Establishing decay rate of infinite-dimensional systems SCINDIS 2026

September 30, 2026

Kirsten Morris

UNIVERSITY OF WATERLOO | FACULTY OF MATHEMATICS
Department of Applied Mathematics

Acknowledgements

Joint work with Anita Gjetesland, Luis Mora & David Del Ray Fernandez (Waterloo)

Funded by NSERC

Establishing decay rate

Controller design

Higher-order approximations

Summary

port-Hamiltonian system with boundary dissipation

$$\frac{\partial z(x,t)}{\partial t} = P_1 \frac{\partial}{\partial x}\left(Qz(x,t)\right), \quad P_1 = P_1^\top, x \in [a,b] \tag{1}$$

$$\mathbf{u}_a(t) = 0 \quad \text{and} \quad \mathbf{u}_b(t) + K\mathbf{y}_b(t) = 0, \quad K = K^\top \geq 0.$$

port-Hamiltonian system with boundary dissipation

$$\frac{\partial z(x,t)}{\partial t} = P_1 \frac{\partial}{\partial x}\left(Qz(x,t)\right), \quad P_1 = P_1^\top, x \in [a,b] \tag{1}$$

$$\mathbf{u}_a(t) = 0 \quad \text{and} \quad \mathbf{u}_b(t) + K\mathbf{y}_b(t) = 0, \quad K = K^\top \geq 0.$$

$$\begin{bmatrix} \mathbf{u}_a(t) \\ \mathbf{y}_a(t) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} W_a \\ \widetilde{W}_a \end{bmatrix}\begin{bmatrix} -P_1 \\ I \end{bmatrix} Qz(a,t) \text{ and } \begin{bmatrix} \mathbf{u}_b(t) \\ \mathbf{y}_b(t) \end{bmatrix} = \frac{1}{2}\underbrace{\begin{bmatrix} W_b \\ \widetilde{W}_b \end{bmatrix}}_{\Psi_b}\begin{bmatrix} P_1 \\ I \end{bmatrix} Qz(b,t)$$

Assume B.C.s so system well-posed on $L^2(a,b),\mathbb{R}^n)$.

Energy Dissipation

Total energy: $\mathcal{H}(t) = \frac{1}{2}\int_a^b z^\top(x,t)Qz(x,t)\mathrm{d}x$

$$\frac{\mathrm{d}\mathcal{H}(t)}{\mathrm{d}t} = \boldsymbol{y}_b(t)^\top \mathbf{u}_b(t) \tag{2}$$

$$= -\boldsymbol{y}_b(t)^\top K \boldsymbol{y}_b(t) \tag{3}$$

$$\leq -\lambda_{min}(K)\|\boldsymbol{y}_b(t)\|^2 \tag{4}$$

Energy Dissipation

Total energy: $\mathcal{H}(t) = \frac{1}{2}\int_a^b z^\top(x,t)Qz(x,t)\mathrm{d}x$

$$\frac{\mathrm{d}\mathcal{H}(t)}{\mathrm{d}t} = \boldsymbol{y}_b(t)^\top \mathbf{u}_b(t) \tag{2}$$

$$= -\boldsymbol{y}_b(t)^\top K \boldsymbol{y}_b(t) \tag{3}$$

$$\leq -\lambda_{min}(K)\|\boldsymbol{y}_b(t)\|^2 \tag{4}$$

Theorem (Villegas, Zwart, Le Gorrec, & Maschke, B. 2009)

If $K > 0$ then there is $M \geq 1, \alpha > 0$ so that

$$\|\mathcal{H}(t)\| \leq Me^{-\alpha t}\|\mathcal{H}(0)\|.$$

Energy Dissipation

Total energy: $\mathcal{H}(t) = \frac{1}{2}\int_a^b z^\top(x,t)Qz(x,t)\mathrm{d}x$

$$\frac{\mathrm{d}\mathcal{H}(t)}{\mathrm{d}t} = \boldsymbol{y}_b(t)^\top \mathbf{u}_b(t) \tag{2}$$

$$= -\boldsymbol{y}_b(t)^\top K \boldsymbol{y}_b(t) \tag{3}$$

$$\leq -\lambda_{min}(K)\|\boldsymbol{y}_b(t)\|^2 \tag{4}$$

Theorem (Villegas, Zwart, Le Gorrec, & Maschke, B. 2009)

If $K > 0$ then there is $M \geq 1, \alpha > 0$ so that

$$\|\mathcal{H}(t)\| \leq Me^{-\alpha t}\|\mathcal{H}(0)\|.$$

Decay rate α?

Outline of proof of result to find decay rate

Basic idea: Modify the Energy function to an equivalent one that has $\dot{V}(t) \leq \delta V(t)$: For some functional w and $\varepsilon_0 > 0$, define $V(t) = \mathcal{H}(t) + \varepsilon w(z,t)$ with $\varepsilon \in [0, \varepsilon_0)$ and show

1. $\left(1 - \frac{\varepsilon}{\varepsilon_0}\right) \mathcal{H}(t) \leq V(t) \leq \left(1 + \frac{\varepsilon}{\varepsilon_0}\right) \mathcal{H}(t)$
2. $\dot{V}(t) \leq -\delta V(t)$ for some $\delta > 0$.

