Characterizations of Input-to-State Stability for Wide Classes of Control Systems

Andrii Mironchenko, Fabian Wirth

Faculty of Mathematics and Computer Science
University of Passau

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Class of systems

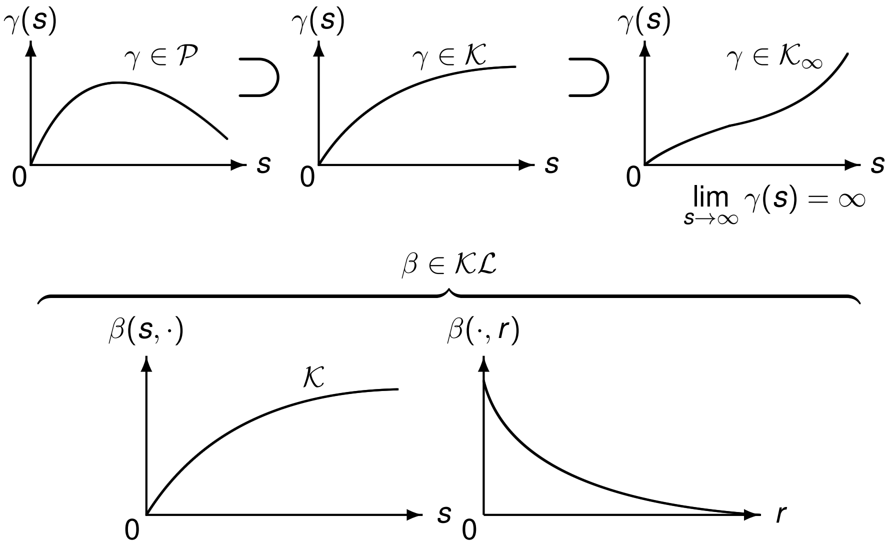
$$\begin{cases} \dot{x}(t) = Ax(t) + f(x(t), u(t)), & x(t) \in D(A) \subset X, \\ x(0) = x_0. \end{cases}$$

- $\mathcal{U} = PC(\mathbb{R}_+, U)$
- $Ax = \lim_{t \to +0} \frac{1}{t}(T(t)x - x)$.
- T is a C_0-semigroup.
- f is a Lipschitz continuous perturbation.

$x \in C([0, T], X)$ is a mild solution iff

$$x(t) = T(t)x_0 + \int_0^t T(t-s) f(x(s), u(s)) ds.$$

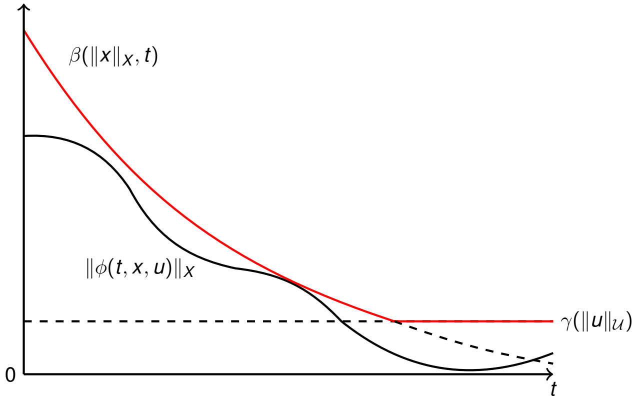
Comparison functions



Definition (ISS)

ISS $:\Leftrightarrow \quad \exists \beta \in \mathcal{KL}, \gamma \in \mathcal{K}_\infty: \quad \forall t \geq 0, \forall x \in X, \forall u \in \mathcal{U}$

$$\|\phi(t,x,u)\|_X \leq \max\left\{\beta(\|x\|_X, t), \gamma\Big(\sup_{s\in[0,t]} \|u(s)\|_U\Big)\right\}.$$



Motivation

Why ISS?

