

Real and complex stability radii for a class of transport networks

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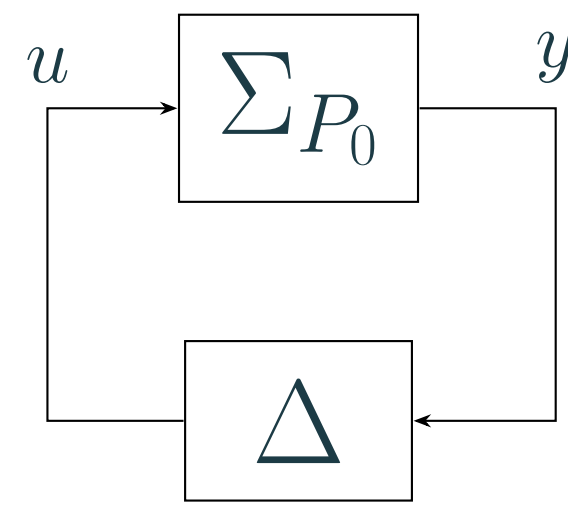
The class of transport networks

$$\begin{aligned} \frac{\partial x}{\partial t}(\xi, t) &= -\frac{\partial}{\partial \xi}(\lambda_0(\xi)x(\xi, t)) + P_0(\xi)\lambda_0(\xi)x(\xi, t), \quad x(\xi, 0) = x_0(\xi), \\ \begin{bmatrix} 0 \\ I \end{bmatrix} u(t) &= -K\lambda_0(0)x(0, t) - L\lambda_0(1)x(1, t), \\ y(t) &= -K_y\lambda_0(0)x(0, t) - L_y\lambda_0(1)x(1, t), \end{aligned} \quad (\Sigma_{P_0})$$

with

- $t \geq 0, \xi \in [0, 1]$,
 - initial condition $x_0 \in L^2([0, 1]; \mathbb{K}^n)$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$,
 - $\lambda_0, \lambda_0^{-1} \in L^\infty([0, 1]; (0, \infty))$ and $P_0 \in L^\infty([0, 1]; \mathbb{K}^{n \times n})$,
 - inputs $u(t) \in \mathbb{K}^m$, $m \leq n$, and outputs $y(t) \in \mathbb{K}^k$,
 - $K, L \in \mathbb{K}^{n \times n}$, $K_y, L_y \in \mathbb{K}^{k \times n}$,
 - K invertible, which is equivalent to well-posedness, see [Zwa+10].
- (Σ_{P_0}) is a port-Hamiltonian system in the sense of [JZ12].

Perturbed system



(Σ_{P_0}) with the output feedback $u(t) = \Delta y(t)$, $\Delta \in \mathbb{K}^{m \times k}$, yields

$$\begin{aligned} \frac{\partial x}{\partial t}(\xi, t) &= -\frac{\partial}{\partial \xi}(\lambda_0(\xi)x(\xi, t)) + P_0(\xi)\lambda_0(\xi)x(\xi, t), \quad x(\xi, 0) = x_0(\xi), \\ 0 &= -(K - \begin{bmatrix} 0 \\ I \end{bmatrix} \Delta K_y) \lambda_0(0)x(0, t) - (L - \begin{bmatrix} 0 \\ I \end{bmatrix} \Delta L_y) \lambda_0(1)x(1, t), \end{aligned} \quad (\Sigma_{P_0}^\Delta)$$

which is well-posed iff $K - \begin{bmatrix} 0 \\ I \end{bmatrix} \Delta K_y$ is invertible.

Definition of stability radii

Q: How *large* can the perturbation Δ be while the perturbed system $(\Sigma_{P_0}^\Delta)$ remains stable?

$$r_{\mathbb{K}}(\Sigma_{P_0}) := \inf \left\{ \|\Delta\| : \Delta \in \mathbb{K}^{m \times k}, K - \begin{bmatrix} 0 \\ I \end{bmatrix} \Delta K_y \text{ is not invertible or } (\Sigma_{P_0}^\Delta) \text{ is not stable} \right\}.$$

$(\Sigma_{P_0}^\Delta)$ is *stable* if it is asymptotically or exponentially stable, which are equivalent concepts here, see [HJW26, Prop. 1.2].

Simplification

(Σ_{P_0}) is equivalent to

$$\begin{aligned} \frac{\partial \tilde{x}}{\partial t}(\xi, t) &= -\frac{\partial}{\partial \xi}(\lambda_0(\xi)\tilde{x}(\xi, t)), \quad \tilde{x}(\xi, 0) = \tilde{x}_0(\xi), \\ Bu(t) &= \lambda_0(0)\tilde{x}(0, t) - A\lambda_0(1)\tilde{x}(1, t), \\ y(t) &= C\lambda_0(1)\tilde{x}(1, t) + Du(t) \end{aligned} \quad (\Sigma_0)$$

with matrices

$$\begin{aligned} A &:= -K^{-1}LP(1) \in \mathbb{K}^{n \times n}, \quad B := -K^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} \in \mathbb{K}^{n \times m}, \\ C &:= (K_y K^{-1}L - L_y)P(1) \in \mathbb{K}^{k \times n}, \quad D := K_y K^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix} \in \mathbb{K}^{k \times m}. \end{aligned}$$

(Σ_{P_0}) and (Σ_0) are equivalent in the sense that x and $\tilde{x}$ are such that $\tilde{x}(\xi, t) = P^{-1}(\xi)x(\xi, t)$ where P is the solution of $P'(\xi) = P_0(\xi)P(\xi)$, $P(0) = I$, and using the invertibility of K .