Outline of proof of result to find decay rate

Basic idea: Modify the Energy function to an equivalent one that has $\dot{V}(t) \leq \delta V(t)$: For some functional w and $\varepsilon_0 > 0$, define $V(t) = \mathcal{H}(t) + \varepsilon w(z,t)$ with $\varepsilon \in [0, \varepsilon_0)$ and show

1. $\left(1 - \frac{\varepsilon}{\varepsilon_0}\right) \mathcal{H}(t) \leq V(t) \leq \left(1 + \frac{\varepsilon}{\varepsilon_0}\right) \mathcal{H}(t)$
2. $\dot{V}(t) \leq -\delta V(t)$ for some $\delta > 0$.

$$
\begin{aligned}
V(t) &\leq e^{-\delta t} V(0) \\
&\leq e^{-\delta t} \left(1 + \frac{\varepsilon}{\varepsilon_0}\right) \mathcal{H}(0) \\
\left(1 - \frac{\varepsilon}{\varepsilon_0}\right) \mathcal{H}(t) &\leq e^{-\delta t} \left(1 + \frac{\varepsilon}{\varepsilon_0}\right) \mathcal{H}(0) \\
\mathcal{H}(t) &\leq \frac{1 + \frac{\varepsilon}{\varepsilon_0}}{1 - \frac{\varepsilon}{\varepsilon_0}} e^{-\delta t} \mathcal{H}(0).
\end{aligned}
$$

$V(t) = \mathcal{H}(t) + \varepsilon w(z,t)$. How to choose w?

Standard multiplier function (eg. Tucsnak & Weiss (2009))

$$w(t) = \frac{1}{2}\int_a^b m(x) z^\top(x,t) P_1^{-1} z(x,t)\mathrm{d}x, \qquad \boxed{\text{m(x)=x-a}}$$

- Show existence of $c_w > 0$, $\varepsilon_0 > 0$ and $\varepsilon_1 > 0$ so that

$$|w(t)| \leq \frac{1}{\varepsilon_0}\mathcal{H}(t)$$
$$\frac{\mathrm{d}w}{\mathrm{d}t} \leq -\frac{1}{\varepsilon_1}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} - c_w\mathcal{H}(t)$$

$V(t) = \mathcal{H}(t) + \varepsilon w(z,t)$. How to choose w?

Standard multiplier function (eg. Tucsnak & Weiss (2009))

$$w(t) = \frac{1}{2}\int_a^b m(x) z^\top(x,t) P_1^{-1} z(x,t)\mathrm{d}x, \qquad \boxed{\text{m(x)=x-a}}$$

- Show existence of $c_w > 0$, $\varepsilon_0 > 0$ and $\varepsilon_1 > 0$ so that

$$|w(t)| \leq \frac{1}{\varepsilon_0}\mathcal{H}(t) \quad \checkmark$$

$$\frac{\mathrm{d}w}{\mathrm{d}t} \leq -\frac{1}{\varepsilon_1}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} - c_w\mathcal{H}(t) \quad \Longleftarrow$$

Show $\frac{\mathrm{d}w}{\mathrm{d}t} \leq -\frac{1}{\varepsilon_1}\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} - c_w\mathcal{H}(t)$:

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_a^b m(x)z^\top(x,t)P_1^{-1}z(x,t)\mathrm{d}x = \int_a^b m(x)z^\top(x,t)P_1^{-1}\frac{\partial z(x,t)}{\partial t}\,\mathrm{d}x$$

$$= \int_a^b \underbrace{m(x)z^\top(x,t)\frac{\partial z(x,t)}{\partial x}}_{A}\,\mathrm{d}x \quad (Q = I)$$

Use $\frac{1}{2}\frac{\partial}{\partial x}\left(m(x)z(x)^T z(x)\right) = \frac{1}{2}z^\top(x,t)\left(\frac{\partial m(x)}{\partial x}\right)z(x,t) + \underbrace{m(x)z^\top(x,t)\frac{\partial z(x)}{\partial x}}_{A}$,

$$\frac{\mathrm{d}w(t)}{\mathrm{d}t} = \frac{1}{2}m(x)z^\top(x,t)z(x,t)\Big|_a^b - \frac{1}{2}\int_a^b z^\top(x,t)z(x,t)\,\mathrm{d}x$$

$$= \frac{1}{2}m(b)z^\top(b,t)z(b,t) - \underbrace{\frac{1}{2}\int_a^b z^\top(x,t)z(x,t)\,\mathrm{d}x}_{\mathcal{H}(t)}.$$

This leads to:

Theorem (Mora & KM 2022)

Define

$$\lambda_\Psi = \max \operatorname{eig}\left(\begin{bmatrix} -K \\ I \end{bmatrix}^\top \Psi_b^{-\top} Q^{-1} \Psi_b^{-1} \begin{bmatrix} -K \\ I \end{bmatrix} \right),$$

$$\eta_K = \min \operatorname{eig}(K),\ \eta_Q = \min \operatorname{eig}(Q),\ \lambda_P^2 = \max \operatorname{eig}(P_1^{-2}),$$

$$\varepsilon_0 = \frac{\eta_Q}{(b-a)\lambda_P}, \quad \varepsilon_1 = \frac{\eta_K}{2(b-a)\lambda_\Psi}.$$

For any $\varepsilon \in [0, \min\{\varepsilon_0, \varepsilon_1\})$, *define* $M(\varepsilon) = \frac{\varepsilon_0+\varepsilon}{\varepsilon_0-\varepsilon}$, $\alpha(\epsilon) = \frac{\varepsilon\varepsilon_0}{\varepsilon_0+\varepsilon}$

$$\mathcal{H}(t) \leq M(\varepsilon) e^{-\alpha(\varepsilon)t} \mathcal{H}(0)$$

This leads to:

