Mathematical Modeling of the Agrocoenosis

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Outline

Modeling approaches

Two modeling approaches

- **Mathematical** - building of the model on the basis of the mathematical correlations and its analysis with the analytical or numeric methods.
- **Simulational** - the building of the material or programming analog of the investigating object, and observation of its behaviour.

Mathematical modeling works better, when the system is:

1. Hard connected inhomogenous system, that in interaction with other systems behaves itself as the whole.
2. System, that consists of vast amount of homogeneous particles (or subsystems)

Plants Population

Supposition about plants

We will assume, that all characteristics of plants are functions only of age.

Equation of Plant Population

Let $P(x, y, t, \tau)$ - the density of the plants of the age τ in the point (x, y) at the moment t.

$$\frac{\partial P(x, y, t, \tau)}{\partial t} + \frac{\partial P(x, y, t, \tau)}{\partial \tau} = -Tod\,(x, y, t, Tiere)$$

Initial/boundary conditions

$\mathrm{P}(x, y, 0, \tau) = g(x, y, \tau)$
$P(x, y, t, \tau) = 0, \tau \geq \tau_T$ - death-condition
$\mathrm{P}(x, y, t, 0) =$ density of seeds, grown up in the point (x, y) at the moment t

Transport, Creation and Germination of Seeds

We'll suppose that:

- The seeds are transported by the wind
- Direction of the wind in every point at every time instant is random, and its velocity is constant.
- There are a vast amount of seeds, and they are very small.

Let $u(x, y, t)$ be the density of seeds in point (x, y) at moment t

Processes in the point

1. Diffusion. Flowdensity of the distributing seeds $\vec{q} = -D\nabla u$
2. Germination. In every point holds $\frac{du}{dt} = -\alpha u$
3. Seedscreation. Density of seeds, created in the point (x, y) at the moment t is:

$$\int_{\tau_G}^{\tau_T} P(x, y, t, \tau)\, f(\tau)\, d\tau$$

Equation of Seeds Distribution

Balance equation in element dS on timespan h

$$(u(x,y,t+h)-u(x,y,t))\,dS = \left(\int\limits_{\tau_G}^{\tau_T} P(x,y,t,\tau)\,f(\tau)\,d\tau\cdot h-\alpha u\cdot h\right)dS-\nabla\cdot(\vec{q})\,dS\cdot h$$

Equation of Seeds Distribution

$$\frac{\partial u(x,y,t)}{\partial t} = c^2\Delta u-\alpha u+\int\limits_{\tau_G}^{\tau_T} P(x,y,t,\tau)\,f(\tau)\,d\tau$$

The boundary conditions:

- If the boundary is the sea: $u|_{\partial\Omega} = 0$
- If the boundary is the rock: $\frac{\partial u}{\partial n}\Big|_{\partial\Omega} = 0$

General model of agrocoenosis

If the boundary is the sea, then the agrocoenosis can be modelled as follows:

$$\begin{cases} \frac{\partial u(x,y,t)}{\partial t} = c^2 \Delta u - \alpha u + \int\limits_{\tau_G}^{\tau_T} P(x,y,t,\tau) f(\tau)\, d\tau \\ \frac{\partial P(x,y,t,\tau)}{\partial t} + \frac{\partial P(x,y,t,\tau)}{\partial \tau} = -Tod(x,y,t,Tiere), \tau < \tau_T \\ P(x,y,t,0) = \alpha u(x,y,t) \\ P(x,y,t,\tau) = 0, \tau \geq \tau_T \\ P(x,y,0,\tau) = g(x,y,\tau) \\ u(x,y,0) = \varphi(x,y) \\ u|_{\partial\Omega} = 0 \end{cases}$$

+ realization of the animals' development

Argocoenosis without animals

IBV-problem for the Plantspopulation

$$\begin{cases} \frac{\partial P(x,y,t,\tau)}{\partial t} + \frac{\partial P(x,y,t,\tau)}{\partial \tau} = 0, \tau < \tau_T \\ P(x,y,0,\tau) = g(x,y,\tau) \\ P(x,y,t,0) = \alpha u(x,y,t) \\ P(x,y,t,\tau) = 0, \tau \geq \tau_T \end{cases}$$

Solution of this problem

$$P(x,y,t,\tau) = \begin{cases} g(x,y,\tau - t) + \alpha u(x,y,t-\tau), \tau < \tau_T \\ 0, \tau \geq \tau_T \end{cases}$$

