

Well-Posedness For a Class Of Input-Output Second-Order Linear Thermoelastic Systems

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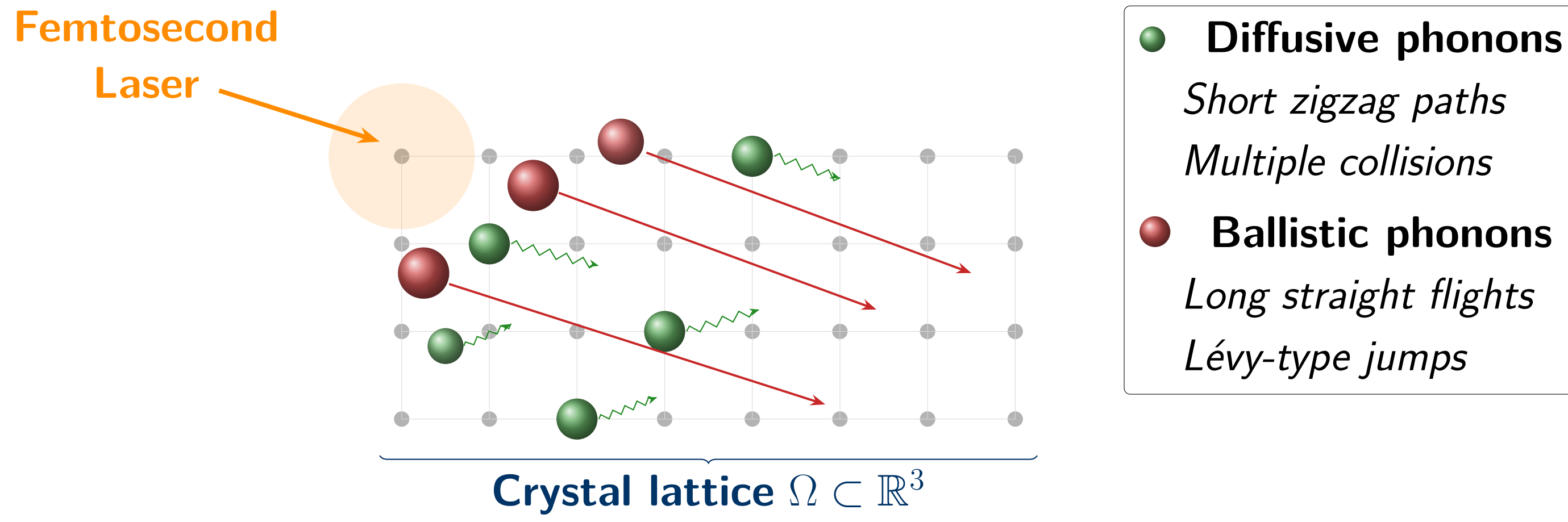
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Motivation: Transcending Fourier's Law

At the nanoscale (e.g., nanostructures irradiated by femtosecond lasers), thermal transport exhibits a dual nature that classical Fourier's law cannot capture:

- ☛ **Diffusive Phonons:** Characterized by short paths and multiple collisions. Modeled by a local **Green-Naghdi (Type III)** flux.
- ☛ **Ballistic Phonons:** Characterized by long-range jumps (Lévy flights). Modeled by a nonlocal **fractional Gurtin-Pipkin** flux.

Hybrid Model: By superposing these two laws ($\mathbf{q}^T = \mathbf{q}_l + \mathbf{q}_{nl}$) into the linearized energy balance equation, we derive our final integro-differential PDE.