Theorem (Mora & KM 2022)

Define

$$\lambda_\Psi = \max \operatorname{eig}\left(\begin{bmatrix} -K \\ I \end{bmatrix}^\top \Psi_b^{-\top} Q^{-1} \Psi_b^{-1} \begin{bmatrix} -K \\ I \end{bmatrix} \right),$$

$$\eta_K = \min \operatorname{eig}(K),\ \eta_Q = \min \operatorname{eig}(Q),\ \lambda_P^2 = \max \operatorname{eig}(P_1^{-2}),$$

$$\varepsilon_0 = \frac{\eta_Q}{(b-a)\lambda_P}, \quad \varepsilon_1 = \frac{\eta_K}{2(b-a)\lambda_\Psi}.$$

For any $\varepsilon \in [0, \min\{\varepsilon_0, \varepsilon_1\})$, *define* $M(\varepsilon) = \frac{\varepsilon_0+\varepsilon}{\varepsilon_0-\varepsilon}$, $\alpha(\epsilon) = \frac{\varepsilon\varepsilon_0}{\varepsilon_0+\varepsilon}$

$$\mathcal{H}(t) \leq M(\varepsilon) e^{-\alpha(\varepsilon)t} \mathcal{H}(0)$$

Example: Piezoelectric beam with magnetic effects

$v(x,t)$ is longitudual displacement, $p(x,t)$ electric charge

$$\rho \frac{\partial^2 v(x,t)}{\partial t^2} = \xi \frac{\partial}{\partial x}\frac{\partial v(x,t)}{\partial x} - \gamma\beta \frac{\partial}{\partial x}\frac{\partial p(x,t)}{\partial x}$$
$$\mu \frac{\partial^2 p(x,t)}{\partial t^2} = \beta \frac{\partial}{\partial x}\frac{\partial p(x,t)}{\partial x} - \gamma\beta \frac{\partial}{\partial x}\frac{\partial v(x,t)}{\partial x}$$

with boundary conditions

$$v(a,t) = p(a,t) = 0,$$
$$\beta \frac{\partial p(b,t)}{\partial x} - \gamma\beta \frac{\partial v(b,t)}{\partial x} = u_{b1}(t) \quad \textit{voltage}$$
$$\xi \frac{\partial v(b,t)}{\partial x} - \gamma\beta \frac{\partial p(b,t)}{\partial x} = u_{b2}(t) \quad \textit{mechanical force}$$

Piezoelectric beam (pH form)

PHS

$$\frac{\partial z(x,t)}{\partial t} = P_1 \frac{\partial z(x,t)}{\partial x},$$

$$\mathbf{u}_a(t) = \begin{bmatrix} z_1(a,t) \\ z_3(a,t) \end{bmatrix} = 0,$$

$$\boldsymbol{y}_b(t) = \begin{bmatrix} \frac{1}{\sqrt{\mu}} z_3(b,t) \\ \frac{1}{\sqrt{\rho}} z_1(b,t) \end{bmatrix}$$

$$\mathbf{u}_b(t) = \begin{bmatrix} \sqrt{\beta} z_4(b,t) \\ \sqrt{\xi_1} z_2(b,t) - \gamma\sqrt{\beta} z_4(b,t) \end{bmatrix}$$

$$\mathbf{u}_b(t) = K\boldsymbol{y}_b(t)$$

State variables and total energy

$$z(x,t) = \begin{bmatrix} z_1(x,t) \\ z_2(x,t) \\ z_3(x,t) \\ z_4(x,t) \end{bmatrix} = \begin{bmatrix} \sqrt{\rho}\frac{\partial v(x,t)}{\partial t} \\ \sqrt{\xi_1}\frac{\partial v(x,t)}{\partial x} \\ \sqrt{\mu}\frac{\partial p(x,t)}{\partial t} \\ \sqrt{\beta}\left(\frac{\partial p(x,t)}{\partial x} - \gamma\frac{\partial v(x,t)}{\partial x}\right) \end{bmatrix}$$

$$\mathcal{H}(t) = \int_a^b \frac{1}{2} z^\top(x,t) z(x,t) \mathrm{d}x$$

Decay rate for piezoelectric beam

Param.	Description	Value	Units
ξ_1	Elastic stiffness	7.99	N/m^2
β	Impermittivity	1.54e7	m/F
γ	Piezoelectric coefficient	1.72e-7	C/m^3
ρ	Mass density	6.83e3	kg/m^3
μ	Magnetic permeability	1e-2	H/m
ℓ	Beam length	6.67e-2	m
h	Beam thickness	1.8e-3	m

$$\varepsilon_0 = 0.5128,$$
$$\varepsilon_1 = \frac{29.9850K}{6830 + 0.1252K^2}.$$

$$\max \varepsilon_1 = 0.4956$$

$$Me^{-\alpha t} = \left(\frac{54553 + 467.0375K + K^2}{54553 - 467.0375K + K^2}\right) e^{-\left(\frac{239.4968K}{54553 + 467.0375K + K^2}\right)t}$$