Germination Equation

Putting the solution $P(x, y, t, \tau)$ from the previous slide into the equation of seeds distribution, we'll have:

$$\frac{\partial u\,(x,y,t)}{\partial t} = c^2 \Delta u - \alpha u + \int_{\tau_G}^{\tau_T} \left(g\,(x,y,\tau - t) + \alpha u\,(x,y,t-\tau)\right) f\,(\tau)\, d\tau$$

Now, let:

- There aren't any plants: $g \equiv 0$
- $f\,(\tau) = const = k$

Then we can write the previous equation in the following form:

$$\frac{\partial u\,(x,y,t)}{\partial t} = c^2 \Delta u - \alpha u + k \int_{t-\tau_T}^{t-\tau_G} \alpha u\,(x,y,s)\, ds$$

This equation we will call **germination equation**

Basis of the model

Phases of plant's growth

1. Seed
2. Sprout
3. Stage of roots' growth
4. Stage of leaves' growth
5. Stage of generation
6. Senile stage

Main processes in a plant

- Photosynthesis
  $H_2O + CO_2 \rightarrow org.subst. + O_2 \uparrow$
- Breath
  $org.subst. + O_2 \rightarrow H_2O + CO_2$
- Self-control
  Mass of the seeds must be maximal

Mass and Self-control

Mass

- Mass of the roots $M_W(t)$
- Mass of the vegetative parts: $M_V(t)$
- Mass of the seeds $M_{Fr}(t)$
- Mass of the starch $M_S(t)$

Energy exchange

Intensity of energy production one can evaluate as follows:

- $\dot{E}(t) = \dot{E}_W(t) + \dot{E}_V(t) + \dot{E}_{Fr}(t) = FS_N(t) - GI_L(t) - \dot{M}_S(t)$

Total energy one can divide into 3 parts:

- $E_W(t)$ - energy for the building of roots
- $E_V(t)$ - energy for the building of leaves
- $E_{Fr}(t)$ - energy for the building of seeds

Model of the one-year plant

One-year plant life cycle is modelled with the following system of ODE, that is controlled with the max-seeds principle:

$$\begin{cases} \dot{H_2O}(t) = Nehm(t) - k_{Tr} H_2O(t) \cdot offen(t) \\ \dot{E}(t) = FS_N(t) - Gl_L(t) - \dot{M}_S(t) \\ \dot{M}_W(t) = \frac{1}{Gl_W} \dot{E}_W(t) \\ \dot{M}_V(t) = \frac{1}{Gl_V} \dot{E}_V(t) \\ \dot{M}_{Fr}(t) = \frac{1}{Gl_{Fr}} \dot{E}_{Fr}(t) \\ \dot{M}_S(t) = -k_S H_2O(t) \end{cases}$$

Max-seeds principle:

$$\max_{E_W(t), E_V(t), E_{Fr}(t)} M_{Fr}(t)$$

$$0 \le E_W(t), E_V(t), E_{Fr}(t) \le E(t)$$

General model

Agrocoenosis

Seeds

$$\frac{\partial u(x,y,t)}{\partial t} = c^2 \Delta u - \alpha u + \int_{\tau_\sigma}^{\tau_T} P(x,y,t,\tau) f(\tau)\, d\tau$$

Model of a plant

Plants Population

$$P(x,y,t,0) = \alpha u(x,y,t)$$

$$\frac{\partial P(x,y,t,\tau)}{\partial t} + \frac{\partial P(x,y,t,\tau)}{\partial \tau} = -Tod(x,y,t,Tiere)$$

Model of an animal

Existence and Uniqueness theorem for GE

Model we need to analyze:

$$\begin{cases} \frac{\partial u(x,y,t)}{\partial t} = c^2 \Delta u - \alpha u + k \int\limits_{t-\tau_T}^{t-\tau_G} \alpha u\,(x,y,s)\,ds,\ (x,y,t) \in G \times [0,T] \\ u\,(x,y,t)|_{G\times[-\tau_T,0]} = \varphi\,(x,y,t) \\ u\,(x,y,t)|_{\partial G\times[0,T]} = \psi\,(x,y,t) \end{cases}$$

Existence and Uniqueness theorem

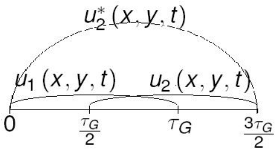
Let for the IBVP for heat equation

$$\begin{cases} \frac{\partial u(x,y,t)}{\partial t} = c^2 \Delta u - \alpha u + f,\ (x,y,t) \in G \times [0,T] \\ u\,(x,y,0)|_{G} = \varphi\,(x,y) \\ u\,(x,y,t)|_{\partial G\times[0,T]} = \psi\,(x,y,t) \end{cases}$$

there exists unique solution in the class K for every function $f \in F$. Then there exists unique solution $u \in K$ for the IBVP for germination equation.

