Design of saturated controls for an unstable parabolic PDE

Andrii Mironchenko[1], Christophe Prieur[2], Fabian Wirth[1]

[1] Faculty of Mathematics and Computer Science
University of Passau

[2] CNRS, Grenoble INP, GIPSA-lab
Université Grenoble Alpes

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Problem formulation

Consider a linear heat equation with controls $v_1, \ldots, v_m$:

$$w_t(t,x) = w_{xx}(t,x) + c(x)w(t,x) + \sum_{k=1}^{m} b_k(x)v_k(t), \;\; t>0, \;\; x \in (0,L),$$

$$w(t,0) = w(t,L) = 0, \quad t > 0,$$

$$w(0,x) = w^0(x), \quad x \in (0,L).$$

- The state space of this system is $X := L_2(0,L)$ and $c, b_k \in X$.

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'All control actuation devices are subject to amplitude saturation. Force, torque, thrust, stroke, voltage, current, flow rate, and every conceivable physical input in every conceivable application of control technology is ultimately limited.'

D. Bernstein, A. Michel. *A chronological bibliography on saturating actuators*, 1995.

Realistic control schemes have to take into account the saturations of the control input: $v_k(t) := \mathrm{sat}\big(u_k(t)\big)$

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- The state space of this system is $X := L_2(0,L)$ and $c, b_k \in X$.
- Here sat is a component-wise saturation function, i.e., for all k, v we have:

$$\mathrm{sat}(v)_k := \begin{cases} v_k & \text{if } |v_k| \le \ell, \\ \frac{\ell}{|v_k|} v_k & \text{if } |v_k| \ge \ell. \end{cases}$$

Goal: design controls, locally stabilizing above PDE and estimate the attraction region of the closed-loop system

Although our PDE system is linear, the closed-loop system is nonlinear due to saturations.

Orthogonal decomposition of the solutions

Every mild solution $w(t,\cdot) \in D(A)$ can be expanded as a series in the eigenfunctions of $A : w \mapsto w_{xx} + c(\cdot)w$:

$$w(t,\cdot) = \sum_{j=1}^{\infty} w_j(t) e_j(\cdot), \quad w_j(t) := \big\langle w(t,\cdot), e_j(\cdot)\big\rangle_{L_2(0,L)}, \; j \in \mathbb{N}^* = \mathbb{N} \cup \{0\}.$$

Analogously, we can expand the coefficients b_k in the series

$$b_k(\cdot) = \sum_{j=1}^{\infty} b_{jk} e_j(\cdot), \quad b_{jk} = \big\langle b_k(\cdot), e_j(\cdot)\big\rangle_{L_2(0,L)}, \; j \in \mathbb{N}^*.$$

PDE Reformulation I:

$$\begin{aligned} \dot{w}_j(t) &= \lambda_j w_j(t) + \sum_{k=1}^{m} b_{jk} \mathrm{sat}(u_k(t)) \\ &= \lambda_j w_j(t) + \mathbf{b}_j \cdot \mathrm{sat}(u(t)), \qquad j \in \mathbb{N}^*, \end{aligned}$$

$\mathbf{b}_j$ is the row vector with entries b_{jk}, $k = 1, \ldots, m$.

Mild solution w is only continuous, but w_j are absolutely continuous!

Let $n \in \mathbb{N}^*$ be the number of nonnegative eigenvalues of A and $\eta > 0$ be such that

$$j > n \quad \Rightarrow \quad \lambda_j < -\eta < 0.$$

We introduce the matrix notation:

$$z := \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}, \quad \mathbf{A} := \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}, \quad \mathbf{B} := \begin{pmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nm} \end{pmatrix}.$$

PDE Reformulation II:

$$\begin{aligned} \dot{z}(t) &= \mathbf{A}z(t) + \mathbf{B}\mathrm{sat}(u(t)) \\ \dot{w}_j(t) &= \lambda_j w_j(t) + \mathbf{b}_j \cdot \mathrm{sat}(u(t)), \qquad j > n. \end{aligned}$$

Proposition

If $(\mathbf{A}, \mathbf{B})$ is stabilizable, then there is $\mathbf{K}$ so that $u(t) := \mathbf{K}z(t)$ locally exponentially stabilizes the whole system.

Proof.

