

Erlang ODE approximation of differential equations with distributed time delays

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Distributed time delay systems

Delay differential equations with *distributed* time delay

$$\dot{x}(t) = f(x(t), z(t)), \quad z(t) = \int_{-\infty}^t \alpha(t-s)x(s) \, ds \quad (1)$$

Distribution of time delays: α is the distribution's *density*

- ▶ Animal population models: Variations in gestational periods
- ▶ Molten salt nuclear fission reactors: Non-uniform flow velocity
- ▶ Absorption of medicine/nutrients: Absorption across the intestines
- ▶ Spread of infectious diseases: Variations in incubation times
- ▶ Large interconnected networks: Many different communication delays

Erlang ODE approximation

Original system

$$\dot{x}(t) = f(x(t), z(t)), \quad z(t) = \int_{-\infty}^t \alpha(t-s)x(s) \, ds \quad (2)$$

Erlang ODE approximation

$$\dot{\hat{x}}(t) = f(\hat{x}(t), \hat{z}(t)), \quad \hat{z}(t) = CZ(t), \quad (3a)$$

$$\dot{Z}(t) = AZ(t) + B\hat{x}(t) \quad (3b)$$

Auxiliary memory states and system matrices

$$Z = \begin{bmatrix} Z_0 \\ Z_1 \\ \vdots \\ Z_M \end{bmatrix}, \quad A = a \begin{bmatrix} -I & & & \\ I & -I & & \\ & \ddots & \ddots & \\ & & I & -I \end{bmatrix}, \quad B = a \begin{bmatrix} I \\ \\ \\ \end{bmatrix}, \quad (4)$$

$$C = [c_0 I \quad c_1 I \quad \cdots \quad c_M I] \quad (5)$$

Parameters

$$c_m = \int_{m/a}^{(m+1)/a} \alpha(s) \, ds, \quad a > 0 \quad (6)$$

Initial value problems

Original initial value problem

$$x(t) = x_0(t), \quad t \in (-\infty, t_0], \quad (7a)$$

$$\dot{x}(t) = f(x(t), z(t)), \quad z(t) = \int_{-\infty}^t \alpha(t-s)x(s) \, ds \quad (7b)$$

Erlang ODE approximation

$$\hat{x}(t_0) = x_0(t_0), \quad (8a)$$

$$Z_m(t_0) = \int_{-\infty}^{t_0} \ell_m(t-s)x_0(s) \, ds, \quad m = 0, \dots, M, \quad (8b)$$

$$\dot{\hat{x}}(t) = f(\hat{x}(t), \hat{z}(t)), \quad \hat{z}(t) = CZ(t), \quad (8c)$$

$$\dot{Z}(t) = AZ(t) + B\hat{x}(t) \quad (8d)$$

Basis functions

$$\ell_m(t) = \frac{a^{m+1}}{m!} t^m e^{-at} \quad (9)$$

Main result: Convergence

The approximate state, $\hat{x}$, and memory state, $\hat{z}$, converge uniformly to x and z for $t \in [t_0, t_f]$ as $a \rightarrow \infty$ provided that $M/a \rightarrow \infty$ as well.

Technical assumptions

- ▶ The memory kernel, α , is exponentially bounded and continuous
- ▶ $\alpha \in L_1([0, \infty))$
- ▶ The right-hand side function, f , is differentiable
- ▶ The state, x , is bounded
- ▶ The initial state function, x_0 , is continuous and exponentially bounded
- ▶ The rate parameter in the exponential bound on x_0 is strictly smaller than the one for α

Ritschel, T.K.S., 2026. On Erlang ODE Approximations of Differential Equations with Distributed Time Delays. SIAM Journal on Applied Dynamical Systems. Accepted.

N. Guglielmi and E. Hairer, Applying stiff integrators for ordinary differential equations and delay differential equations to problems with distributed delays, SIAM Journal on Scientific Computing, 47 (2025), pp. A102–A123.