Proof of Existence and Uniqueness theorem

1. On the timespan $Z_1 = [0, \tau_G]$ the germination equation reduces to the heat equation
2. So, on Z_1 there exists unique solution $u_1\,(x, y, t)$ of GE
3. Now we analyse the IBVP for GE on timespan $[\frac{\tau_G}{2}, T]$
4. On $Z_2 = [\frac{\tau_G}{2}, \frac{3\tau_G}{2}]$ there exists unique solution $u_2\,(x, y, t)$ of GE.
5. We can build solution $u_2^*\,(x, y, t)$ on $[0, \frac{3\tau_G}{2}]$



6. After finite number of steps we'll build the unique solution on the timespan $[0, T]$

Reducing of the dimension

In the germination equation

$$\frac{\partial u(x,y,t)}{\partial t} = c^2 \Delta u - \alpha u + k \int\limits_{t-\tau_T}^{t-\tau_G} \alpha u(x,y,s)\, ds$$

We will make substitution:

$$\frac{\partial u}{\partial t} = \frac{u(x,t) - u(x,t-h)}{h}$$

The Germination Equation will become the Helmholz' equation:

$$-\Delta u(x,y,t) + qu(x,y,t) = f(x,y,t)$$

Where

$$q = \frac{(\alpha h + 1)}{hc^2} > 0$$

Function Spaces

Let $U \subset R^n$ - open connected set.

- $C_c^\infty(U)$ - set of all infinitely differentiable functions $\varphi : U \to R^n$ with compact support in U
- $L^p(U)$ - set of all summable functions u, that have $\|u\|_{L^p(U)} < \infty$
- $L^1_{loc}(U) = \{u \in L^1(V), \forall V \subset\subset U\}$
- Let $u, v \in L^1_{loc}(U)$. v is weak derivative of u: $D^\alpha u = v$ if $\int\limits_U uD^\alpha \varphi dx = (-1)^{|\alpha|} \int\limits_U v\varphi dx$ for all $\varphi \in C_c^\infty(U)$
- $W^{k,p}(U)$ - set of all locally summable functions $u : U \to R$, that for every multyindex $\alpha : |\alpha| \leq k\ \exists D^\alpha u \in L^p(U)$
- $W_0^{k,p}(U)$ - the closure of $C_c^\infty(U)$ in $W^{k,p}(U)$.
- $H_0^k(U) = W_0^{k,2}(U)$
- $H^k(U) = W^{k,2}(U)$

Variation formulation of the boundary problem

We will investigate the following problem:

$$\begin{cases} Lu = f, x \in U \\ u = 0, x \in \partial U \end{cases} \quad (1)$$

where $Lu = -\sum_{i,j=1}^{n} \left(a^{ij}(x)\, u_{x_i}\right)_{x_j} + \sum_{i=1}^{n} b^i(x)\, u_{x_i} + c(x)\, u$

- Operator L is elliptic, if there exists $\theta > 0$: $\sum_{i,j=1}^{n} a_{ij}(x)\, s_i s_j \geq \theta\, |s|^2\ \forall x \in U$ and $\forall s \in R^n$
- $u \in H_0^1(U)$ is the weak solution of the problem (1), if

$$B[u, v] = (f, v)\ \forall v \in H_0^1(U), \text{ where}$$

$$B[u, v] := \int_U \sum_{i,j=1}^{n} a^{ij}(x)\, u_{x_i} v_{x_j} + \sum_{i=1}^{n} b^i(x)\, u_{x_i} v + c(x)\, uv dx$$

Lax-Milgram theorem and Energy Estimates

Lax-Milgram theorem

Let H be real Hilbert space, $B : H \times H \to R$ - bilinear functional, $f : H \to R$ - bounded linear functional on H and there exist $\alpha, \beta > 0$, such that:

$$|B[u, v]| \leq \|u\| \cdot \|v\| \, \forall u, v \in H$$
$$\beta \|u\|^2 \leq B[u, u] \, \forall u \in H$$

