

Introduction

Anisotropic, quasilinear Maxwell system for **electric field** $E(t, x) \in \mathbb{R}^3$, **magnetic field** $H(t, x) \in \mathbb{R}^3$ on a bounded, smooth domain $\Omega \subseteq \mathbb{R}^3$ with an **interface** $\Gamma = \partial \text{supp } \sigma \cap \Omega$:

$$\begin{aligned} \partial_t(\varepsilon(x, E)E) &= \text{curl } H - \sigma(x)E, & t \geq 0, x \in \Omega \setminus \Gamma, \\ \partial_t(\mu(x, H)H) &= -\text{curl } E, & t \geq 0, x \in \Omega \setminus \Gamma, \\ \text{div}(\varepsilon(x, E)E)|_{\Omega \setminus \text{supp } \sigma} &= 0, & \text{div}(\mu(x, H)H) = 0, & t \geq 0, x \in \Omega \setminus \Gamma, \\ E(0, x) &= E_0(x), & H(0, x) &= H_0(x), & x \in \Omega \setminus \Gamma, \end{aligned} \quad (\text{ME})$$

where ε is the **permittivity**, μ is the **permeability**, and σ is the **conductivity** of a material.

Perfectly conducting boundary condition: Let ν denote the outer unit normal. Assume

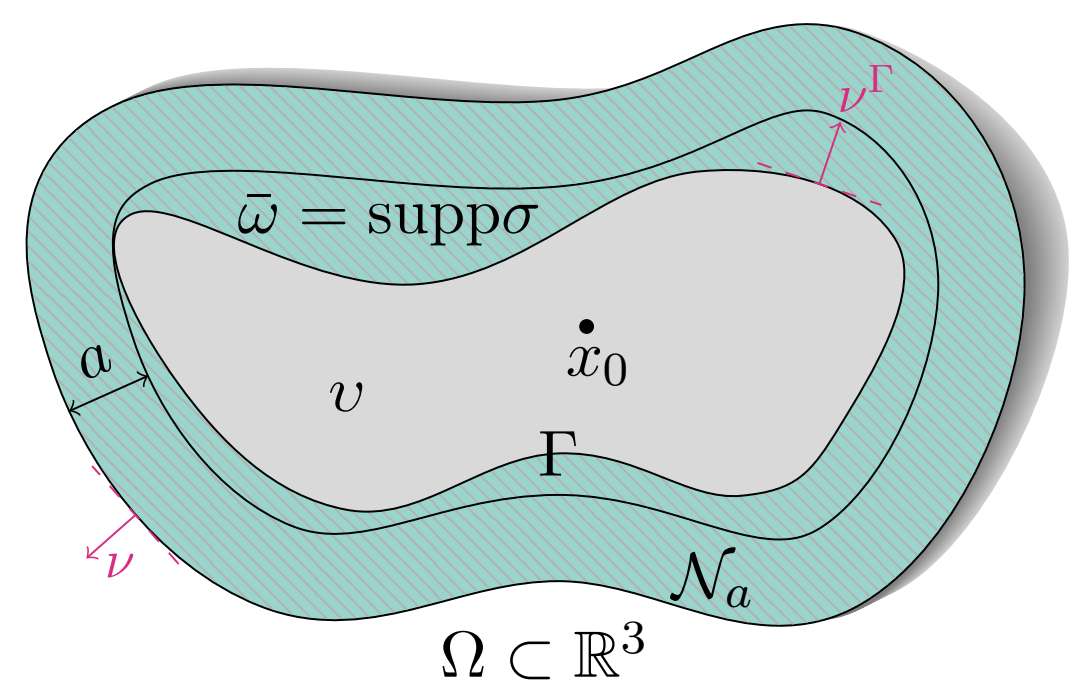
$$E \times \nu(x) = 0, \quad \mu H \cdot \nu(x) = 0, \quad x \in \partial\Omega. \quad (\text{BC})$$