Key prerequisite: Erlang mixture approximation of α

Erlang mixture approximation

$$\hat{\alpha}(t) = \sum_{m=0}^M c_m \ell_m(t), \quad \ell_m(t) = \frac{a^{m+1}}{m!} t^m e^{-at} \quad (10)$$

Convergence: If $a \rightarrow \infty$ and $M/a \rightarrow \infty$ as well

$$\hat{\alpha}(t) \rightarrow \alpha(t), \quad (11a)$$

$$\int_0^t \hat{\alpha}(s) ds \rightarrow \int_0^t \alpha(s) ds, \quad (11b)$$

$$\int_0^\infty |\hat{\alpha}(s) - \alpha(s)| ds \rightarrow 0 \quad (11c)$$

Technical assumptions

- The memory kernel, α , is continuous and bounded

ODE approximation through kernel approximation

DDE with distributed time delay

$$\dot{x}(t) = f(x(t), z(t)), \quad z(t) = \int_{-\infty}^t \alpha(t-s)x(s) \, ds$$

Approximate kernel

$$\alpha(t) \approx \hat{\alpha}(t) = \sum_{m=0}^M c_m \ell_m(t), \quad \ell_m(t) = \frac{a^{m+1}}{m!} t^m e^{-at}$$

Derivatives of basis functions

$$\dot{\ell}_m(t) = a(\ell_{m-1}(t) - \ell_m(t)), \quad \dot{\ell}_0 = -a\ell_0(t)$$

Substitute into integral

$$z(t) \approx \hat{z}(t) = \sum_{m=0}^M c_m \int_{-\infty}^t \ell_m(t-s)x(s) \, ds = \sum_{m=0}^M c_m Z_m(t)$$

Differentiate z_m

$$\dot{Z}_m(t) = \ell_m(0)x(t) + \int_{-\infty}^t \dot{\ell}_m(t-s)x(s) \, ds = \begin{cases} a(x(t) - Z_0(t)), & m = 0, \\ a(Z_{m-1}(t) - Z_m(t)), & m \geq 1 \end{cases}$$

What about linearized stability?

Steady state

$$0 = f(x_s, z_s), \quad z_s = x_s \int_0^{\infty} \alpha(t) dt \quad (12)$$

Approximate steady state

$$0 = f(\hat{x}_s, \hat{z}_s), \quad \hat{z}_s = \hat{x}_s \sum_{m=0}^M c_m \quad (13)$$

Sum of coefficients

$$\sum_{m=0}^M c_m = \sum_{m=0}^M \int_{m/a}^{(m+1)/a} \alpha(t) dt = \int_0^{(M+1)/a} \alpha(t) dt \quad (14)$$

Characteristic equations

Characteristic equation ($F = \frac{\partial f}{\partial x}(x_s, z_s)$, $G = \frac{\partial f}{\partial z}(x_s, z_s)$)

$$P(\lambda) = \det \left(F - \lambda I + G \int_0^\infty e^{-\lambda t} \alpha(t) dt \right) = 0 \quad (15)$$

Approximate characteristic equation

$$\det \left(\begin{bmatrix} \hat{F} & \hat{G}C \\ B & A \end{bmatrix} - \lambda \begin{bmatrix} I & \\ & I \end{bmatrix} \right) = 0 \quad (16)$$

Rewrite for $\lambda \neq a$

$$\det(\hat{F} - \lambda I - \hat{G}C(A - \lambda I)^{-1}B) \det(A - \lambda I) = 0, \quad (17a)$$

$$\underbrace{\det \left(\hat{F} - \lambda I + \hat{G} \int_0^\infty e^{-\lambda t} \hat{\alpha}(t) dt \right)}_{= \hat{P}(\lambda)} \det(A - \lambda I) = 0 \quad (17b)$$

Linearized stability

For every isolated root, $\lambda \in \mathbb{C}$, of P , there are roots of $\hat{P}$ that converge to λ as $a \rightarrow \infty$ provided that $M/a \rightarrow \infty$ and $\operatorname{Re} \lambda > -\rho$ where ρ is the rate parameter in the exponential bound on α .

