mathcal{X}, \mathcal{Y}$.

$$E\dot{x}(t) = Ax(t), \quad x(0) = x_0,$$

$E \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, $A : \mathcal{D}(A) \subset \mathcal{X} \rightarrow \mathcal{Y}$, densely defined and closed.

- ▶ Nonlinear (finite dim.) dissipative Hamiltonian ODE

$$\dot{x} = (J(x) - R(x))\nabla H(x), \quad x(0) = x_0.$$

Here $J, R : U \rightarrow \mathbb{R}^{n \times n}$ are real matrix functions on open set $U \subseteq \mathbb{R}^n$ satisfying $J(x)^\top = -J(x)$ and $R(x)^\top = R(x) = G(x)G(x)^\top \geq 0$ for all $x \in U$ pointwise and G has constant rank r . Here $H : U \rightarrow \mathbb{R}$ is given *Hamiltonian* (energy function), and $x_0 \in U$.



Long- and short-time decay

Goals: Characterize asymptotic/exponential stability and derive

1. Optimal **long-time** decay estimate.

- ▶ exponential decay: $\|x(t) - x_\infty\| \leq ce^{-\lambda t} \|x(0) - x_\infty\|, t > 0.$
- ▶ (sharp) maximal rate $\lambda > 0$ and minimal $c \geq 1$ (uniform in $x(0)$).

2. Optimal **short-time** decay estimate.

- ▶ $\|x(t) - x_\infty\| \leq (1 - ct^\alpha + \mathcal{O}(t^{\alpha+1})) \|x(0) - x_\infty\|, t > 0.$
- ▶ (sharp) optimal decay and $c \geq 1$ (uniform in $x(0)$).

Transform or find norm for which one has (optimal) globally monotone decay and no transients.

- ▷ F. Achleitner, A. Arnold, and E.A. Carlen. *The hypocoercivity index for the short time behavior of linear time-invariant ODE systems*. J. Differential Equations 371 (2023): 83–115.
- ▷ F. Achleitner, A. Arnold, and V. M. *Hypocoercivity and controllability in linear semi-dissipative ODEs and DAEs*. Vol. 103, ZAMM, Zeitschrift f. Angewandte Mathematik und Mechanik, e202100171, 2021. <https://doi.org/10.1002/zamm.202100171>
- ▷ F. Achleitner, A. Arnold, and V. M. *Hypocoercivity in algebraically constrained partial differential equations with application to Oseen equations*. J. Dynamics Differential Eq., 1–40, 2023. <http://arxiv.org/2212.06631>
- ▷ F. Achleitner, A. Arnold, V. M., E. Nigsch. *Hypocoercivity in Hilbert spaces*. J. Functional Analysis, Vol. 288, 110691, 2024. <http://arxiv.org/2307.08280>
- ▷ F. Achleitner, A. Arnold, V. M., and E. Nigsch. *Long- and short-time behavior of hypocoercive evolution equations via modal decompositions* Operator Theory: Advances and Applications, 2025. <http://arxiv.org/abs/2501.15879>
- ▷ A. Arnold, S. Egger, V. M., and E. Nigsch, *Connection of hypocoercivity and hypocontractivity via the Cayley transform*. Submitted 2026. <http://arxiv.org/abs/2601.09656>
- ▷ A. Arnold, S. Egger, V. M., and E. Nigsch, *Hypocoercivity for nonlinear semi-dissipative Hamiltonian systems*. In preparation, 2026.
- ▷ M. Erbay, B. Jacob, K. Morris, T. Reis, and C. Tischendorf. *Index concepts for linear differential-algebraic equations in finite and infinite dimensions*. arXiv preprint, 2024.
- ▷ H. Gernandt, F. E. Haller, T. Reis. *A linear relation approach to port-Hamiltonian differential-algebraic equations*. SIAM J. Matrix Anal. Appl. 2021;42(2):1011–1044.
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- ▷ H. Gernandt, T. Reis. *A pseudo-resolvent approach to abstract differential-algebraic equations*. arXiv. 2023.
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- ▷ B. Jacob, K. Morris, H. Zwart. C_0 -semigroups for hyperbolic partial differential equations on a one-dimensional spatial domain. Journal of Evolution Equations. 2015;15(2):493 –502.
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- ▷ M. Roschkowski, H. Gernandt. Long-and short-time behavior of hypocoercive evolution equations with higher index via modal decompositions. arXiv preprint arXiv:2512.07617. 2025
- ▷ H. Zwart and V. Mehrmann. *Abstract dissipative Hamiltonian differential-algebraic equations are everywhere*. DAE Panel, 2024. <https://arxiv.org/5223640>