Interface conditions: Solutions of the Maxwell equations then satisfy in the absence of surface currents

$$\begin{aligned} \llbracket E \times \nu^\Gamma \rrbracket &= 0, & \llbracket H \times \nu^\Gamma \rrbracket &= 0, \\ \llbracket \varepsilon E \cdot \nu^\Gamma \rrbracket &= \llbracket \varepsilon E(t_0) \cdot \nu^\Gamma \rrbracket - \int_{t_0}^t \llbracket \sigma E \cdot \nu^\Gamma \rrbracket d\tau, & \llbracket \mu H \cdot \nu^\Gamma \rrbracket &= 0. \end{aligned} \quad (\text{IC})$$

where $\llbracket \cdot \rrbracket = \text{tr}_\nu - \text{tr}_\omega$ denotes the jump at the interface Γ .

Problems: Time integral of $\text{div}(\varepsilon E) = \text{div}(\varepsilon E_0) - \int_0^t \text{div}(\sigma E) d\tau$ complicates estimates, huge Kernel of curl, highly coupled due to anisotropy (cf. ε_d for isotropic case), interface, ...



- ▶ Let $\varepsilon, \partial_{\xi_k} \varepsilon, \mu, \partial_{\xi_k} \mu \in C^3(\overline{\Omega \setminus \Gamma} \times \mathbb{R}^3, \mathbb{R}^{3 \times 3}_{\text{sym}})$,
- ▶ ε, μ **uniformly positive definite** on either side of the interface,
- ▶ $\sigma \in C^3(\text{supp } \sigma, \mathbb{R}^{3 \times 3}_{\text{sym}})$ with $\mathcal{N}_a = \{x \in \Omega \mid \text{dist}(x, \partial\Omega) < a\} \subseteq \text{supp } \sigma$ and σ **uniformly positive on its support** and $\partial \text{supp } \sigma$ smooth,
- ▶ ε, μ satisfying '**non-trapping condition**' ($\varepsilon(\cdot, 0) + ((x - x_0) \cdot \nabla) \varepsilon(\cdot, 0) \geq \eta \varepsilon(\cdot, 0)$, resp. μ),
- ▶ as well as '**non-trapping interface conditions**' ($((x - x_0) \cdot \nu^\Gamma) \llbracket \varepsilon(\cdot, 0) \rrbracket < 0$, resp. μ),
- ▶ $(E_0, H_0) \in H^3_1(\Omega)$, i.e. H^k on either side of the interface, satisfying

$$\mu(H_0)H_0 \in H^{\Sigma}_{n0}(\text{div } 0) \quad \text{and} \quad \varepsilon(E_0)E_0|_{\Omega \setminus \text{supp } \sigma} \in H^\Gamma(\text{div } 0, \Omega \setminus \text{supp } \sigma),$$

where $H^{\Sigma}_{n0}(\text{div } 0) = H_{n0}(\text{div } 0)$ for simply connected Ω and $H^\Gamma(\text{div } 0, \Omega \setminus \text{supp } \sigma) = H(\text{div } 0, \Omega \setminus \text{supp } \sigma)$ for connected Γ (see e.g. [1] for further details on spaces),

- ▶ (E_0, H_0) satisfying **compatibility conditions of order 3** of the Maxwell system.

→ By Theorem 7.3 of [9] have unique solutions of Maxwell equations

$$(E, H) \in \mathcal{G}^3_\Gamma = \cap_{k=0}^3 C^k((T_{\min}, T_{\max}), H^{3-k}_1(\Omega)).$$

We set

$$\begin{aligned} e(t) &:= \frac{1}{2} \max_{0 \leq j \leq 3} \left(\|\partial_t^j E(t)\|_{L^2(\Omega)}^2 + \|\partial_t^j H(t)\|_{L^2(\Omega)}^2 \right), \\ d(t) &:= \max_{0 \leq j \leq 3} \|\sigma^{1/2} \partial_t^j E(t)\|_{L^2(\Omega)}^2, \\ z(t) &:= \max_{0 \leq j \leq 3} \left(\|\partial_t^j E(t)\|_{H^{3-j}_1(\Omega)}^2 + \|\partial_t^j H(t)\|_{H^{3-j}_1(\Omega)}^2 \right). \end{aligned}$$