Abstract framework

Let H, U , and Y be Hilbert spaces. We consider the input-output second-order Volterra integro-differential problem:

$$\ddot{x}(t) + \nu \dot{x}(t) + \int_0^t a(t-s) \ddot{x}(s) ds = Ax(t) + \gamma A \dot{x}(t) + \int_0^t b(t-s) Kx(s) ds + Bu(t), \quad (1)$$

$$y(t) = C \dot{x}(t), \quad t \geq 0. \quad (2)$$

Key assumptions & Notations:

- ☛ A is strictly negative on H (generates an analytic C_0 -semigroup).
- ☛ $\gamma > 0$ and $\nu \geq 0$ are constants.
- ☛ $B \in \mathcal{L}(U, H_{-1/2})$ and $C \in \mathcal{L}(H_{1/2}, Y)$ are potentially unbounded operators.
- ☛ $a(\cdot), b(\cdot) \in L^2(\mathbb{R}_+)$, $u(\cdot) \in L^2_{loc}(\mathbb{R}_+; U)$ and $K \in \mathcal{L}(H_{1/2}, H)$.
- ☛ For $C \in \mathcal{L}(H_1, Y)$ we define the Yosida extension of C with respect to A by:

$$C^A x := \lim_{s \rightarrow +\infty} s C R(s, A) x, \quad \mathcal{D}(C^A) := \{x \in H : \text{limit exists in } Y\}.$$

- ☛ $\mathcal{M}^2$ denotes the space of 2×2 real-valued matrices.

First transformation

The main input system (1) can be written (formally) in the state space $\mathcal{H} = H_{1/2} \times H$ as the following first-order integro-differential system

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\dot{x}}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & I \\ A & \gamma A - \nu I \end{bmatrix}}_{\mathcal{A}_{\gamma\nu}} \underbrace{\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}}_{z(t)} + \int_0^t \underbrace{\begin{bmatrix} 0 & 0 \\ a(t-s) & b(t-s) \end{bmatrix}}_{\mathcal{G}(t-s)} \underbrace{\begin{bmatrix} 0 & I \\ K & 0 \end{bmatrix}}_{\mathcal{R}} z(s) ds + \underbrace{\begin{bmatrix} 0 \\ B \end{bmatrix}}_{\mathcal{B}} u(t), \quad (3)$$

$$y(t) = \underbrace{\begin{bmatrix} 0 & C \end{bmatrix}}_{\mathcal{C}} \underbrace{\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}}_{z(t)}, \quad (4)$$

with, $\mathcal{A}_{\gamma\nu} \in \mathcal{L}(H_1 \times H_1, \mathcal{H})$, $\mathcal{G} \in L^2(\mathbb{R}_+, \mathcal{M}^2)$, $\mathcal{R} \in \mathcal{L}(\mathcal{H})$ and $\mathcal{B} \in \mathcal{L}(U, H_{1/2} \times H_{-1/2})$.

In the sequel, $\mathcal{T}_{\gamma\nu}$ will denotes the C_0 -sg generated by $\mathcal{A}_{\gamma\nu}$ on $\mathcal{H}$ and $\tilde{\mathcal{T}}_{\gamma\nu}$ is it's extension to $H_{1/2} \times H_{-1/2}$.

Definition: Mild solutions

Let $\begin{bmatrix} x_0 \\ x_1 \end{bmatrix} \in \mathcal{H}$. We said that $z(\cdot)$ is a mild solution of (3) if $z(\cdot) \in C(\mathbb{R}_+; \mathcal{H})$, $z(t) \in \mathcal{D}(\mathcal{R} \mathcal{A}_{\gamma\nu})$ for almost all $t \geq 0$, and $z(\cdot)$ satisfies

$$z(t) = \mathcal{T}_{\gamma\nu}(t) \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} + \int_0^t \mathcal{T}_{\gamma\nu}(t-s) \int_0^s \mathcal{G}(s-\sigma) \mathcal{R} \mathcal{A}_{\gamma\nu} z(\sigma) d\sigma ds + \int_0^t \tilde{\mathcal{T}}_{\gamma\nu}(t-s) \mathcal{B} u(s) ds. \quad (5)$$

Moreover, when $z(\cdot)$ is a mild solution of (3) given by (5), its first component $x(\cdot)$ is defined as the mild solution of (1) provided that the second component of $z(\cdot)$ is exactly the time derivative of $x(\cdot)$.