With $K = 100$,

$$M = 6.2, \quad \alpha = 0.22.$$

More general pHS

$$\frac{\partial z(x,t)}{\partial t} - \left[P_1 \frac{\partial}{\partial x} + [P_0 - G_0]\right] Q(x) z(x,t) = 0 \tag{7}$$

(boundary conditions as before)

Choose $m(x) \in C^1([a,b])$ with $m(x) \geq 0$, $m(b) > 0$ and define

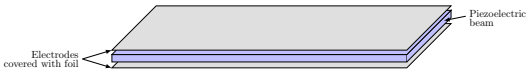
$$\begin{aligned} B(x) =& \frac{\partial Q(x)}{\partial x} - Q(x)(P_0 + G_0)P_1^{-1} + P_1^{-1}(P_0 - G_0)Q(x) \\ A_s(x) =& \frac{\partial m(x)}{\partial x} Q(x) - m(x)B(x) \end{aligned}$$

If there is $\eta_A > 0$ so that

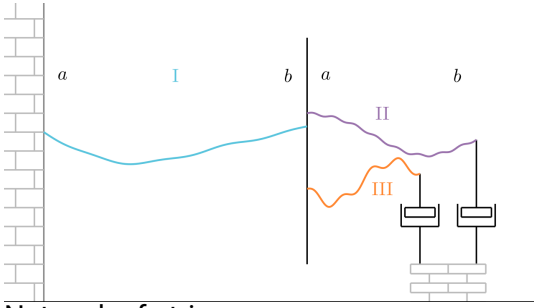
$$\inf_{a \leq x \leq b} \lambda_{min}\left(A_s(x)\right) \geq \eta_A, \tag{8}$$

then (7) is exponentially stable; upper bound on decay rate determined by physical parameters.

What is “enough” boundary dissipation?

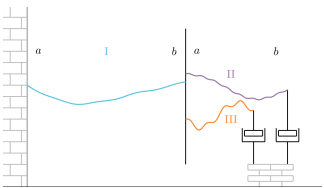


Piezoelectric beam



Network of strings
(Gernandt & Roschkowski (2026))

What is “enough” boundary dissipation?



(Gernandt & Roschkowski (2026))

Gernandt & Roschkowski (2026)

Chose $\gamma > 0$ is large enough that $\gamma Q(x) \geq \pm P_1^{-1}$, for all $x \in [a, b]$. The pH system is exponentially stable if and only for all $z(0) \in \mathrm{dom}(\mathrm{A})$, (1) there exists $0 < \sigma < \tau < T$ such that $\tau - \sigma > 2\gamma(b - a)$ and (2) $k > 0$ such that

$$\|z(T)\|_{\mathcal{X}}^2 - \|z(0)\|_{\mathcal{X}}^2 \leq -k \int_{\sigma}^{\tau} (z^* Q z)(t, b)\, \mathrm{d}t$$

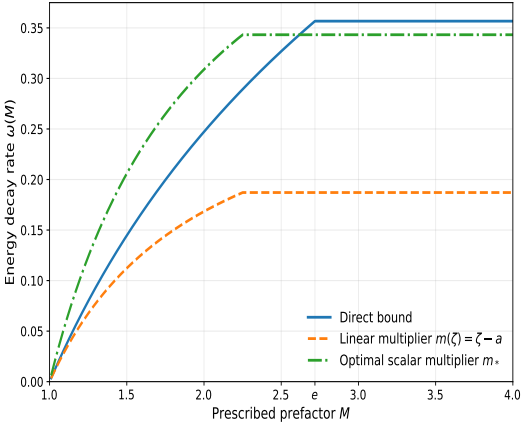
Moreover, for every $M > 1$ a lower bound on the decay rate α can be written in terms of the system parameters.

Choice of multiplier

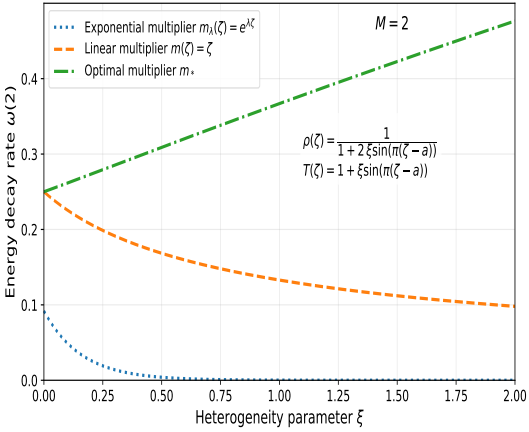
- normalized Timonshenko shown to be exponentially stable using Lyapunov argument in Mattioni et al (2024)
  - can't use linear multiplier; $m(x) = e^{\beta x}$ yields same decay rate as Mattioni et al
- Exponential multiplier does not yield better decay rate than linear for wave equation, or for Timoshenko beam with different parameters
- Gernandt & Roschkowski (2026) investigated matrix multipliers to improve bound and also optimized decay rate over a family of multiplier functions

What is best choice of multiplier for best decay rate?