Lemma 1: Energy Estimate by integrating in time, integration by parts, error estimate

For $0 \leq s \leq t \leq T_{\max}$ the solutions (E, H) of (ME) satisfy

$$e(t) + \int_s^t d(\tau) d\tau \lesssim e(s) + \int_s^t z^{3/2}(\tau) d\tau.$$

→ Can bound the dissipation $\int_s^t d(\tau) d\tau$ by $e(s)$ and error term (absorbed for small data).

Exponential Decay and Global Existence for Small Data

Theorem 1

Let the initial values $(E_0, H_0) \in (H^3_1(\Omega))^6$ be sufficiently small. Then the corresponding solution (E, H) to (ME) exists for all $t \geq 0$. Moreover, there exist constants $M \geq 1$ and $\omega > 0$ such that

$$z(t) \leq M e^{-\omega t} z(0), \quad t \geq 0.$$

- ▶ Shown for constant, scalar coefficient in [7].

- ▶ For linear system in [6].

- ▶ Requires estimates for time derivatives → differentiate (ME):

$\partial_t^k \varepsilon(x, E)$ adds error f_k (and g_k) involving $\partial_{t,x}^{k,\alpha} E$, interferes with divergence conditions:

$$\begin{aligned} \partial_t(\varepsilon_d(x, E) \partial_t^k E) &= \text{curl } \partial_t^k H - \sigma(x) \partial_t^k E - f_k, & t \geq 0, x \in \Omega \setminus \Gamma, \\ \partial_t(\mu_d(x, H) \partial_t^k H) &= -\text{curl } \partial_t^k E - g_k, \end{aligned} \quad (\text{ME}_k)$$

with modified coefficients $\varepsilon_d(x, \xi) := \varepsilon(x, \xi) + \left(\sum_{l=1}^3 \partial_{\xi_j} \varepsilon_{il}(x, \xi) \xi_l \right)_{ij}$ and likewise μ_d .

Main parts of the proof:

- ▶ To obtain global solutions and exponential decay, want to establish the inequality

$$z(t) + \int_s^t z(\tau) d\tau \leq C z(s), \quad (*)$$

for $0 \leq s \leq t < T_* = \max\{T \in [0, T_{\max}] \mid z(\tau) \leq \delta^2 \text{ for } \tau \in [0, T]\}$, suitably small δ and a constant $C \geq 1$. Then, global existence and exponential decay follow as in the proof of Theorem 2.2 in [4].

- Suppose that $T_* < \infty$. $\implies z(T_*) = \delta^2$ by continuity. By (*): $z(t) \leq C z(s)$. For initial values with $\|(E_0, H_0)\|_{H^3_1} \leq r$, one has $z(0) \leq c_0 r^2$. As a result for a fixed, sufficiently small radius $\bar{r} \leq \delta$ we infer $z(T_*) \leq \frac{1}{2} \delta^2$. Therefore, the solution exists and (*) is valid for all times.

- Exponential decay follows by iterating the following: For $T = C(2C - 1)$ we deduce

$$z(T) + \frac{T}{C} z(T) \leq z(T) + \int_0^T z(\tau) d\tau \leq C z(0), \implies z(T) \leq \frac{C}{1 + \frac{T}{C}} z(0) = \frac{1}{2} z(0).$$

- ▶ Key ingredient for (*): *Helmholtz Decomposition* adapted to interface, *Observability Estimate* for divergence-free solutions, *Regularity Analysis* for higher norms.

- ▶ Further one shows $\|(E, H)(t)\|_{L^2(\Omega)}^2 \lesssim \|\partial_t(E, H)(t)\|_{L^2(\Omega)}^2$ (see [6], [7]).

→ Reduce estimate to derivatives of (E, H) .