1. Unified theory of internal and external stability
   - E. D. Sontag. *Input to State Stability: Basic Concepts and Results.* In Nonlinear and Optimal Control Theory, chapter 3, 2008.
2. Robust stabilization of nonlinear systems
   - M. Krstić, I. Kanellakopoulos, P. Kokotović. Nonlinear and adaptive control design, Wiley, 1995.
3. Design of robust nonlinear observers
   - M. Arcak, P. Kokotović. Nonlinear observers: a circle criterion design and robustness analysis, 2001.
4. Stability of networks of nonlinear control systems
   - Z.-P. Jiang, I. Mareels, Y. Wang. A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems, Automatica, 1996.
   - S. Dashkovskiy, B. Rüffer, F. Wirth. Small Gain Theorems for Large Scale Systems and Construction of ISS Lyapunov Functions, SICON, 2010.
5. ...

Directions in ∞D ISS theory

- Linear systems: Characterizations and applications
  F. Bribiesca Argomedo, B. Jacob, I. Karafyllis, M. Krstic, F. Mazenc, AM, R. Nabiullin, J. R. Partington, C. Prieur, F. Schwenninger, F. Wirth, E. Witrant, H. Zwart, ...
- Nonlinear systems: Lyapunov theory, small-gain theorems
  M. Ahmadi, A. Chaillet, S. Dashkovskiy, G. Is. Detorakis, H. Ito, B. Jayawardhana, Z.-P. Jiang, I. Karafyllis, M. Krstic, H. Logemann, F. Mazenc, AM, S. Palfi, A. Papachristodoulou, A. Pisano, C. Prieur, E. P. Ryan, S. Senova, Y. Orlov, A. Tanwani, S. Tarbouriech, G. Valmorbida
- Nonlinear systems: Characterizations
  AM, F. Wirth.

Motivation II

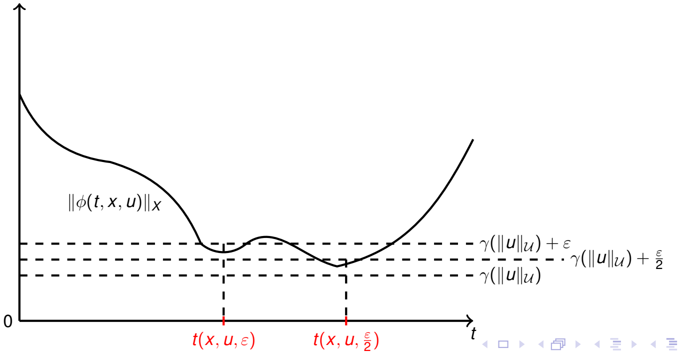
Definition (Stability and Attractivity for zero inputs)

0-ULS $:\Leftrightarrow \quad \exists \sigma \in \mathcal{K}_\infty, \, r > 0$

$$\|x\|_X \le r, \; t \ge 0 \quad \Rightarrow \quad \|\phi(t,x,0)\|_X \le \sigma(\|x\|_X)$$

LIM $:\Leftrightarrow \quad \exists \gamma \in \mathcal{K}_\infty \colon \forall x \in X, \, \forall u \in \mathcal{U}, \, \forall \varepsilon > 0 \; \exists t = t(\varepsilon, x, u) :$

$$\|\phi(t,x,u)\|_X \le \varepsilon + \gamma(\|u\|_\mathcal{U}).$$



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Theorem (Characterizations of ISS (Sontag, Wang))

$$\dot{x} = f(x,u). \qquad \text{(ODE)}$$

For a system (ODE) it holds that

$$\mathit{ISS} \quad \Leftrightarrow \quad \mathit{FC} \wedge \mathit{LIM} \wedge \mathit{0\text{-}ULS}$$

- E. D. Sontag and Y. Wang. On characterizations of the input-to-state stability property. Sys. & Cont. Letters, 1995.
- E. D. Sontag and Y. Wang. New characterizations of input-to-state stability. IEEE TAC, 1996.

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Can we prove a generalization of this theorem to evolution equations?

Thinking about stability theory

Definition

Σ is called forward complete (FC) if for any $x \in X$, any $u \in \mathcal{U}$ the solution $\phi(\cdot, x, u)$ exist and is finite for all times.

Thinking about stability theory

Does forward completeness tell us something beyond mere existence of solutions?