Main theorem

Let $G(z) = C(z - A)^{-1}B + D \in \mathbb{C}^{k \times m}$, $z \in \rho(A)$. Then:

$$r_{\mathbb{C}}(\Sigma_{P_0}) = \left[\max_{z \in \partial \mathbb{D}} \|G(z)\|_{\mathbb{C}^{k \times m}} \right]^{-1}.$$

For $m = 1$ or $k = 1$, we have

$$r_{\mathbb{R}}(\Sigma_{P_0}) = \left[\max \left\{ \|D\|_{\mathbb{R}^{k \times m}}, \max_{z \in \partial \mathbb{D}} \text{dist}(\text{Re}(G(z)), \mathbb{R}\text{Im}(G(z))) \right\} \right]^{-1}$$

and when $m, k > 1$ we obtain

$$r_{\mathbb{R}}(\Sigma_{P_0}) = \left[\max \left\{ \|D\|_{\mathbb{R}^{k \times m}}, \max_{z \in \partial \mathbb{D}} \inf_{\gamma \in (0, 1]} \sigma_2 \left(\begin{bmatrix} \text{Re}(G(z)) & -\gamma \text{Im}(G(z)) \\ \frac{1}{\gamma} \text{Im}(G(z)) & \text{Re}(G(z)) \end{bmatrix} \right) \right\} \right]^{-1}.$$

$\sigma_2(\cdot)$ is the second largest singular value. Convention: $\frac{1}{\infty} = 0$.

For the real stability radius we assume that all norms are induced by the Euclidean norm.

Equivalent discrete-time system

(Σ_0) is equivalent to the infinite-dimensional discrete-time system

$$\begin{aligned} x_d(\tau + 1) &= Ax_d(\tau) + Bu_d(\tau), \quad x_d(0) = x_{d,0}, \\ y_d(\tau) &= Cx_d(\tau) + Du_d(\tau), \end{aligned} \quad (\Sigma_d)$$

with initial condition $x_{d,0} \in L^2([0, 1]; \mathbb{K}^n)$, inputs $u_d(\tau) \in L^2([0, 1]; \mathbb{K}^m)$, outputs $y_d(\tau) \in L^2([0, 1]; \mathbb{K}^k)$, $\tau \in \mathbb{N}_0$, see [HJZ25, Lem. 3.1].

- Solution of (Σ_0) : $\tilde{x}(\xi, t) = \lambda_0(\xi)^{-1}f(k(\xi) + p(1)^{-1}t)$ with some function f and $k(\xi) := 1 - p(1)^{-1}p(\xi)$ with $p(\xi) := \int_0^\xi \lambda_0(\zeta)^{-1}d\zeta$.
- Relation: $x_d(\tau)(\zeta) = f(\tau + \zeta)$, $u_d(\tau)(\zeta) = u((\tau + \zeta)p(1))$, $y_d(\tau)(\zeta) = y((\tau + \zeta)p(1))$, $\tau \in \mathbb{N}_0, \zeta \in [0, 1]$.

Proof sketch

- For $\Delta \in \mathbb{K}^{m \times k}$: (Σ_{P_0}) with $u(t) = \Delta y(t)$ is exponentially/asymptotically stable iff (Σ_d) with $u_d(\tau) = \Delta y_d(\tau)$ is exponentially/asymptotically stable.
- Consider the finite-dimensional discrete-time system

$$\begin{aligned} x_f(\tau + 1) &= Ax_f(\tau) + Bu_f(\tau), \quad \tau \in \mathbb{N}_0, \quad x_d(0) = x_{f,0} \in \mathbb{K}^n, \\ y_f(\tau) &= Cx_f(\tau) + Du_f(\tau), \quad \tau \in \mathbb{N}_0. \end{aligned} \quad (\Sigma_f)$$

(Σ_d) with $u_d(\tau) = \Delta y_d(\tau)$ is exponentially stable iff it is asymptotically stable iff (Σ_f) with $u_f(\tau) = \Delta y_f(\tau)$ is stable.

- The formulas for the stability radii are obtained from the known formulas for the stability radii for finite-dimensional systems, which can be found, e.g., in [HP05, Sec. 5.3].

Example: A network of two transport equations

$$\begin{aligned} \frac{\partial x}{\partial t}(\xi, t) &= -\frac{\partial}{\partial \xi}(\lambda_0(\xi)x(\xi, t)), \quad x(\xi, 0) = x_0(\xi), \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t) &= -\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \lambda_0(0)x(0, t) - \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \lambda_0(1)x(1, t), \\ y(t) &= -\begin{pmatrix} -1 & 0 \end{pmatrix} \lambda_0(1)x(1, t), \end{aligned} \quad (\Sigma_{\text{Ex}})$$

with $\xi \in [0, 1], t \geq 0$, $\mathbb{K} = \mathbb{R}$ and arbitrary $\lambda_0 \in L^\infty([0, 1], (0, \infty))$ such that $\lambda_0^{-1} \in L^\infty([0, 1], (0, \infty))$.

- (Σ_{P_0}) with $m = 1, k = 1$ and $n = 2$ and $K = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, $L = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$, $K_y = 0$, $L_y = \begin{pmatrix} -1 & 0 \end{pmatrix}$ as well as $P_0 \equiv 0$.
- Associated matrices: $A = \begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 \end{pmatrix}$ and $D = 0$.
- $G(z) = C(z - A)^{-1}B + D = \frac{z}{z^2 + 1}$ on $\partial \mathbb{D}$.
- Application of the main theorem: $r_{\mathbb{R}}(\Sigma_{\text{Ex}}) = \frac{5}{4}$ and $r_{\mathbb{C}}(\Sigma_{\text{Ex}}) = \frac{3}{4}$.

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