Helmholtz Decomposition

Split solution $\partial_t^k(E, H) \in \mathcal{G}^{3-k}_\Gamma$ into a $\text{div}_{\varepsilon_d}$ -free and a curl-free part. Adjust the interface at Γ and boundary conditions at $\partial\Omega = \cup_{\ell=0}^K (\partial\Omega)_\ell$ such that the curl operator behaves well. For $k \in \{1, 2, 3\}$ we solve the elliptic interface problem

$$\begin{aligned} \text{div}(\varepsilon_d \nabla p^k) &= \text{div}(\varepsilon_d \partial_t^k E) = -\text{div}(\sigma \partial_t^{k-1} E + f_{k-1}) & \text{on } \Omega_\pm, \\ \llbracket p^k \rrbracket &= 0 & \text{on } \Gamma, \\ \llbracket \text{tr}_n(\varepsilon_d \nabla p^k) \rrbracket &= \llbracket \text{tr}_n(\varepsilon_d \partial_t^k E) \rrbracket & \text{on } \Gamma, \\ p^k &= 0 & \text{on } (\partial\Omega)_0, \\ p^k &\text{ constant on } (\partial\Omega)_\ell, \text{ and } \langle \text{tr}_n \varepsilon_d(\nabla p^k - \partial_t^k E), \mathbf{1} \rangle_{H^{-1/2}((\partial\Omega)_\ell)} = 0, & \forall \ell \in \{1, \dots, K\}. \end{aligned}$$

- ▶ Decompose $\partial_t^k E = V^k + \nabla p^k$ with $\text{div}(\varepsilon_d V^k) = 0$, $\text{curl } \nabla p^k = 0$ (and $\partial_t^k H = W^k + \nabla q^k$).

- ▶ Suitable core estimates for $(\nabla p^k, \nabla q^k)$ relating to dissipation: $\|\partial_t(\nabla p^k, \nabla q^k)\|_{L^2} \lesssim d + z^2$.

- ▶ $\varepsilon_d V^k \in H(\text{div } 0, \Omega)$ and $\mu_d W^k \in H_0(\text{div } 0, \Omega)$.

Observability Estimate

Lemma 2

Let $\kappa \in (0, 1]$. For $k \in \{1, 2, 3\}$ and $0 \leq s \leq t < T_*$, we have

$$\begin{aligned} \int_s^t \int_\Omega \left(|V^k|^2 + |W^k|^2 \right) dx d\tau &\lesssim \int_s^t \left(\kappa z(\tau) + c_\kappa(z^{3/2}(\tau) + d(\tau)) \right) dx d\tau \\ &\quad + \kappa(z(s) + z(t)) + c_\kappa(e(s) + z^2(s) + z^2(t)). \end{aligned}$$

- ▶ For scalar coefficients in [5].

- ▶ Use *Morawetz multiplier*:

Multiply first line of (ME_k) with $\chi \mu_d W^k \times (x - x_0)$ for some $\chi \in C^1(\Omega)$. Integration by parts, vector calculus identities and further calculations (...) lead to the estimate.

- ▶ Non-trapping conditions enter in the proof.