(Gernandt & Roschkowski 2026), variable coefficient wave equation



α vs M



α vs heterogeneity ξ

Outline

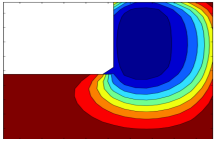
Establishing decay rate

Controller design

Higher-order approximations

Summary

Controller design for DPS





- generally a lumped approximation used in controller (and estimator) design
- not every design leads to closed loop performance predicted by the discretization
- uniform stability, or more generally, uniform stabilizability, key condition
- uniform stability or preservation of decay rate important for realistic simulations
- known conditions for parabolic systems
- conditions for hyperbolic systems are challenging

Closed-loop convergence

Theorem

If the approximations are uniformly exponentially stable, the controllers K_n obtained by solving the finite-dimensional linear quadratic problem converge to the optimal control. Also, $u(t) = -K_n z(t)$ with the original DPS leads to a sequence of uniformly exponentially stable systems and performance that converges to the optimal performance.

Closed-loop convergence

Theorem

If the approximations are uniformly exponentially stable, the controllers K_n obtained by solving the finite-dimensional linear quadratic problem converge to the optimal control. Also, $u(t) = -K_n z(t)$ with the original DPS leads to a sequence of uniformly exponentially stable systems and performance that converges to the optimal performance.

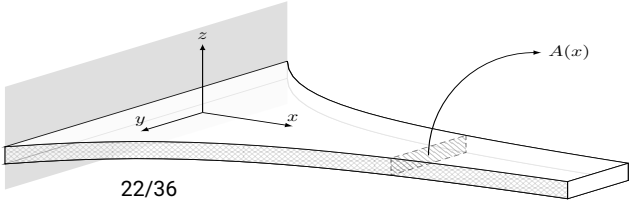
(Outline of key steps of proof)
Known that approximating semigroups converge strongly in the initial condition, uniformly on finite time intervals, so do approximate control operators $\mathcal{B}_n$.
Convergence of the adjoint systems: standard calculation shows that they have the same structure, implying also convergence
Use then results of [Gibson 1987], [Ito 1987]

Boundary-damped non-uniform port-Hamiltonian system

$$Q^{-1}(x)\frac{\partial}{\partial t}\begin{bmatrix}\mathbf{e}^q(x,t)\\ \mathbf{e}^p(x,t)\end{bmatrix} = P_1\frac{\partial}{\partial x}\begin{bmatrix}\mathbf{e}^q(x,t)\\ \mathbf{e}^p(x,t)\end{bmatrix} + \begin{bmatrix}B_q(x)\\ B_p(x)\end{bmatrix}\boldsymbol{u}(t),$$

$$A^\top \mathbf{e}^q(b,t) = -K\mathbf{e}^p(b,t), \quad \mathbf{e}^p(a,t) = 0,$$

- $P_1 = \left[\begin{smallmatrix} 0 & A \\ A^\top & 0\end{smallmatrix}\right]$ is invertible, $A \in \mathbb{R}^{n\times n}$, $Q(x) = \begin{bmatrix} Q_q(x) & 0 \\ 0 & Q_p(x)\end{bmatrix}$, $K = diag\,(\kappa_1, \ldots, \kappa_n) > 0$
- $B_q(x), B_p(x) \in \mathcal{L}\left(\mathbb{R}^m, L^2(a,b;\mathbb{R}^n)\right)$, $\mathbf{u}(t) \in \mathbb{R}^m$ is the control signal



UNIVERSITY OF WATERLOO | FACULTY OF MATHEMATICS Department of Applied Mathematics

Boundary-damped non-uniform port-Hamiltonian systems

$$Q^{-1}(x)\frac{\partial}{\partial t}\begin{bmatrix}\mathbf{e}^q(x,t)\\ \mathbf{e}^p(x,t)\end{bmatrix} = P_1\frac{\partial}{\partial x}\begin{bmatrix}\mathbf{e}^q(x,t)\\ \mathbf{e}^p(x,t)\end{bmatrix} + \begin{bmatrix}B_q(x)\\ B_p(x)\end{bmatrix}\boldsymbol{u}(t),$$

$$A^\top \mathbf{e}^q(b,t) = -K\mathbf{e}^p(b,t), \quad \mathbf{e}^p(a,t) = 0,$$

- $P_1 = \left[\begin{smallmatrix} 0 & A \\ A^\top & 0\end{smallmatrix}\right]$ is invertible, $A \in \mathbb{R}^{n\times n}$, $Q(x) = \left[\begin{smallmatrix} Q_q(x) & 0 \\ 0 & Q_p(x)\end{smallmatrix}\right]$, $K = diag\,(\kappa_1, \ldots, \kappa_n) > 0$
- $B_q(x), B_p(x) \in \mathcal{L}\left(\mathbb{R}^m, L^2(a,b;\mathbb{R}^n)\right)$, $\mathbf{u}(t) \in \mathbb{R}^m$ is the control signal

Assumption: there is $\delta^c > 0$ such that for every $i = 1 \ldots n,\ a \le x \le b,$

$$\theta_i^z(x) - (x-a)(\theta_i^z)'(x) > \delta^c\theta_i^z(x), \quad z = \{p, q\} \tag{A1}$$

Stability of boundary-damped port-Hamiltonian systems

Theorem

Defining $\epsilon_0 = \frac{\eta_Q}{\ell \mu_{P_1}}$ *and* $\epsilon_1 = \frac{2\eta_K}{\ell \mu_\Psi}$, $\eta_K = \min eig(K)$, $\mu_\Psi = \max eig(\Psi)$, $\Psi = \begin{bmatrix} A^{-\top}K \\ I_n \end{bmatrix}^\top Q^{-1}(b) \begin{bmatrix} A^{-\top}K \\ I_n \end{bmatrix}$, $\eta_Q = \min_{x\in[a,b]} eig(Q(x))$, *and* $\mu_{P_1} = \sqrt{\max eig(P_1^{-2})}$, *the uncontrolled system is exponentially stable with a decay rate bounded by*

$$\alpha = \frac{\delta^c \epsilon \epsilon_0}{\epsilon + \epsilon_0}, \quad \forall \epsilon \in (0, \min\{\epsilon_0, \epsilon_1\}).$$