Then $\exists$ unique element $u \in H$, that: $B[u, v] = \langle f, v \rangle \, \forall v \in H$

Energy estimates

For the bilinear form, defined earlier, the following statement holds: $\exists \alpha, \beta > 0$ and $\gamma \geq 0$, such that $\forall u, v \in H_0^1(U)$:

$$|B[u, v]| \leq \|u\|_{H_0^1(U)} \|v\|_{H_0^1(U)}$$
$$\beta \|u\|^2_{H_0^1(U)} \leq B[u, u] + \gamma \|u\|^2_{L^2(U)}$$

Existence and Uniqueness

Lemma

Let $Lu = -\sum_{i,j=1}^{n} \left(a^{ij} u_{x_i}\right)_{x_j} + cu$ *elliptic and* $c \geq 0$. *Then for* $B[u,v] = (Lu, v)$ *holds:* $\beta \|u\|^2_{H^1_0(U)} \leq B[u,u]$

Proof

1. From ellipticity follows: $B[u,u] \geq \theta \|Du\|^2_{L^2(U)} + c\|u\|^2_{L^2(U)}$
2. Using Poincare inequality $\|u\|^2_{L^2(U)} \leq C\|Du\|^2_{L^2(U)}$, we derive: $\|u\|^2_{H^1_0(U)} \leq \frac{1}{K}\|Du\|^2_{L^2(U)}$
3. Combining 1 and 2, we have: $B[u,u] \geq K\theta \|u\|^2_{H^1_0(U)}$

Existence and Uniqueness for Helmholz equation

Operator $L = -\Delta + c$ is elliptic, so if $c \geq 0$ then for $B[u,v]$ statement of Lax-Milgram Theorem holds.

Finite Elements for the Germination Equation

We want to solve the Helmholz equation in the variation form:

$$B\left[u,v\right]+q\cdot(u,v)=(f_n,v)$$

We construct it as

$$u=\sum_{i=1}^{n}\alpha_i\varphi_i,$$

where $\{\varphi_i\}|_1^n$ is the basis of $V^n\subset H^1$. Also we put

$$v=\varphi_j, j=\overline{1,n}$$

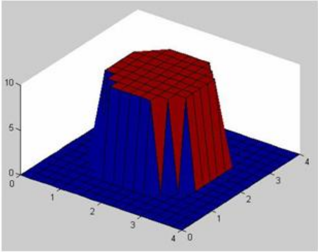
We will have the following system of equations:

$$\sum_{i=1}^{n}\alpha_i B[\varphi_i,\varphi_j]+q\sum_{i=1}^{n}\alpha_i(\varphi_i,\varphi_j)=(f_n,\varphi_j), j=\overline{1,n}$$

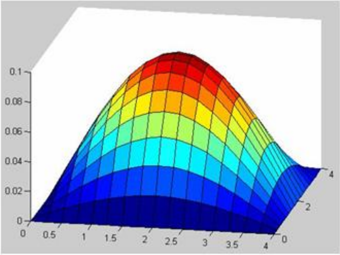
In matrix form:

$$(A+qG)\,\alpha=f_n$$

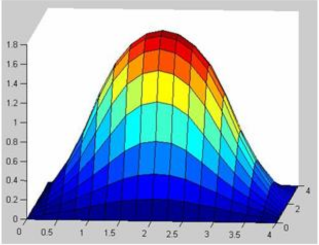
Results, where $\tau_G = 3, \tau_T = 5, \alpha = 1, k = 2$



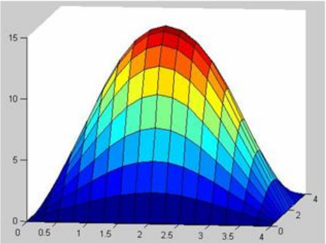
Initial distribution of seeds (t=0)



Plants begin to produce seeds t=3



Plants have already created some seeds, and their density increases (t=3.5)



Distrubution of seeds, t=8

Summary

Main results

- There were built models of agrocoenosis and one-year plant.
- It was proved the uniqueness and existence theorems for the main IBVPs.
- IBVP for Germination Equation was solved numerically.

Literature

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