- ▷ port-Hamiltonian systems in general.
- ▷ Gas transport networks.
- ▷ District heating networks.
- ▷ Electrical circuits.
- ▷ Flow problems, Navier-Stokes, Oseen.
- ▷ Hydraulic systems.
- ▷ Magneto-hydrodynamics.
- ▷ Hybrid multimodal power networks.
- ▷ Reactive flow control.
- ▷ Poro-elastic networks.
- ▷ Multi-body dynamics.
- ▷



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Classical evolution equations

Definition

Consider evolution equation $\dot{x} = -Bx$ in Hilbert space $\mathcal{H}$. B is **coercive** if $\langle x, Bx \rangle \geq \kappa \|x\|_{\mathcal{H}}^2$ for all $x \in \mathcal{H}$ and some $\kappa > 0$.

Classical: If B is coercive then solution exponentially decays (is asymptotically stable).

But this is not necessary:

Example: $B = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \in \mathbb{C}^{2,2}$, Evs $\frac{1}{2}(-1 \pm \sqrt{3}i)$, decay rate $\frac{1}{2}$.

However, $-B = (J - R)$ with $J = -J^*$, $R = R^* \geq 0$ is not coercive, since R is only semidefinite.

To deal with this case one can employ the concept of **hypocoercivity**.



Analysis via Hypocoercivity

- ▶ Notion **hypocoercivity** was introduced for $\dot{x} = -Bx$, $x(0) = x_0$ on Hilbert space $\mathcal{H}$, where linear operator B is not coercive, but solutions still decay exponential in time.
- ▶ More precisely, for hypocoercive operators B , there exist constants $\lambda > 0$ and $c \geq 1$, such that

$$\|e^{-Bt}x_0\|_{\tilde{\mathcal{H}}} \leq c e^{-\lambda t} \|x_0\|_{\tilde{\mathcal{H}}} \quad \text{for all } x_0 \in \tilde{\mathcal{H}}, \quad t \geq 0,$$

where $\tilde{\mathcal{H}}$ is Hilbert space, densely embedded in $(\ker B)^\perp \subset \mathcal{H}$.

- ▶ C. Villani. Hypocoercivity. *Mem. Amer. Math. Soc.*, 202, 2009.



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Hypocoercivity and controllability

Consider $\dot{x} = Ax = -Bx = (J - R)x$. $A \in \mathbb{C}^{n,n}$

Lemma (Achleitner, Arnold, M. 2021)

Let $J, R \in \mathbb{C}^{n,n}$ with $J^H = -J$ and $R^H = R \geq 0$. T.f.a.e.

1. There exists integer $m \geq 0$ such that $\text{rank}[R, JR, \dots, J^m R] = n$.
2. There exists integer $m \geq 0$ such that $\mathcal{T}_m := \sum_{j=0}^m J^j R (J^H)^j > 0$.
3. No eigenvector of J lies in the kernel of R .
4. $\text{rank}[\lambda I - J, R] = n$ for every $\lambda \in \mathbb{C}$.

Moreover, the smallest possible m in 1. and 2. coincide.

- F. Achleitner, A. Arnold, and V. Mehrmann. Hypocoercivity and controllability in linear semi-dissipative ODEs and DAEs. Vol. 103, *ZAMM, Zeitschrift f. Angewandte Mathematik und Mechanik*, e202100171, 2021. <http://arxiv.org/abs/2104.07619>
<https://doi.org/10.1002/zamm.202100171>



Definition

Let $J, R \in \mathbb{C}^{n,n}$ with $J^H = -J$ and $R^H = R \geq 0$.

The **hypocoercivity index (HC-index)** $m_{HC}(B)$ of $-B = A = J - R$ is the smallest integer m such that $\sum_{j=0}^m J^j R (J^H)^j > 0$. If m is finite then we call B **hypocoercive**.

Numerically feasible way to check hypocoercivity via controllability staircase from.

- ▶ F. Achleitner, A. Arnold, and V. Mehrmann. Hypocoercivity and controllability in linear semi-dissipative ODEs and DAEs. Vol. 103, *ZAMM, Zeitschrift f. Angewandte Mathematik und Mechanik*, e202100171, 2021. <http://arxiv.org/abs/2104.07619>
<https://doi.org/10.1002/zamm.202100171>



Global monotone decay for any initial value and worst case.