Proposition ([1])

Let $\gamma > 0$, $\nu \geq 0$. Then:

- ☛ $\mathcal{R}$ is infinite-time admissible observation operator for $\mathcal{A}_{\gamma\nu}$ on $\mathcal{H}$.
- ☛ For each $\begin{bmatrix} x_0 \\ x_1 \end{bmatrix} \in \mathcal{H}$, the free problem (3) has a mild solution $z(\cdot) \in C(\mathbb{R}_+, \mathcal{H})$ given by :

$$z(t) = \mathcal{T}_{\gamma\nu}(t) \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} + \int_0^t \mathcal{T}_{\gamma\nu}(t-s) \int_0^s \mathcal{G}(s-\sigma) \mathcal{R} \mathcal{A}_{\gamma\nu} z(\sigma) d\sigma ds$$

with $\mathcal{D}(\mathcal{R} \mathcal{A}_{\gamma\nu}) = H_{1/2} \times H_{1/2}$ and $\mathcal{R} \mathcal{A}_{\gamma\nu} = \begin{bmatrix} 0 & I_{H_{1/2}} \\ K & 0 \end{bmatrix}$.

Second transformation

We recast (3)-(4) as an input-output abstract Cauchy problem in the Banach space $\mathcal{H} := \mathcal{H} \times L^2(\mathbb{R}_+; \mathcal{H})$ as follows:

$$\underbrace{\begin{bmatrix} \dot{z}(t) \\ \dot{f}(t) \end{bmatrix}}_{\dot{\mathcal{Z}}(t)} = \underbrace{\begin{bmatrix} \mathcal{A}_{\gamma\nu} & \delta_0 \\ \mathcal{G}(\cdot) \mathcal{R} & Q \end{bmatrix}}_{\mathcal{A}} \underbrace{\begin{bmatrix} z(t) \\ f(t) \end{bmatrix}}_{\mathcal{Z}(t)} + \underbrace{\begin{bmatrix} \mathcal{B} \\ 0 \end{bmatrix}}_{\mathcal{B}} u(t), \quad y(t) = \underbrace{\begin{bmatrix} \mathcal{C} \\ 0 \end{bmatrix}}_{\mathcal{C}} \mathcal{Z}(t), \quad (6)$$

where $\mathcal{A}$ with $\mathcal{D}(\mathcal{A}) = \mathcal{D}(\mathcal{A}_{\gamma\nu}) \times W^{1,2}(\mathbb{R}_+; \mathcal{H})$ generates a C_0 -semigroup $\mathcal{T}$ on $\mathcal{H}$, $\mathcal{B} \in \mathcal{L}(U, \mathcal{H}_{-1})$, $\mathcal{C} \in \mathcal{L}(\mathcal{H}, \mathcal{H})$ and $Q = \frac{d}{ds}$ generates the left shift semigroup and $\delta_0(f) = f(0)$.

Theorem: Existence & Uniqueness Of solution ([1])

Let $\gamma > 0$, $\nu \geq 0$. For any $\begin{bmatrix} x_0 \\ x_1 \end{bmatrix} \in \mathcal{H}$ and $u(\cdot) \in L^2_{loc}(\mathbb{R}_+; U)$ there exists $x(\cdot) \in C(\mathbb{R}_+, H_{1/2}) \cap C^1(\mathbb{R}_+, H)$ such that:

- ☛ The input-output Cauchy system (6) is regular (Salamon-Weiss sense) in $\mathcal{H}$.

$$\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} = \mathcal{T}_{\gamma\nu}(t) \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} + \int_0^t \mathcal{T}_{\gamma\nu}(t-s) \int_0^s \mathcal{G}(s-\sigma) \mathcal{R} \mathcal{A}_{\gamma\nu} \begin{bmatrix} x(\sigma) \\ \dot{x}(\sigma) \end{bmatrix} d\sigma ds + \int_0^t \tilde{\mathcal{T}}_{\gamma\nu}(t-s) \mathcal{B} u(s) ds,$$

- ☛ **Continuous dependence:** $\forall \tau > 0$, $\exists \kappa_\tau > 0$:

$$\|x(\tau)\|_{H_{1/2}}^2 + \|\dot{x}(\tau)\|_H^2 + \|y\|_{L^2([0,\tau]; Y)}^2 \leq \kappa_\tau (\|x_0\|_{H_{1/2}}^2 + \|x_1\|_H^2 + \|u\|_{L^2([0,\tau]; U)}^2)$$