Technical assumptions

- ▶ Same as for the state convergence
- ▶ There exists a steady state, $\hat{x}_s$, of the Erlang ODE approximation

Numerical example: Convergence rate

Numerical example: Method of manufactured solutions

System

$$\dot{N}(t) = \kappa N(t) \left(1 - \frac{1}{K} \int_{-\infty}^t \alpha(t-s) N(s) \, ds \right) + Q(t) \quad (18)$$

Kernel

$$\alpha(t) = \frac{2}{\sqrt{\pi}} e^{-t^2} \quad (19)$$

Choose the artificial forcing term, Q , such that the solution is

$$x(t) = 1 + e^{-\left(\frac{t}{\gamma}\right)^2} \quad (20)$$

Erlang mixture approximations

Domain of approximation

$$\int_{t_h}^{\infty} |\alpha(t)| dt < \epsilon \quad (21)$$

Rate parameter

$$a = \frac{M+1}{t_h}. \quad (22)$$

Theoretical coefficients

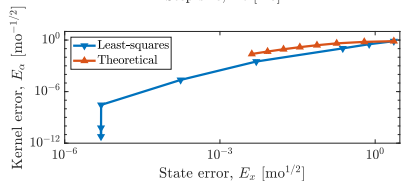
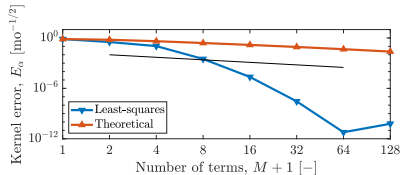
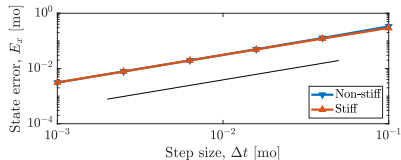
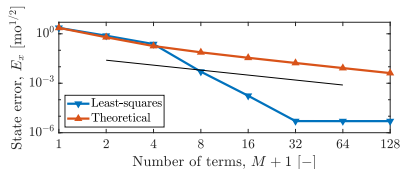
$$c_m = \int_{m/a}^{(m+1)/a} \alpha(t) dt \quad (23)$$

Least-squares coefficients

$$\min_{\{c_m\}_{m=0}^M} \phi = \frac{1}{2} \sum_{k=0}^{N-1} (\alpha(k\Delta t) - \hat{\alpha}(k\Delta t))^2 \Delta t + \frac{w}{2} \sum_{m=0}^M c_m^2, \quad (24a)$$

$$\text{subject to } \sum_{m=0}^M c_m = \beta(t_h), \quad (24b)$$

Numerical example: Empirical convergence rates



Numerical example: Bifurcation analysis

Numerical example

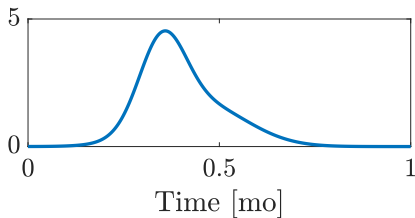
► System

$$\dot{N}(t) = \kappa N(t) \left(1 - \frac{1}{K} \int_{-\infty}^t \alpha(t-s) N(s) \, ds \right)$$

► Kernel

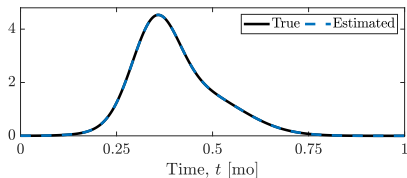
$$\alpha(t) = \gamma_1 F(t; \mu_1, \sigma_1) + \gamma_2 F(t; \mu_2, \sigma_2),$$

$$F(t; \mu, \sigma) = \frac{\exp\left(-\frac{1}{2} \left(\frac{t-\mu}{\sigma}\right)^2\right) + \exp\left(-\frac{1}{2} \left(\frac{t+\mu}{\sigma}\right)^2\right)}{\sqrt{2\pi}\sigma}$$