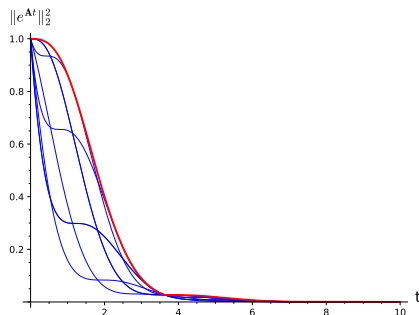


Figure: For the ODE with $A = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$, the squared norm $(\|e^{At}\|_2^2 \sim 1 - t^3/6 + \mathcal{O}(t^4)$ for $t \rightarrow 0+$), (red line) and the squared norms of a family of solutions (blue lines) are plotted.



HC-index and short-time decay

Relation between HC-index and **short-time decay** of $\|e^{-Bt}\|_2$.

Theorem

Consider a **semi-dissipative ODE** $\dot{x} = Ax = -Bx = (J - R)x$ where B has finite HC-index. HC-index is m_{HC} if and only if

$$\|e^{-Bt}\|_2 = 1 - ct^a + \mathcal{O}(t^{a+1}) \quad \text{for } t \rightarrow 0+,$$

where $c > 0$ and $a = 2m_{HC}(B) + 1$.

**But: HC index not invariant under state space transformations.
Can we get optimal local and global decay?**



Optimal local and global decay

Lemma

Consider **semi-dissipative ODE** $\dot{x} = -Bx = Ax = (J - R)x$, with finite HC-index. Let $-\mu < 0$ be maximal real part of evs of A .

1. If every ev with real part $-\mu$ is semi-simple, then exists $E_\mu > 0$ s.t.

$$-E_\mu A - A^H E_\mu \geq 2\mu E_\mu,$$

and inequality is not strict.

2. If at least one ev with real part $-\mu$ is defective, i.e. has Jordan block larger than 1 to λ , then for every $\epsilon > 0$ exists $E_\mu > 0$ s.t.

$$-E_\mu A - A^H E_\mu \geq 2(\mu - \epsilon)E_\mu.$$

- ▶ F. Achleitner, A. Arnold, and E. Carlen. *The hypocoercivity index for the short time behavior of linear time-invariant ODE systems*. Journal of Differential Equations 371 (2023): 83-115.
- ▶ A. Arnold, S. Egger, V. M., and E. Nigsch, *Connection of hypocoercivity and hypocontractivity via the Cayley transform*. Submitted 2026. <http://arxiv.org/abs/2601.09656>
- ▶ C. Beattie, V.M., H. Xu, and H. Zwart, *Optimally robust representations of linear time-invariant asymptotically stable and strictly passive systems*. In preparation.



Coercive representations

Use E_μ to represent the ODE as **dissipative Hamiltonian DAE**

$E_\mu \dot{x} = E_\mu Ax$ or equivalently set $\xi = E_\mu^{-\frac{1}{2}} x$ to obtain

semi-dissipative (dissipative) ODE $\dot{\xi} = \left(E_\mu^{-\frac{1}{2}} A E_\mu^{-\frac{1}{2}} \right) \xi = (\tilde{J} - \tilde{R}) \xi$.

Theorem

*The transformation to dissipative form with such an E_μ yields a **coercive system**, i.e. the Hermitian part is positive definite and the **local and global decay rate** is bounded by $-\mu t$.*

Moreover, the field of values is to the left of the line $-\mu + i\mathbb{R}$.

Thus, no transient growth and optimal decay with constant μ .

Similar results in inf. dim case.

Advantage of DAE formulation: limiting cases with singular E_μ can be considered.

- ▶ A. Arnold, S. Egger, V. M., and E. Nigsch, *Connection of hypocoercivity and hypocontractivity via the Cayley transform*. Submitted 2026. <http://arxiv.org/abs/2601.09656>
- ▶ C. Beattie, V.M., H. Xu, and H. Zwart, *Optimally robust representations of linear time-invariant asymptotically stable and strictly passive systems*. In preparation.



Consider **semi-dissipative homogeneous DAE**

$$E\dot{x} = (J - R)x$$

with $E = E^H \geq 0$, $J = -J^H$ and $R = R^H \geq 0$

Can we extend hypocoercivity analysis to this case?