- ☛ $\dot{x}(t) \in \mathcal{D}(C)$ for a.e. $t \geq 0$, and $y(\cdot) = C \dot{x}(\cdot) \in L^2_{loc}(\mathbb{R}_+; Y)$

Proposition: Transfer Function ([1])

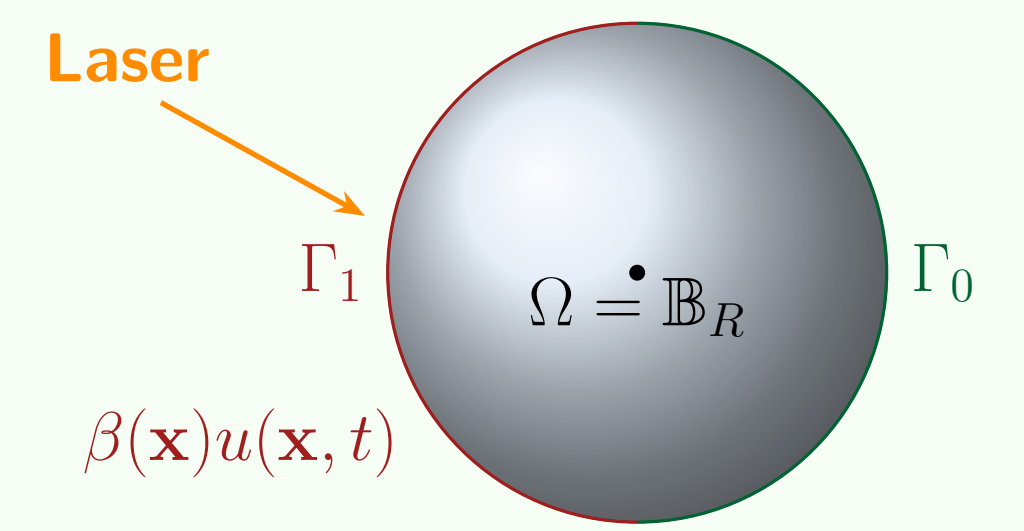
If K has an extension $\tilde{K} \in \mathcal{L}(H, H_{-1/2})$, then the transfer function of (1)-(2) is given by

$$\mathbb{H}(\lambda) = \lambda C (\lambda^2 + \lambda \nu - (1 + \gamma \lambda) \tilde{A} - \hat{b}(\lambda) \tilde{K} - \lambda \hat{a}(\lambda))^{-1} B, \quad \text{Re } \lambda >> 0,$$

where $\tilde{A} \in \mathcal{L}(H_{1/2}, H_{-1/2})$ is the extension of A in $H_{-1/2}$.

Application: Thermoelastic Wave Equation with Kelvin-Voigt Damping

Physical Model: We consider non-Fourier heat propagation in rigid conductors (e.g., semiconducting nanostructures) defined on a spherical domain $\Omega = \mathbb{B}_R \subset \mathbb{R}^3$. The boundary is partitioned into a passive region Γ_0 and an active control interface Γ_1 with a shape function $\beta \in L^\infty(\Gamma_1)$.



Thermoelastic System with Collocated Observation:

$$\begin{aligned} \ddot{\theta}(\mathbf{x}, t) + 2\dot{\theta}(\mathbf{x}, t) + \int_0^t a(t-s) \dot{\theta}(\mathbf{x}, s) ds &= \Delta \theta(\mathbf{x}, t) + \Delta \dot{\theta}(\mathbf{x}, t) + \int_0^t b(t-s) (-\Delta)^{1/2} \theta(\mathbf{x}, s) ds, \\ \theta(\mathbf{x}, t) &= 0 \quad \text{on } \Gamma_0 \times \mathbb{R}_+, \quad \theta(\mathbf{x}, t) = \beta(\mathbf{x}) u(\mathbf{x}, t) \quad \text{on } \Gamma_1 \times \mathbb{R}_+, \\ y(t) &= -\beta(\mathbf{x}) \frac{\partial}{\partial \nu} (\Delta^{-1} \dot{\theta}(\mathbf{x}, t)) \quad \text{on } \Gamma_1 \times \mathbb{R}_+ \end{aligned}$$

Abstract Formulation & State Spaces:

We recast the system into an abstract form (1)-(2) using the state space $H = \mathcal{H}^{-1}(\Omega)$ and the input/output space $U = Y = L^2(\Gamma_1)$.