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Definition

FC Σ has bounded reachability sets (BRS) $:=$

$:= \quad \forall R > 0 \; \forall \tau > 0 \; \exists M = M(R,\tau)$:

$$\sup_{\|x\|_X \leq R, \; \|u\|_{\mathcal{U}} \leq R, \; t \in [0,\tau]} \|\phi(t,x,u)\|_X \leq M(R,\tau) < \infty.$$

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Proposition (Lin, Sontag, Wang, 1996)

$$\Sigma_{ODE} : \dot{x} = f(x,u), \qquad x(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^m.$$

$$\Sigma_{ODE} \text{ is FC} \quad \Leftrightarrow \quad \Sigma_{ODE} \text{ is BRS.}$$

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BRS is a "bridge" between solution theory and stability theory

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Compactness of closed bounded balls is needed in the proof

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Nonlinear ∞-dim FC systems are not necessarily BRS!

Nonlinear undisturbed systems: examples

Consider systems

$$\dot{x} = Ax + f(x)$$

- FC $\wedge$ 0-GAS $\not\Rightarrow$ BRS

$$X = l_2 := \{(z_i)_{i=1}^{\infty} : \sum_{i=1}^{\infty} |z_i|^2 < \infty\}, \quad z_i = (x_i, y_i) \in \mathbb{R}^2.$$

$$\Sigma : \begin{cases} \Sigma_i : \begin{cases} \dot{x}_i = -x_i + x_i^2 y_i - \frac{1}{i^2} x_i^3, \\ \dot{y}_i = -y_i. \end{cases} \\ i = 1, \ldots, \infty \end{cases}$$

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- FC $\wedge$ BRS $\wedge$ 0-GAS $\not\Rightarrow$ 0-UGS

Moral

What we know:

- a closed unit ball is never compact
- FC $\wedge$ 0-GAS $\quad\not\Rightarrow\quad$ BRS
- FC $\wedge$ BRS $\wedge$ 0-GAS $\quad\not\Rightarrow\quad$ 0-UGS

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What we know:

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Generalizing results from ODE theory is not straightforward!

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Non-Lyapunov characterizations due to Sontag, Wang do not hold for in ∞-dim!

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- a closed unit ball is never compact
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do not hold for in ∞-dim!

Key: Uniform weak attractivity!

Definition

ULS $:\Leftrightarrow \quad \exists \sigma, \gamma \in \mathcal{K}_\infty,\ r > 0$:

$$t \geq 0, \|x\|_X \leq r, \|u\|_{\mathcal{U}} \leq r \ \Rightarrow \ \|\phi(t,x,u)\|_X \leq \sigma(\|x\|_X) + \gamma(\|u\|_{\mathcal{U}}).$$

LIM $:\Leftrightarrow \quad \exists \gamma \in \mathcal{K}_\infty$: $\forall x \in X, \forall u \in \mathcal{U}, \forall \varepsilon > 0\ \exists T = T(\varepsilon, x, u)$:

$$\|\phi(t,x,u)\|_X \leq \varepsilon + \gamma(\|u\|_{\mathcal{U}}).$$

ULIM $:\Leftrightarrow \quad \exists \gamma \in \mathcal{K}_\infty \cup \{0\}$: $\forall \varepsilon, \delta > 0\ \exists T = T(\varepsilon, \delta)$:

$$u \in \mathcal{U},\ \|x\|_X \leq \delta \quad \Rightarrow \quad \exists t \leq T : \|\phi(t,x,u)\|_X \leq \varepsilon + \gamma(\|u\|_{\mathcal{U}}).$$

Definition

$$\text{ULS} \;:\Leftrightarrow\; \exists \sigma,\gamma \in \mathcal{K}_\infty,\ r>0:$$

$$t \geq 0, \|x\|_X \leq r, \|u\|_\mathcal{U} \leq r \;\Rightarrow\; \|\phi(t,x,u)\|_X \leq \sigma(\|x\|_X) + \gamma(\|u\|_\mathcal{U}).$$

$$\text{LIM} \;:\Leftrightarrow\; \exists \gamma \in \mathcal{K}_\infty: \forall x \in X, \forall u \in \mathcal{U}, \forall \varepsilon > 0\ \exists T = T(\varepsilon, x, u):$$

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$$u \in \mathcal{U},\ \|x\|_X \leq \delta \quad \Rightarrow \quad \exists t \leq T: \|\phi(t,x,u)\|_X \leq \varepsilon + \gamma(\|u\|_\mathcal{U}).$$

Proposition (AM, F. Wirth, 2017)

For forward complete ODEs it holds that:

$$\text{LIM} \quad \Leftrightarrow \quad \text{ULIM}$$

Proof.