Lemma (Structured Kronecker form)

Consider semi-dissipative DAE. There exist nonsingular L, Z such that

$$\hat{E} := L E Z =: \begin{bmatrix} E_{1,1} & 0 & 0 & 0 & 0 \\ 0 & E_{2,2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \hat{A} := L A Z =: \begin{bmatrix} 0 & 0 & 0 & I & 0 \\ 0 & A_{2,2} & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ -I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Matrices are partitioned conformably, with (square) diagonal blocks of sizes $n_1, n_2, n_3, n_4 = n_1, n_5$. $E_{1,1}$ and $E_{2,2}$ Hermitian positive definite. Hermitian part of $A_{2,2}$ positive semidefinite.



Definition

Consider **semi-dissipative homogeneous DAE**

$$E\dot{x} = (J - R)x$$

with $E = E^H \geq 0$, $J = -J^H$ and $R = R^H \geq 0$. **HC-index** m_{HC} of $\lambda E - (J - R)$ is defined as the HC-index of $(E_{2,2}^{1/2})^{-1} A_{2,2} (E_{2,2}^{1/2})^{-1}$, otherwise it is defined as 0. If the HC-index is finite then the DAE is called **hypocoercive**.

Open problem: Define HC-index for DAEs without Kronecker form.



Consider semi-norm $\|x(t)\|_E = \langle x, Ex \rangle^{\frac{1}{2}}$, let $S(t)$ be the evolution operator of the DAE $E\dot{x} = Ax = (J - R)x$ with $E = E^H \geq 0$, $J = -J^H$ and $R = R^H \geq 0$.

Theorem

Consider regular hypocoercive semi-dissipative DAE with non-trivial dynamics, consistent initial condition $x(0)$. The HC-index is m_{HC} if and only if

$$\|S(t)\|_E = 1 - ct^a + \mathcal{O}(t^{a+1}) \quad \text{for } t \rightarrow 0+,$$

where $c > 0$ and $a = 2m_{HC} + 1$, and the semi-norm is

$$\|S(t)\|_E := \sup_{\|x(0)\|_E=1, \text{ for consistent } x(0)} \|x(t)\|_E, \quad t \geq 0.$$



Problem with semi-norm

For $\varepsilon > 0$, consider linear semi-dissipative DAE system with

$$E = \begin{bmatrix} 5 & 3 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1 & 1 & \varepsilon \\ -1 & 0 & 1 & 0 \\ -1 & -1 & 0 & 0 \\ -\varepsilon & 0 & 0 & 0 \end{bmatrix}, \quad R := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In almost Kronecker form with $n_1 = n_2 = n_3 = n_4 = 1$.

For given $y_2(0) \in \mathbb{R}$, the solution is

$$y_1(t) = 0, \quad y_2(t) = y_2(0) e^{-t}, \quad y_3(t) = -y_2(0) e^{-t}, \quad y_4(t) = -\frac{3}{\varepsilon} y_2(0) e^{-t},$$

and $y_4(0) = -3y_2(0)/\varepsilon$ can be arbitrarily large for small $\varepsilon > 0$.
In contrast, the squared weighted semi-norm of this solution satisfies $\|y(t)\|_E^2 = 2(y_2(0))^2 e^{-2t}$ for $t \geq 0$.



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Lemma (Achleitner, Arnold, M., Nigsch 2023)

Let $J = -J^* \in \mathcal{B}(\mathcal{H})$, $0 \leq R = R^* \in \mathcal{B}(\mathcal{H})$. Consider conditions:

1. There exists $m \in \mathbb{N}_0 \cup \{\infty\}$ s.t. $\overline{\text{Span}}(\bigcup_{j=0}^m \text{Im}(J^j \sqrt{R})) = \mathcal{H}$.
2. There exists $m \in \mathbb{N}_0 \cup \{\infty\}$ s.t. $\bigcap_{j=0}^m \ker(\sqrt{R}(J^*)^j) = \{0\}$.
3. There exists $m \in \mathbb{N}_0 \cup \{\infty\}$ s.t. for all $x \in \mathcal{H} \setminus \{0\}$ there exists $j \leq m$ if m is finite or $j \in \mathbb{N}$ if $m = \infty$ s. t. $\langle J^j R (J^*)^j x, x \rangle > 0$.
4. If a closed subspace V of $\ker R$ is invariant under J then $V = \{0\}$.
5. No eigenvector of J lies in the kernel of R .

Then the implications $1. \Leftrightarrow 2. \Leftrightarrow 3. \Leftrightarrow 4. \Rightarrow 5.$ hold.

The implication $5. \Rightarrow 4.$ holds if and only if $\dim \ker R < \infty$.