- ☛ **System Operator:** The extension $A\phi = \Delta\phi$ with domain $\mathcal{D}(A) = \mathcal{H}_0^1(\Omega)$.
- ☛ **Control & Observation:** The control operator $B \in \mathcal{L}(U, [\mathcal{D}(A_D)]')$ and its collocated observation adjoint $C = B^*$ given by:

$$B^* = - \left(\beta \frac{\partial}{\partial \nu} (A_D^{-1} \cdot) \right)_{\Gamma_1} \in \mathcal{L}(L^2(\Omega), U)$$

- ☛ **Relaxation Kernels:** $a(t) = \varepsilon(t)$ and $b(t) = -c_1^{-1} \dot{\mu}(t)$, derived from the explicit heat flux $\varepsilon(t)$ and energy-temperature $\mu(t)$ relaxation functions.

Main Results: i) Well-Posedness (via Theorem)

For any finite time $\tau > 0$, initial data $(\theta_0, \theta_1) \in L^2(\Omega) \times \mathcal{H}^{-1}(\Omega)$, and input $u \in L^2([0, \tau]; L^2(\Gamma_1))$, the system admits a **unique mild solution** $\theta \in C(\mathbb{R}_+; L^2(\Omega)) \cap C^1(\mathbb{R}_+; \mathcal{H}^{-1}(\Omega))$ satisfying the continuous dependence estimate:

$$\begin{aligned} \|\theta(\cdot, \tau)\|_{L^2(\Omega)}^2 + \|\dot{\theta}(\cdot, \tau)\|_{\mathcal{H}^{-1}(\Omega)}^2 + \|y(\cdot)\|_{L^2([0,\tau]; U)}^2 \\ \leq \kappa_\tau \left(\|\theta_0(\cdot)\|_{L^2(\Omega)}^2 + \|\theta_1(\cdot)\|_{\mathcal{H}^{-1}(\Omega)}^2 + \|u(\cdot)\|_{L^2([0,\tau]; U)}^2 \right), \end{aligned}$$

where $\kappa_\tau > 0$ is a constant depending only on the geometry Ω, Γ_1 , and τ .

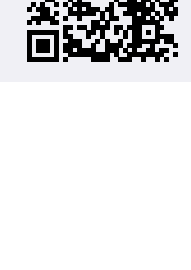
ii) Transfer Function (via Proposition): For $u \in L^2(\Gamma_1)$ and $s > 0$ large enough

$$(\mathbb{H}(s)u)(\theta, \phi) = \frac{\sqrt{2}}{R^2} \beta(\theta, \phi) \sum_{n,l,m} \frac{(-1)^n I_{nl} \left(\sqrt{\frac{s(s+2)}{1+s}} \right) Z_{nl}(s)}{\mu_{nl} i_l \left(R \sqrt{\frac{s(s+2)}{1+s}} \right)} \langle \beta u, Y_l^m \rangle_{L^2(\Gamma_1)} Y_l^m(\theta, \phi).$$

Where:

- ☛ $\Gamma_1 = \{(R, \theta, \phi) \mid 0 \leq \theta \leq \pi, \frac{\pi}{2} < \phi < \frac{3\pi}{2}\}$.
- ☛ $Y_l^m(\theta, \phi)$ denotes the spherical harmonics.
- ☛ $i_l(\cdot)$ represents the modified spherical Bessel functions.
- ☛ $I_{nl}(\cdot)$ and $Z_{nl}(\cdot)$ are known specific functions depending on the system parameters.

References

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