- $u(t) := \mathbf{K}z(t)$ locally stabilizes z-subsystem
- as $\lambda_j < -\eta < 0$, $\quad (w_j)_{j>n}$-subsystem is input-to-state stable
- The whole system is locally exp. stable

Region of attraction

$$\dot{z}(t) = \mathbf{A}z(t) + \mathbf{B}\mathrm{sat}(\mathbf{K}z(t)). \tag{1}$$

Definition

We say that S is a region of attraction of 0 if

(i) $0 \in \operatorname{int} S$;

(ii) for any $z_0 \in S$ the solution of (1) satisfies $z(t; z_0) \to 0$ as $t \to \infty$;

(iii) S is forward invariant, i.e. for any $z_0 \in S$ it holds that $z(t; z_0) \in S$ for all $t \geq 0$.

The largest such set is called the maximal region of attraction.

Definition

(1) is locally exponentially stable with region of attraction S, if:

(i) there exist $\varepsilon, M, a > 0$ such that $\forall z(0) \in X$ satisfying $|z(0)| < \varepsilon$, it holds

$$|z(t)| \leq Me^{-at}|z(0)| \quad \forall t \geq 0.$$

(ii) $B_\varepsilon(0) \subset S$ and S is a region of attraction of (1).

Discretized system with a feedback $u(t) = \mathbf{K}z(t)$:

$$\begin{aligned} \dot{z}(t) &= \mathbf{A}z(t) + \mathbf{B}\mathrm{sat}(\mathbf{K}z(t)) \\ \dot{w}_j(t) &= \lambda_j w_j(t) + \mathbf{b}_j \cdot \mathrm{sat}(\mathbf{K}z(t)), \qquad j > n. \end{aligned}$$

Let ι be the isomorphism between the spaces X and ℓ_2.

Proposition (Attraction region for the whole system)

Assume $\mathbf{K}$ is chosen such that the z-subsystem is locally exponentially stable in 0 with region of attraction $S \subset \mathbb{R}^n$. Then:

(i) 'discretized' system is locally exponentially stable in 0 with region of attraction $S \times \ell_{2,j>n}$.

(ii) 'original' PDE system with the feedback $u(t) := \mathbf{K}z(t)$ is locally exponentially stable in 0 with region of attraction $\iota(S) \times X_n^{\perp}$.

The next aim:
Approximate the attraction region for the z-subsystem.

$$\dot{z}(t) = \mathbf{A}z(t) + \mathbf{B}\mathrm{sat}(\mathbf{K}z(t)) \tag{2}$$

Proposition (Estimation of the attraction region)

Consider (2) with $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$ and $\mathbf{K} \in \mathbb{R}^{m\times n}$. Assume that there is a symmetric pdf matrix $P \in \mathbb{R}^{n\times n}$, a diagonal pdf matrix $\mathbf{D} \in \mathbb{R}^{m\times m}$ and a matrix $\mathbf{C} \in \mathbb{R}^{m\times n}$ such that

$$M_1 := \begin{bmatrix} (\mathbf{A}+\mathbf{BK})^\top P + P(\mathbf{A}+\mathbf{BK}) & P\mathbf{B} - (\mathbf{DC})^\top \\ (P\mathbf{B})^\top - \mathbf{DC} & -2\mathbf{D} \end{bmatrix} < 0 \tag{3}$$

$$M_2 := \begin{bmatrix} P & (\mathbf{K}-\mathbf{C})^\top \\ \mathbf{K}-\mathbf{C} & \ell^2 I_m \end{bmatrix} \geq 0. \tag{4}$$

Then (2) is locally asymptotically stable in 0 with a region of attraction given by

$$\mathcal{A} := \{z,\ z^\top P z \leq 1\}. \tag{5}$$

Note that (3), (4) are bilinear matrix inequality due to the term **DC**.

Hence $\imath(\mathcal{A}) \times X_n^\perp$ is an attraction region of the overall system.