Definition (Achleitner, Arnold, M. 2023)

Let $\mathcal{H}$ be a separable Hilbert space. For semi-dissipative operators $A = J - R \in \mathcal{B}(\mathcal{H})$, the (HC-index) m_{HC} of A is defined as the smallest integer $m \in \mathbb{N}_0$ (if it exists) such that

$$\sum_{j=0}^m (A^*)^j R A^j \geq \kappa I$$

for some $\kappa > 0$.

Actually for bounded operators the HC-index can only be finite.

- ▶ F. Achleitner, A. Arnold, and V. M. *Hypocoercivity in algebraically constrained partial differential equations with application to Oseen equations*. J. Dynamics Differential Eq., 1–40, 2023. <http://arxiv.org/2212.06631>
- ▶ F. Achleitner, A. Arnold, V. M., E. Nigsch. *Hypocoercivity in Hilbert spaces*. J. Functional Analysis, Vol. 288, 110691, 2024. <http://arxiv.org/2307.08280>



$E = I$ case: Consider the semigroup $S(t) := e^{-Bt}$ ($t \geq 0$) and its operator norm $\|S(t)\| := \sup\{\|S(t)x\| : \|x\| = 1\}$.

Theorem (Achleitner, Arnold, M., Nigsch 2023)

Let the operator $-B = A = J - R \in \mathcal{B}(\mathcal{H})$ be semi-dissipative.

- a) *If B has HC index m_{HC} , then there exist $c_1 \geq c \geq c_2 > 0$, $C_1, C_2 \geq 0$ and $t_0 > 0$ such that with $a := 2m_{HC} + 1$ we have*

$$1 - c_1 t^a - C_1 t^{a+1} \leq \|S(t)\| \leq 1 - c_2 t^a + C_2 t^{a+1} \quad \text{for } t \in [0, t_0]$$

and

$$\|S(t)\| = 1 - ct^a + o(t^a) \quad \text{for } t \rightarrow 0.$$

- b) *Conversely, if $\|S(t)\|$ satisfies the bounds with some $c > 0$ and $a > 0$, then a is an odd integer and B has HC index $m_{HC} = \frac{a-1}{2}$.*



- ▶ If $\|S(t)\|$ is sufficiently often differentiable, then we get the same result as in finite dimensional case.
- ▶ Under some extra assumptions controllability staircase form via a nested sequence of subspaces.
- ▶ Successful approach: Use modal (Fourier) decomposition and apply finite dimensional result to modes.



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Isotropic Oseen on 2D torus

Incompressible Oseen equation with *isotropic* viscosity on 2D torus $\mathbb{T}^2 := (0, 2\pi)^2$,

$$\begin{aligned}u_t &= -(b \cdot \nabla)u - \nabla p + \nu \Delta u, t > 0, \quad \text{on } \mathbb{T}^2, \\0 &= -\operatorname{div} u, t \geq 0,\end{aligned}$$

velocity $u(x, t)$, pressure $p(x, t)$ in $x \in \mathbb{T}^2$ and $t \geq 0$.

Assume periodic boundary conditions in both u and p .

Here $\nu > 0$ is the viscosity and $b \in \mathbb{R}^2$ constant drift velocity.

For any initial condition

$$u(0) \in \{u \in (L^2(\mathbb{T}^2))^2 \mid \operatorname{div} u = 0\},$$

the equation has a unique smooth solution for $t > 0$.



Consider Fourier expansion with

$$u(x, t) = \sum_{k \in \mathbb{Z}^2} \phi_k(t) e^{ik \cdot x}, \quad p(x, t) = \sum_{k \in \mathbb{Z}^2} p_k(t) e^{ik \cdot x}.$$

Fourier coefficients $\phi_k(t) \in \mathbb{C}^2$, $p_k(t) \in \mathbb{C}$, $k \in \mathbb{Z}^2$, satisfy decoupled DAEs

$$\begin{cases} \frac{d}{dt} \phi_k &= -ikp_k - i(b \cdot k)\phi_k - \nu|k|^2\phi_k, & t > 0, \\ 0 &= -ik \cdot \phi_k. \end{cases}$$

For $k \neq 0$, DAEs exhibit non-trivial dynamics with HC-index 0. The solution $(u(\cdot, t), p(\cdot, t))$ of the Oseen equation converges, as $t \rightarrow \infty$, to the constant (in x and t) equilibrium $(\phi_0, p_0) \in \mathbb{R}^3$, for $p_0 = 0$ with the exponential decay rate $\mu = \min_{k \neq 0} (\nu|k|^2) = \nu$.