Example: LQ control of anistropic wave equation

wave equation on $[0,1]$ with spatially varying physical parameters.

$$\left(\begin{bmatrix} \theta^q(x) & 0 \\ 0 & \theta^p(x) \end{bmatrix}\right)^{-1} \frac{\partial}{\partial t}\begin{bmatrix} e^q(x,t) \\ e^p(x,t) \end{bmatrix} = P_1 \frac{\partial}{\partial x}\begin{bmatrix} e^q(x,t) \\ e^p(x,t) \end{bmatrix} + \begin{bmatrix} 0 \\ b(x) \end{bmatrix} u(t),$$

$$e^p(0,t) = 0, \quad e^q(1,t) = -\kappa_1 e^p(1,t)$$

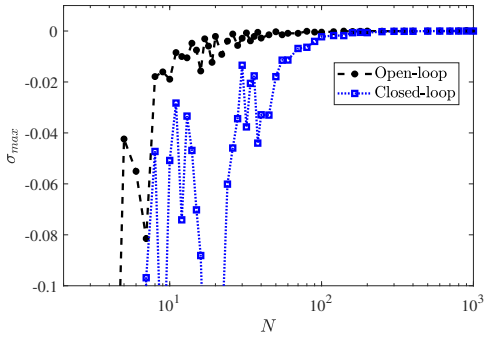
$P_1 = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, $\rho_0 = \tau_0 = 1$, $\kappa_1 = 0.5$, $\theta(x) = \frac{10-x}{10}$, $\theta^q(x) = \theta(x)\tau_0$, $\theta^p(x) = \frac{1}{\theta(x)\rho_0}$

$$b(x) = \begin{cases} 3 \times 10^4 x^2 (x-0.1)^2, & x \in [0, 0.1], \\ 0, & \text{otherwise} \end{cases}.$$

Previous result shows this system is exponentially stable

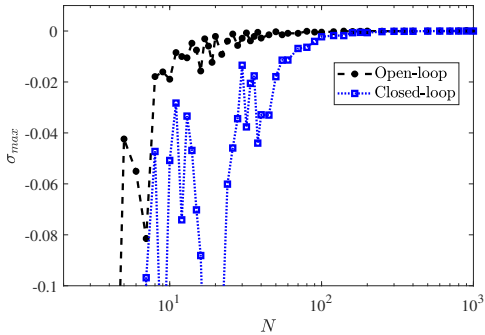
UNIVERSITY OF WATERLOO | FACULTY OF MATHEMATICS Department of Applied Mathematics

Standard FEM approximation for LQ control of non-isotropic wave equation

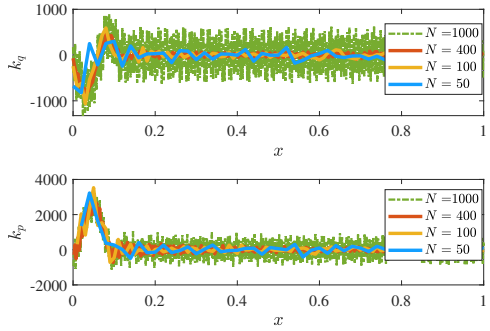


Decay rate vs discretization order *N*
controlled & uncontrolled

Standard FEM approximation for LQ control of non-isotropic wave equation



Decay rate vs discretization order *N*
controlled & uncontrolled



Control gains for different discretizations

Mixed FEM approximation

Theorem

Consider $z \in \{p, q\}$ *and* $i \in \{1, \ldots, n\}$. *Let* Q_{zi} *and* O_{zi} *be matrices constructed from the values of* $\theta_i^z(x)$ *at the nodes. If there exists a positive constant* δ^d, *independent of N, such that for every i*

$$Q_{zi} - O_{zi} \geq \delta^d Q_{zi}, \tag{9}$$

then the mixed FEM approximation is a pHS and the energy $\mathcal{H}_d(t)$ *decays exponentially with a decay rate of at least* $\alpha^d = \dfrac{\delta^d \epsilon \epsilon_0}{\epsilon + \epsilon_0}$ *for any* $\epsilon \in (0, \min\{\epsilon_0, \epsilon_1\})$, *where* $\epsilon_0 = \dfrac{\eta_Q}{\ell \mu_{P_1}}$ *and* $\epsilon_1 = \dfrac{2\eta_K}{\ell \mu_\Psi}$.

UNIVERSITY OF WATERLOO | FACULTY OF MATHEMATICS Department of Applied Mathematics

Mixed FEM approximation

Theorem

Consider $z \in \{p, q\}$ *and* $i \in \{1, \ldots, n\}$. *Let* Q_{zi} *and* O_{zi} *be matrices constructed from the values of* $\theta_i^z(x)$ *at the nodes. If there exists a positive constant* δ^d, *independent of N, such that for every i*

$$Q_{zi} - O_{zi} \geq \delta^d Q_{zi}, \tag{9}$$

then the mixed FEM approximation is a pHS and the energy $\mathcal{H}_d(t)$ *decays exponentially with a decay rate of at least* $\alpha^d = \dfrac{\delta^d \epsilon \epsilon_0}{\epsilon + \epsilon_0}$ *for any* $\epsilon \in (0, \min\{\epsilon_0, \epsilon_1\})$, *where* $\epsilon_0 = \dfrac{\eta_Q}{\ell \mu_{P_1}}$ *and* $\epsilon_1 = \dfrac{2\eta_K}{\ell \mu_\Psi}$.