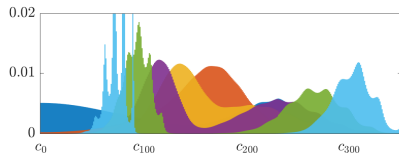
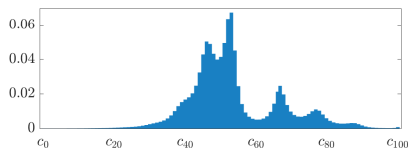
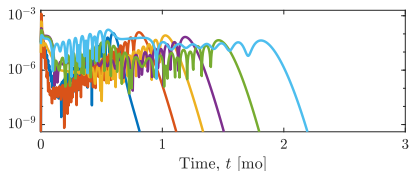
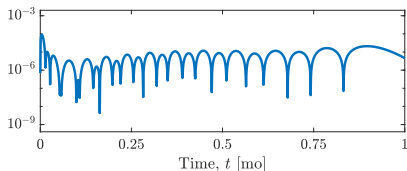
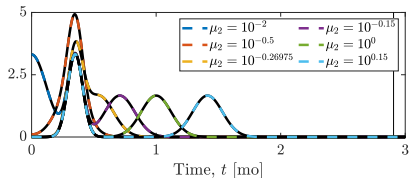


Numerical example: Bifurcation analysis

Model parameter

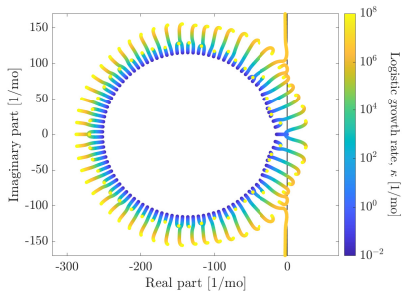


Kernel parameter

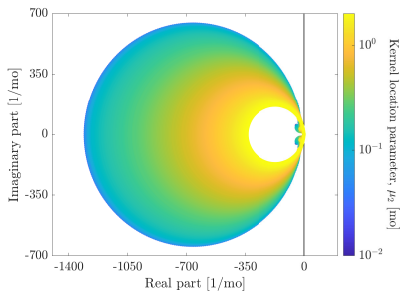


Numerical example: Bifurcation analysis

Model parameter

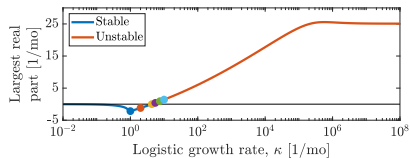


Kernel parameter

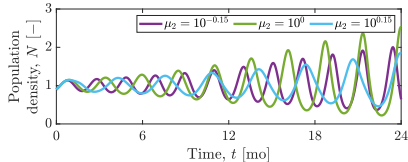
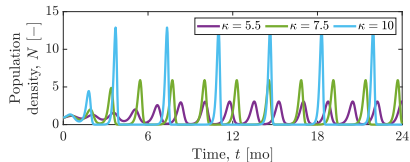
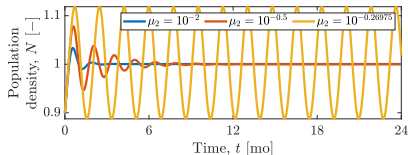
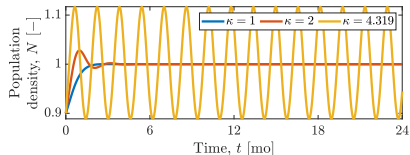
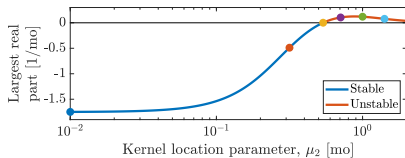


Numerical example: Bifurcation analysis

Model parameter



Kernel parameter

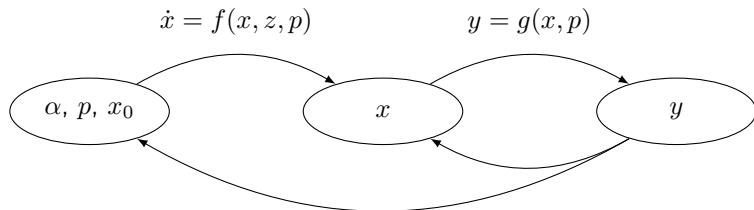


The numerical simulations are obtained with Euler's implicit method and a right rectangle rule for approximating the integral.