Anisotropic Oseen equations on 2D torus.

$$\begin{aligned}u_t &= -(b(x) \cdot \nabla)u - \nabla p + \nu \partial_{x_2}^2 u, t > 0, \text{ on } \mathbb{T}^2, \\0 &= -\operatorname{div} u, t \geq 0,\end{aligned}$$

subject to periodic boundary conditions.

Drift velocity vector $b(x) \in \mathbb{R}^2$ may depend on $x \in \mathbb{T}^2$, and diffusion happens only in x_2 .



Take $b_1 = \sin(x_2)$ and perform a modal decomposition for $k \in \mathbb{Z}^2$. Due to the non-constant coefficient $b_1(x_2)$, these modes decouple now only w.r.t. k_1 , but not w.r.t. k_2 and for each mode $k \in \mathbb{Z}^2$ we have the DAE system

$$\begin{cases} \frac{d}{dt}\phi_k &= \frac{k_1}{2}(\phi_{k+e_2} - \phi_{k-e_2}) - ikp_k - \nu k_2^2 \phi_k, & t > 0, \\ 0 &= -ik \cdot \phi_k, \end{cases}$$

Proposition

Let $b_1 = \sin(x_2)$. Then for all $k_1 \in \mathbb{Z} \setminus \{0\}$, the modal dynamics is hypocoercive in $\ell^2(\mathbb{Z}; \mathbb{C})$ with HC-index $m_{HC} = 1$.



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Abstract dissipative Hamiltonian DAEs

Abstract linear differential-algebraic equation (DAE)

$$\mathcal{E}\dot{x}(t) = \mathcal{A}\mathcal{Q}x(t)$$

on infinite-dimensional Hilbert space $\mathbb{X}$ with inner product $\langle \cdot, \cdot \rangle$.

Assume $\mathcal{A} : D(\mathcal{A}) \subseteq \mathbb{X} \rightarrow \mathbb{X}$ is a **(semi)-dissipative linear operator**, i.e., $\langle \mathcal{A}x, x \rangle + \langle x, \mathcal{A}x \rangle \leq 0$ f.a. x in domain of $\mathcal{A}$.

Operators $\mathcal{E} : \mathbb{X} \rightarrow \mathbb{X}$ and $\mathcal{Q} : \mathbb{X} \rightarrow \mathbb{X}$ are assumed to be bounded and define a nonnegative *Hamiltonian* via

$$\mathcal{H}(x) := \langle \mathcal{E}x, \mathcal{Q}x \rangle,$$

i.e., $\mathcal{H}(x) \geq 0$ for all $x \in \mathbb{X}$.

- ▶ M. Erbay, B. Jacob, K. Morris, T. Reis, and C. Tischendorf. Index concepts for linear differential-algebraic equations in finite and infinite dimensions. arXiv preprint, 2024.
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- ▶ H. Zwart and V.M. *Abstract dissipative Hamiltonian differential-algebraic equations are everywhere*. DAE Panel 2024. <https://arxiv.org/5223640>.



Vibrating string example

Vibrating string in 1D domain Ω can be modelled by PDE

$$\rho \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial \zeta} \left(T \frac{\partial w}{\partial \zeta} \right),$$

ρ mass density, w vertical displacement, T is Young modulus.
First order in time formulation: Introducing

$$x(t) = \begin{bmatrix} \frac{\partial w}{\partial t} \\ \frac{\partial w}{\partial \zeta} \end{bmatrix},$$

gives **ODAE representation** in $\mathbb{X} = L^2(\Omega; \mathbb{R}^2)$

$$\underbrace{\begin{bmatrix} \rho & 0 \\ 0 & 1 \end{bmatrix}}_{\mathcal{E}} \dot{x}(t) = \underbrace{\begin{bmatrix} 0 & \frac{\partial}{\partial \zeta} \\ \frac{\partial}{\partial \zeta} & 0 \end{bmatrix}}_{\mathcal{A}} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & T \end{bmatrix}}_{\mathcal{Q}} x(t),$$
$$\mathcal{E} \quad \dot{x}(t) = \mathcal{A} \quad \mathcal{Q} \quad x(t).$$

Both limits $\rho \rightarrow 0$, and $T \rightarrow 0$ make sense giving $\mathcal{E}$ or $\mathcal{Q}$ singular.