Proof uses multiplier method

Mixed FEM approximation

Theorem

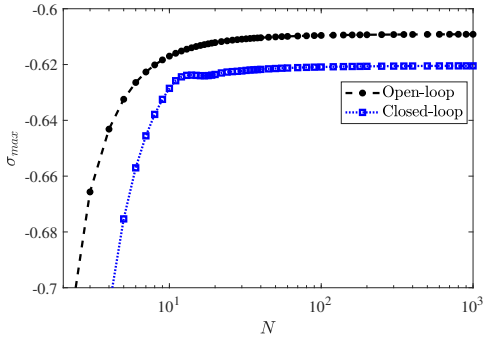
Consider $z \in \{p, q\}$ *and* $i \in \{1, \ldots, n\}$. *Let* Q_{zi} *and* O_{zi} *be matrices constructed from the values of* $\theta_i^z(x)$ *at the nodes. If there exists a positive constant* δ^d, *independent of N, such that for every i*

$$Q_{zi} - O_{zi} \geq \delta^d Q_{zi}, \tag{9}$$

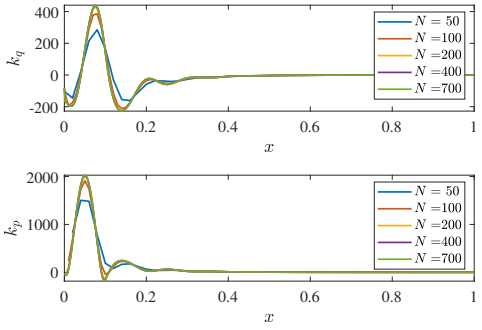
then the mixed FEM approximation is a pHS and the energy $\mathcal{H}_d(t)$ *decays exponentially with a decay rate of at least* $\alpha^d = \dfrac{\delta^d \epsilon \epsilon_0}{\epsilon + \epsilon_0}$ *for any* $\epsilon \in (0, \min\{\epsilon_0, \epsilon_1\})$, *where* $\epsilon_0 = \dfrac{\eta_Q}{\ell \mu_{P_1}}$ *and* $\epsilon_1 = \dfrac{2\eta_K}{\ell \mu_\Psi}$.

Condition satisfied for constant parameters and for fine enough discretization (and in some other situations).

Mixed FEM approximation for LQ control of non-isotropic wave equation



Decay rate vs discretization order *N*
controlled & uncontrolled



Control gains for different discretizations

Outline

Establishing decay rate

Controller design

Higher-order approximations

Summary

More general approximations: Summation-by-parts (SBP) operators

Definition

Let the interval $[a, b]$ be discretized into a set of N nodes, $\boldsymbol{x} = \left[x_1, x_2, \ldots, x_N\right]^\top$ let $f(x)$ be a polynomial of degree less than or equal to k, and let $\boldsymbol{f}^\top = \left[f(x_1), f(x_2), \ldots, f(x_N)\right]$ be the vector representation of f on the nodes $\boldsymbol{x}$. The matrix $\bar{\mathrm{D}}$ is a degree k SBP approximation of $\frac{\partial}{\partial x}$ if

1. $\bar{\mathrm{D}} = \bar{\mathrm{M}}^{-1}\bar{\mathrm{K}}$ where $\bar{\mathrm{M}} = \bar{\mathrm{M}}^\top > 0$,
2. $\bar{\mathrm{D}}\boldsymbol{f} = \boldsymbol{f}'$ where $\boldsymbol{f}' = \left[\frac{\mathrm{d}f(x_1)}{\mathrm{d}x}, \frac{\mathrm{d}f(x_2)}{\mathrm{d}x}, \ldots, \frac{\mathrm{d}f(x_N)}{\mathrm{d}x}\right]^\top$, and
3. $\bar{\mathrm{K}} + \bar{\mathrm{K}}^\top = \bar{\mathbf{t}}_R\bar{\mathbf{t}}_R^\top - \bar{\mathbf{t}}_L\bar{\mathbf{t}}_L^\top$, where $\bar{\mathbf{t}}_R^\top\boldsymbol{f} = f(b)$, $\bar{\mathbf{t}}_L^\top\boldsymbol{f} = f(a)$.