Kernel identification

Kernel identification (with John Wyller, NMBU, Norway)

Forward problem



Inverse problem

Ritschel, T.K.S., Wyller, J., 2025. An Algorithm for Distributed Time Delay Identification without A Priori Knowledge of the Kernel. *Automatica* 178, pp. 112382. DOI: 10.1016/j.automatica.2025.112382.

Example

► System

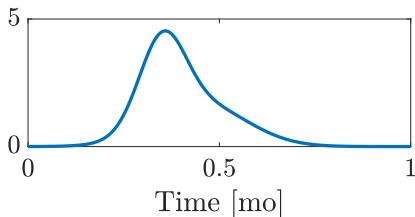
$$\dot{N}(t) = \kappa N(t) \left(1 - \frac{1}{K(t)} \int_{-\infty}^t \alpha(t-s) N(s) \, ds \right),$$

$$K(t) = (1 + A_1 \sin(2\pi\omega_1 t) + A_2 \sin(2\pi\omega_2 t)) \bar{K}$$

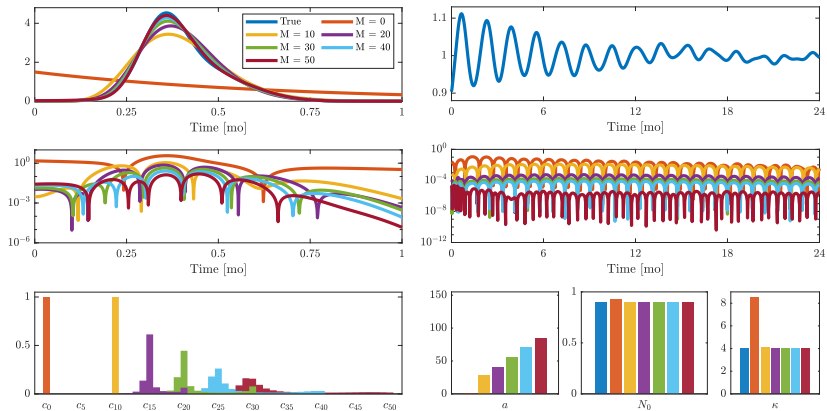
► Kernel

$$\alpha(t) = \gamma_1 F(t; \mu_1, \sigma_1) + \gamma_2 F(t; \mu_2, \sigma_2),$$

$$F(t; \mu, \sigma) = \frac{\exp\left(-\frac{1}{2} \left(\frac{t-\mu}{\sigma}\right)^2\right) + \exp\left(-\frac{1}{2} \left(\frac{t+\mu}{\sigma}\right)^2\right)}{\sqrt{2\pi}\sigma}$$



Example



Summary

Summary

Original system of delay differential equations

$$\dot{x}(t) = f(x(t), z(t)), \quad z(t) = \int_{-\infty}^t \alpha(t-s)x(s) \, ds \quad (29)$$

Erlang ODE approximation

$$\dot{\hat{x}}(t) = f(\hat{x}(t), \hat{z}(t)), \quad \hat{z}(t) = CZ(t), \quad (30a)$$

$$\dot{Z}(t) = AZ(t) + Bx(t) \quad (30b)$$

- ▶ Uniform convergence of $\hat{x}$ and $\hat{z}$ over finite time intervals
- ▶ Convergence of steady state
- ▶ Convergence of roots, λ , of characteristic equation

Future work

- ▶ Approximate systems with finite memory
- ▶ Approximate solution of optimal control problems
- ▶ Approximate feedback control laws
- ▶ Approximate delay-adaptive control laws

Bibliography

- [1] T. K. S. Ritschel and J. Wyller, “An algorithm for distributed time delay identification without a priori knowledge of the kernel,” *Automatica*, vol. 178, p. 112382, 2025.
- [2] T. K. S. Ritschel, “On Erlang ODE approximations of differential equations with distributed time delays,” *SIAM Journal on Applied Dynamical Systems*, 2026. Accepted.
- [3] N. Guglielmi and E. Hairer, “Applying stiff integrators for ordinary differential equations and delay differential equations to problems with distributed delays,” *SIAM Journal on Scientific Computing*, vol. 47, no. 1, pp. A102–A123, 2025.

<https://tobiasritschel.github.io/mixbox>

Thank you!