Proved with the help of Corollary III.3, Sontag & Wang 1996. □

Theorem (AM, F. Wirth, 2017)

Let Σ be BRS. Then the following statements are equivalent:

(i) Σ *is ISS*

(ii) Σ *is BRS* $\wedge$ *ULIM* $\wedge$ *ULS*

Characterizations of ISS

Theorem (AM, F. Wirth, 2017)

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But in ODE case we have had:

Theorem (Sontag, Wang, 1996)

$$\dot{x} = f(x, u). \qquad \text{(ODE)}$$

For a system (ODE) it holds that

$$\textit{ISS} \quad \Leftrightarrow \quad \textit{FC} \wedge \textit{LIM} \wedge \textit{0-ULS}$$

ISS Lyapunov functions

$$\dot{x}(t) = Ax(t) + f(x(t), u(t)),$$

Definition

$V : X \to \mathbb{R}_+$ is a non-coercive ISS-Lyapunov function iff $\exists \psi_2, \sigma, \alpha \in \mathcal{K}_\infty$:

(i) $0 < V(x) \leq \psi_2(\|x\|_X) \quad \forall x \neq 0$

(ii) $\dot{V}_u(x) \leq -\alpha(\|x\|_X) + \sigma(\|u(0)\|_U) \quad \forall x \in X, \forall u \in \mathcal{U}$,

$$\dot{V}_u(x) = \overline{\lim_{t \to +0}} \frac{1}{t}\big(V(\phi(t, x, u)) - V(x)\big).$$

V is called a coercive ISS-Lyapunov function if

$$\exists \psi_1, \psi_2 \in \mathcal{K}_\infty : \quad \psi_1(\|x\|_X) \leq V(x) \leq \psi_2(\|x\|_X), \quad \forall x \neq 0.$$

Theorem (Classical Direct Lyapunov theorem)

$\exists$ a coercive (L)ISS Lyapunov function $\Rightarrow$ (L)ISS.

LISS = 0-UAS = ∃ coercive LISS-LF

$$\dot{x}(t) = Ax(t) + f(x(t), u(t))$$

Theorem (AM, Sys. & Cont. Lett., 2016)

(i) $\forall C > 0 \ \exists K(C) > 0$:

$$\|x\|_X \leq C,\ \|y\|_X \leq C \quad \Rightarrow \quad \|f(y,v) - f(x,v)\|_X \leq L_f(C)\|y - x\|_X.$$

(ii) $f(x,\cdot)$ *be continuous for all* $x \in X$.

(iii) $\exists\ \sigma \in \mathcal{K}$ *and* $\rho > 0$:

$$\|v\|_U \leq \rho,\ \|x\|_X \leq \rho \quad \Rightarrow \quad \|f(x,v) - f(x,0)\|_X \leq \sigma(\|v\|_U).$$

$$\Downarrow$$

0-UAS $\Leftrightarrow$ ∃ *0-UAS LF* $\Leftrightarrow$ ∃ *LISS LF* $\Leftrightarrow$ *LISS*

Counterexample: 0-UGAS but not LISS

$$\dot{x}_k(t) = -\frac{1}{1 + k|u(t)|} x_k(t)$$

- $X = l_1 := \{(x_k)_{k=1}^{\infty} : \sum_{k=1}^{\infty} |x_k| < \infty\}$
- $\mathcal{U} := PC(\mathbb{R}_+, \mathbb{R})$

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- $\forall u, \ \forall x \quad \Rightarrow \quad \|\phi(t, x, u)\|_X \to 0, t \to \infty$

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- Properties (i) and (ii) hold

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But property (iii) does not hold!

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But property (iii) does not hold!

It is not LISS!

Final remarks concerning LISS

$\mathbb{R}^n$-world

1. In $\mathbb{R}^n$ (ii) $\wedge$ (i) $\Rightarrow$ (iii).

Final remarks concerning LISS

$\mathbb{R}^n$-world

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2. Sontag and Wang, 1996: 0-GAS $\Rightarrow$ LISS for ODEs.