- i) The state-space is a Hilbert space $\mathbb{X} = \mathbb{X}_1 \oplus \mathbb{X}_2 \oplus \mathbb{X}_3$.
- ii) $\mathcal{A} = \begin{bmatrix} \mathcal{A}_1 \\ \mathcal{A}_2 \\ \mathcal{A}_3 \end{bmatrix}$ is a *dissipative operator* on $\mathbb{X}$, i.e., $\operatorname{Re}\langle \mathcal{A}x, x \rangle \leq 0$ for all x in the domain $D(\mathcal{A})$.
- iii) $\mathcal{E}$ and $\mathcal{Q}$ are block-operators of the form

$$\mathcal{E} = \begin{bmatrix} \mathcal{E}_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathcal{E}_3 \end{bmatrix}, \quad \mathcal{Q} = \begin{bmatrix} \mathcal{Q}_1 & 0 & 0 \\ 0 & \mathcal{Q}_2 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where $\mathcal{E}_1, \mathcal{E}_3, \mathcal{Q}_1$ and $\mathcal{Q}_2$ are bounded and boundedly invertible. Furthermore, $\mathcal{E}_1^* \mathcal{Q}_1$ is self-adjoint and (strictly) positive, and defines a Hamiltonian $\mathcal{H}(x) := \langle \mathcal{E}x, \mathcal{Q}x \rangle$.

- iv) There exists $s \in \mathbb{C}^+ := \{s \in \mathbb{C} \mid \operatorname{Re}(s) > 0\}$ such that $s\mathcal{E} - \mathcal{A}\mathcal{Q}$ with domain $\{x \in \mathbb{X} \mid \mathcal{Q}x \in D(\mathcal{A})\}$ is boundedly invertible.



Theorem

Consider aHDAE satisfying assumptions. Then operator defined by

$$\tilde{\mathcal{A}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} := \begin{bmatrix} \mathcal{A}_1 \\ \mathcal{A}_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix},$$

$$D(\tilde{\mathcal{A}}) = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{X}_1 \oplus \mathbb{X}_2 \mid \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} \in D(\mathcal{A}) \right\}.$$

is dissipative and with

$$\tilde{\mathcal{E}} = \begin{bmatrix} \mathcal{E}_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{\mathcal{Q}} = \begin{bmatrix} \mathcal{Q}_1 & 0 \\ 0 & \mathcal{Q}_2 \end{bmatrix},$$

the pair $(\tilde{\mathcal{E}}, \tilde{\mathcal{A}}\tilde{\mathcal{Q}})$ is regular.



Theorem

Consider regular aHDAE $(\mathcal{E}, \mathcal{A}\mathcal{Q})$ satisfying assumptions and, whenever $\begin{bmatrix} 0 \\ x_2 \end{bmatrix} \in D(\mathcal{A})$ is such that $\mathcal{A}_2 \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = 0$, then $x_2 = 0$. Then operator $\mathcal{A}_{red} : D(\mathcal{A}_{red}) \subset \mathbb{X}_{1,\mathcal{E}\mathcal{Q}} \rightarrow \mathbb{X}_{1,\mathcal{E}\mathcal{Q}}$ defined by

$$\mathcal{A}_{red}x_1 = \mathcal{E}_1^{-1}\mathcal{A}_1 \begin{bmatrix} \mathcal{Q}_1x_1 \\ \mathcal{Q}_2x_2 \end{bmatrix}$$

$$D(\mathcal{A}_{red}) = \{x_1 \in \mathbb{X}_{1,\mathcal{E}\mathcal{Q}} \mid \exists x_2 \in \mathcal{X}_2 \text{ s.t. } \begin{bmatrix} \mathcal{Q}_1x_1 \\ \mathcal{Q}_2x_2 \end{bmatrix} \in D(\mathcal{A}), \mathcal{A}_2 \begin{bmatrix} \mathcal{Q}_1x_1 \\ \mathcal{Q}_2x_2 \end{bmatrix} = 0\}$$

generates contraction semigroup on reduced space $\mathbb{X}_{1,\mathcal{E}\mathcal{Q}}$.

More results and applications in:

- ▶ H. Zwart and V. M. *Abstract dissipative Hamiltonian differential-algebraic equations are everywhere*. DAE Panel 2024. <https://arxiv.org/5223640>.



- ▶ Theory of hypocoercivity for abstract dissipative Hamiltonian DAEs.
- ▶ What does Galerkin projection do to the HC index?
- ▶ Unbounded operators?



- 1 Introduction
- 2 Classical evolution equations
- 3 Finite dimensional case
- 4 Infinite dimension
- 5 Example: Oseen equations
- 6 Abstract dissipative Hamiltonian DAEs
- 7 Hypocoercivity for nonlinear systems**



- ▷ Which classes of nonlinear problems allow hypocoercivity theory?
- ▷ HC index definition.
- ▷ HC index of the system and its linearization?
- ▷ Local in initial value and local near equilibrium.
- ▷ Global monotone decay.