More general approximations: Summation-by-parts (SBP) operators

Definition

Let the interval $[a, b]$ be discretized into a set of N nodes, $\boldsymbol{x} = \left[x_1, x_2, \ldots, x_N\right]^\top$ let $f(x)$ be a polynomial of degree less than or equal to k, and let $\boldsymbol{f}^\top = \left[f(x_1), f(x_2), \ldots, f(x_N)\right]$ be the vector representation of f on the nodes $\boldsymbol{x}$. The matrix $\bar{\mathrm{D}}$ is a degree k SBP approximation of $\frac{\partial}{\partial x}$ if

1. $\bar{\mathrm{D}} = \bar{\mathrm{M}}^{-1}\bar{\mathrm{K}}$ where $\bar{\mathrm{M}} = \bar{\mathrm{M}}^\top > 0$,
2. $\bar{\mathrm{D}}\boldsymbol{f} = \boldsymbol{f}'$ where $\boldsymbol{f}' = \left[\frac{\mathrm{d}f(x_1)}{\mathrm{d}x}, \frac{\mathrm{d}f(x_2)}{\mathrm{d}x}, \ldots, \frac{\mathrm{d}f(x_N)}{\mathrm{d}x}\right]^\top$, and
3. $\bar{\mathrm{K}} + \bar{\mathrm{K}}^\top = \bar{\mathbf{t}}_R\bar{\mathbf{t}}_R^\top - \bar{\mathbf{t}}_L\bar{\mathbf{t}}_L^\top$, where $\bar{\mathbf{t}}_R^\top\boldsymbol{f} = f(b)$, $\bar{\mathbf{t}}_L^\top\boldsymbol{f} = f(a)$.

The operator $\bar{\mathrm{D}}$ will approximate the spatial derivative: if $\boldsymbol{f}$ is the vector approximating $f(x)$ then $\boldsymbol{f}' \approx \bar{\mathrm{D}}\boldsymbol{f}$

Uniform stability of polynomial-based approximations

Theorem

Consider an approximation scheme based on Lagrange polynomials and weak imposition of boundary conditions (Simultaneous Appproximation Terms) If the original system is exponentially stable, then the approximation has uniform exponential stability with a lower bound for the decay rate and amplitude determined by system parameters.

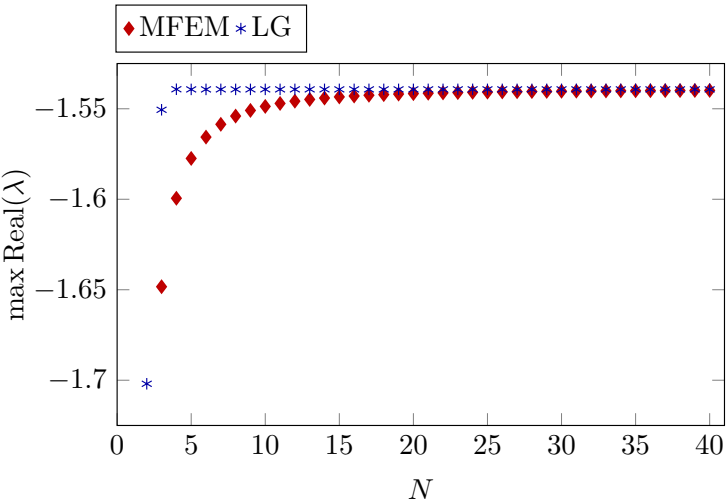
Uniform stability of polynomial-based approximations

Theorem

Consider an approximation scheme based on Lagrange polynomials and weak imposition of boundary conditions (Simultaneous Appproximation Terms) If the original system is exponentially stable, then the approximation has uniform exponential stability with a lower bound for the decay rate and amplitude determined by system parameters.

Outline of proof: Summation-by-Parts operator is discrete analogue of integration-by-parts
Show that discrete analogue of multiplier function leads to an Lyapunov function

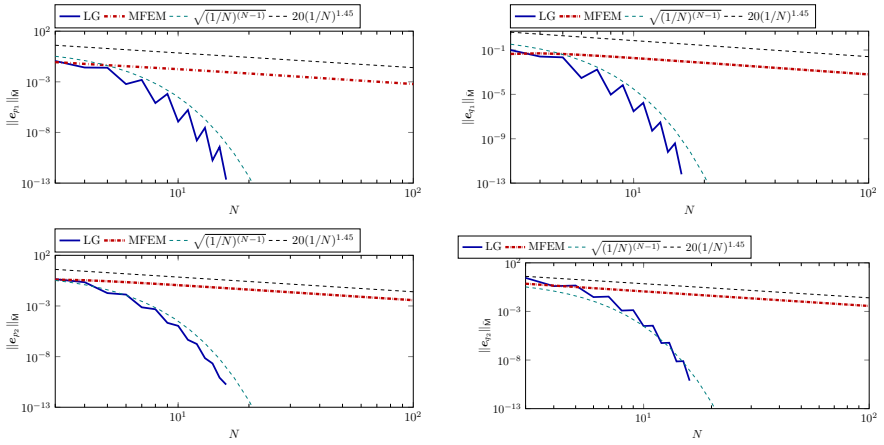
Decay rates of polynomial and MFEM approximations of piezoelectric beam



(Maximum real part of the eigenvalues vs *N*)

UNIVERSITY OF WATERLOO | FACULTY OF MATHEMATICS Department of Applied Mathematics

Error convergence: piezoelectric beam state variables p_i, q_i



(Method of manufactured solutions)

UNIVERSITY OF WATERLOO | FACULTY OF MATHEMATICS Department of Applied Mathematics

Outline

Establishing decay rate

Controller design

Higher-order approximations

Summary

Summary

- Have decay rate of boundary-damped pHS in terms of system parameters
- Uniform stability of approximations important for realistic simulations and reliable controller design
- Failure jeopardizes controller design
- Result on uniform stability of approximated port-Hamiltonian systems
  - mixed finite-element
  - single element polynomial approximations
  - multi-element approximations

Perspectives

- Choice of multiplier function affects bound on decay rate
- How much boundary damping ensures stability?
- Higher-dimensional problems