Final remarks concerning LISS

$\mathbb{R}^n$-world

1. In $\mathbb{R}^n$ (ii) $\wedge$ (i) $\Rightarrow$ (iii).
2. Sontag and Wang, 1996: 0-GAS $\Rightarrow$ LISS for ODEs.
3. An adaptation of the argument by Sontag and Wang would give 0-AS = LISS.

Final remarks concerning LISS

$\mathbb{R}^n$-world

1. In $\mathbb{R}^n$ (ii) $\wedge$ (i) $\Rightarrow$ (iii).
2. Sontag and Wang, 1996: 0-GAS $\Rightarrow$ LISS for ODEs.
3. An adaptation of the argument by Sontag and Wang would give 0-AS = LISS.
4. Our result is more general and uses other technique.

Characterizations of ISS

Theorem (AM, F. Wirth, 2017)

Let Σ be BRS. Then the following statements are equivalent:

(i) Σ *is ISS*

(ii) Σ *is BRS* $\wedge$ *ULIM* $\wedge$ *ULS*

If in addition $\exists\, \sigma \in \mathcal{K}$ *and* $\rho > 0$*:*

$$\|v\|_U \leq \rho,\ \|x\|_X \leq \rho \quad \Rightarrow \quad \|f(x,v) - f(x,0)\|_X \leq \sigma(\|v\|_U).$$

then the following notion is also equivalent to ISS:

(iii) Σ *is BRS* $\wedge$ *ULIM* $\wedge$ *0-ULS*

Results of Sontag and Wang are a special case of our results!

Characterizations of ISS

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then the following notion is also equivalent to ISS:

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- A.M. *Local input-to-state stability: Characterizations and counterexamples.* Sys. & Con. Lett., 2016
- A.M., F. Wirth. *Characterizations of input-to-state stability for infinite-dimensional systems.* Provisionally accepted to TAC, 2017
- A.M. *Criteria for input-to-state practical stability.* Submitted, 2017.

Extensions and Open problems

What we could discuss

- Generalizations of these results to wide classes of systems, including:
  - DEs in Banach spaces
  - Time-delay systems
  - Switched systems
- Characterizations of practical ISS

Open problems

- Can these results be tightened for special classes of systems, as e.g. time-delay systems.
- Can we obtain the characterizations of ISS in terms of non-coercive ISS Lyapunov function

Comment on non-coercive Lyapunov functions

$$\dot{x}(t) = Ax(t) + f(x(t)),$$

$V : X \to \mathbb{R}_+$ is a non-coercive Lyapunov function iff $\exists \psi_2, \sigma, \alpha \in \mathcal{K}_\infty$:

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Theorem (AM, F. Wirth, 2017, submitted)

BRS $\wedge$ $\exists$ *a noncoercive Lyapunov function* $\Rightarrow$ *UGAS*

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Can we extend this result to ISS case?

2-nd Workshop "Stability and Control of Infinite-Dimensional Systems" (SCINDIS 2018)

Scope

- Stability and control of partial differential equations
- Stability and control of time-delay systems
- Input-to-state stability of infinite-dimensional systems
- Stabilizability of infinite-dimensional systems
- Semigroup and admissibility theory

- Venue: University of Würzburg, Germany
- Date: 10–12 October, 2018.
- Organisers: S. Dashkovskiy, B. Jacob, AM, F. Wirth.

Previous workshop: Passau, Germany, 12–14 October, 2016

- 47 Participants from 11 countries; 22 Invited speakers
- http://www.fim.uni-passau.de/en/dynamical-systems/workshop/

Characterizations of ISS

Theorem (AM, F. Wirth, 2017)

Let Σ be BRS. Then the following statements are equivalent:

(i) Σ *is ISS*

(ii) Σ *is BRS* $\wedge$ *ULIM* $\wedge$ *ULS*

If in addition $\exists\ \sigma \in \mathcal{K}$ *and* $\rho > 0$*:*

$$\|v\|_U \leq \rho,\ \|x\|_X \leq \rho \quad \Rightarrow \quad \|f(x,v) - f(x,0)\|_X \leq \sigma(\|v\|_U).$$

then the following notion is also equivalent to ISS:

(iii) Σ *is BRS* $\wedge$ *ULIM* $\wedge$ *0-ULS*

Thank you for attention!

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