Some partial results for semidissipative Hamiltonian ODEs

- ▷ A. Arnold, S. Egger, V. M., and E. Nigsch, *Hypocoercivity for nonlinear semi-dissipative Hamiltonian systems*. In preparation, 2026.



Semidissipative Hamiltonian ODEs

Nonlinear (finite dim.) dissipative Hamiltonian ODE

$$\dot{x} = F(x) = (J(x) - R(x))\nabla H(x), \quad x(0) = x_0.$$

Here $J, R: U \rightarrow \mathbb{R}^{n \times n}$ are real matrix functions on open set $U \subseteq \mathbb{R}^n$ satisfying $J(x)^\top = -J(x)$, $R(x)^\top = R(x) = G(x)G(x)^\top \geq 0$ for all $x \in U$ pointwise and G has constant rank r . $H: U \rightarrow \mathbb{R}$ is given *Hamiltonian* (energy function), and $x_0 \in U$.

Let Φ_t denote the local nonlinear flow, i.e., $\Phi_t(x_0)$ is the value at time t of the solution with initial condition $x(0) = x_0$.



Local approximation

Assume that $X_0 = \nabla^2 H(0) > 0$ is a nondegenerate strict local minimum of H . Hence $H(x) > 0$ near 0 for $x \neq 0$, and Taylor expansion gives

$$H(x) = \frac{1}{2}x^\top X_0 x + O(\|x\|_{\mathbb{R}^n}^3).$$

Thus H corresponds to the energy distance squared, to leading order. For all sufficiently small *amplitudes* $\rho > 0$ the local energy shell

$$\Sigma_\rho := \{x \in U_0 : H(x) = \rho^2/2\}$$

then is compact and

$$M_\rho(t) := \max_{x \in \Sigma_\rho} \left(\frac{H(\Phi_t(x))}{H(x)} \right)^{1/2}$$

exists for all $t \geq 0$.



Local initial decay theorem

Consider a semi-dissipative Hamiltonian system on an open neighborhood of 0, with $R = GG^\top$. Assume that H , J , and G are C^∞ near 0, and that 0 is a nondegenerate local minimum of H , i.e.

$$\nabla H(0) = 0, \quad X_0 := \nabla^2 H(0) > 0.$$

Let $M_\rho(t)$ and $M_{\leq \rho}(t)$ be defined by the local energy shells. Assume that linearized system at equilibrium has HC-index exactly m . Set

$$A_0 := (R(0) - J(0))X_0, \quad \tilde{A}_0 := X_0^{1/2}(R(0) - J(0))X_0^{1/2}$$

so that $\tilde{A}_0 = X_0^{1/2}A_0X_0^{-1/2}$ is the Euclidean norm representative of the linearized operator.



Hypoocoercivity and initial decay

Let $c_0 > 0$ be the corresponding linear short-time constant, i.e.

$$\|e^{-\tilde{A}_0 t}\|_2 = 1 - c_0 t^{2m+1} + O(t^{2m+2}) \quad (t \downarrow 0).$$

Then there are $\rho_0 > 0$, $t_0 > 0$, and $c_-, C_+ > 0$ such that

$$M_\rho(t) \leq 1 - c_- t^{2m+1}, \quad 0 < \rho \leq \rho_0, \quad 0 < t \leq t_0,$$

and

$$1 - C_+ t^{2m+1} \leq M_{\leq \rho_0}(t) \leq 1 - c_- t^{2m+1}, \quad 0 < t \leq t_0.$$

Consequently,

$$0 < c_- \leq \liminf_{t \downarrow 0} \frac{1 - M_{\leq \rho_0}(t)}{t^{2m+1}} \leq \limsup_{t \downarrow 0} \frac{1 - M_{\leq \rho_0}(t)}{t^{2m+1}} \leq C_+ < \infty.$$

In this sense the worst short-time decay is the same exponent $2m + 1$ as in the linearization.



- ▷ Initial and global decay rates via HC-index.
- ▷ In linear evolution equation case local and global optimal decay can be characterized.
- ▷ Results for DAEs in finite dimensions.
- ▷ Results for operator DAEs?
- ▷ First results for special class of nonlinear semidissipative ODEs.
- ▷ Extension to nonlinear infinite dimensional and DAE case?



Thank you very much
for your attention

Details: <http://www.math.tu-berlin.de/?id=76888>