Regularity Analysis

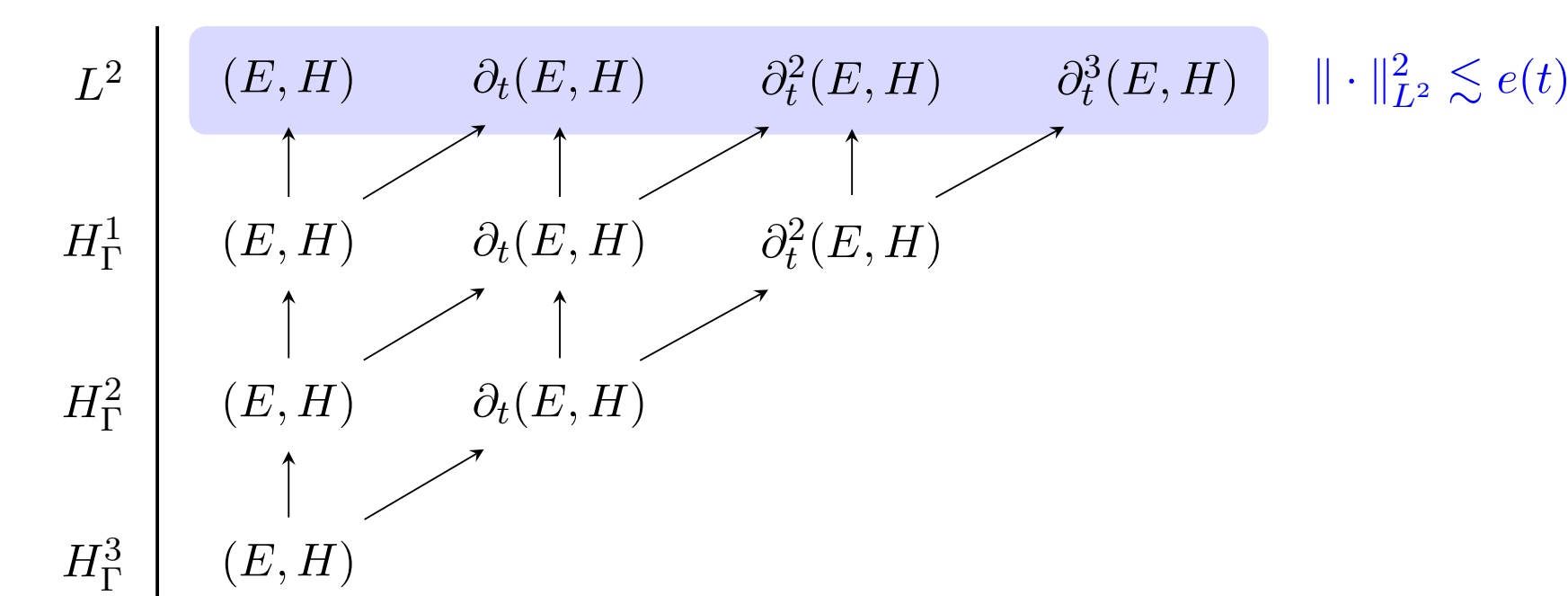
- ▶ Combining estimates of the Helmholtz decomposition, observability estimate, etc.:

$$\int_s^t e(\tau) d\tau \lesssim e(t) + c_\kappa \left(z(s) + z^2(t) + \int_s^t (d(\tau) + z^{3/2}(\tau)) d\tau \right) + \kappa \left(z(t) + \int_s^t z(\tau) d\tau \right).$$

- ▶ Have to control higher-order space derivatives. Idea: Iteratively apply (a version of) the *div-curl estimate*

$$\|u\|_{H^k_\Gamma} \leq \|u\|_{L^2(\Omega)} + \|\text{div}(\alpha u)\|_{H^{k-1}_\Gamma} + \|\text{curl } u\|_{H^{k-1}_\Gamma},$$

for u in $H(\text{div}_\alpha) \cap H_0(\text{curl})$ or $H_0(\text{div}_\alpha) \cap H(\text{curl})$, see [2], [3].



We estimate the H^1_Γ -norm by terms in H^{1-1}_Γ -norm using (ME_k) , finally reaching the L^2 -level. These terms are then bound by $e(t)$.

- ▶ Split solutions using cutoffs around $\partial\Omega$, $\text{supp } \sigma$, Γ , $\Omega \setminus \text{supp } \sigma$ and employ further ideas from [4], [8], [9], and additional arguments at the interface. A sophisticated analysis then shows:

Lemma 3

For $0 \leq s \leq t < T_*$ the solutions (E, H) of (ME) satisfy

$$z(t) + \int_s^t z(\tau) d\tau \lesssim z(s) + z^2(t) + e(t) + \int_s^t (e(\tau) + z^{3/2}(\tau)) d\tau.$$

- ▶ For sufficiently small δ and κ this establishes (*), which implies Theorem 1.

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