Numerical Experiment I

$$
\begin{aligned}
& w_t(t,x) = w_{xx}(t,x) + c(x)w(t,x) + b(x)\mathrm{sat}(u_1(t)),\ t>0,\ x\in(0,L), \\
& w(t,0) = w(t,L) = 0, \quad t>0, \\
& w(0,x) = w^0(x), \quad x\in(0,L).
\end{aligned} \tag{6}
$$

We choose

$$
c(x)\equiv 10, \quad L=2, \quad \ell = 2, \quad b = e_1 + e_2,
$$

We introduce

$$
A := \partial_{xx} + c\,\mathrm{id} : X \to X, \quad D(A) = H^2(0,L)\cap H^1_0(0,L) \tag{7}
$$

and the eigenfunctions e_j of A are given by

$$
e_j(x) := \left(\frac{2}{L}\right)^{1/2} \sin\frac{j\pi x}{L}, \quad j\in\mathbb{N}^*, \quad x\in(0,L). \tag{8}
$$

These parameters lead to the following values for the matrices **A**, **B**:

$$
\mathbf{A} \approx \begin{pmatrix} 7.5325989 & 0 \\ 0 & 0.1303956 \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
$$

- We choose the matrix $\mathbf{K}$ so that $\sigma(\mathbf{A}+\mathbf{BK}) = \{-1\}$.
- Additionally, we require $(\mathbf{K}-\mathbf{C})\cdot(\mathbf{K}-\mathbf{C})^T \to \min$.

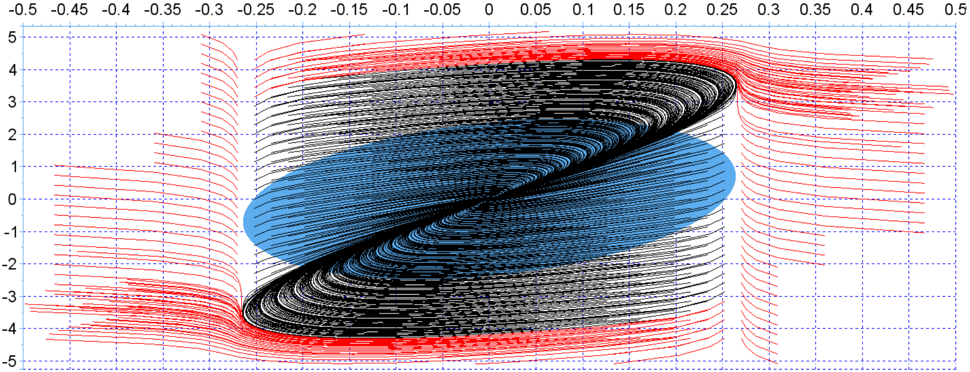
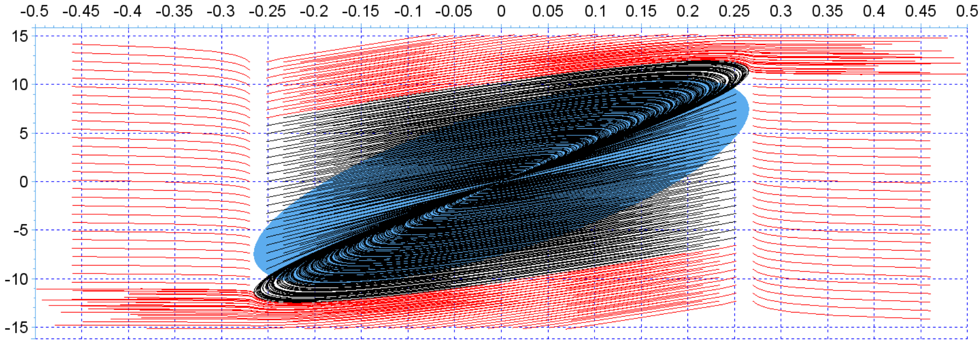


Figure: Region of attraction (in blue) computed via the LMI technique. Trajectories of (2) are computed (0.025 seconds) by direct solution of the ODE, trajectories attracted to the origin are in black, diverging trajectories are in red. (43 seconds with plotting)

Numerical Experiment II

- Let **K** be so that $\sigma(\mathbf{A} + \mathbf{BK}) = \{-0.1, -0.2\}$.
- Let all other parameters be the same.



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Conclusions

We discussed

- Method for construction of saturated controllers for linear parabolic systems
- Estimates for attraction region of the closed-loop system.

Available extensions

- Enlarging the region of attraction using a dynamic controller
- Pointwise saturations
- Applications to boundary control of heat equation subject to control saturations (only under certain restrictive assumptions)

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Further extensions to other types of systems are possible.

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Papers and slides can be found at
www.mironchenko.com