Towards unified input-to-state stability theory

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joint work with

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Class of systems

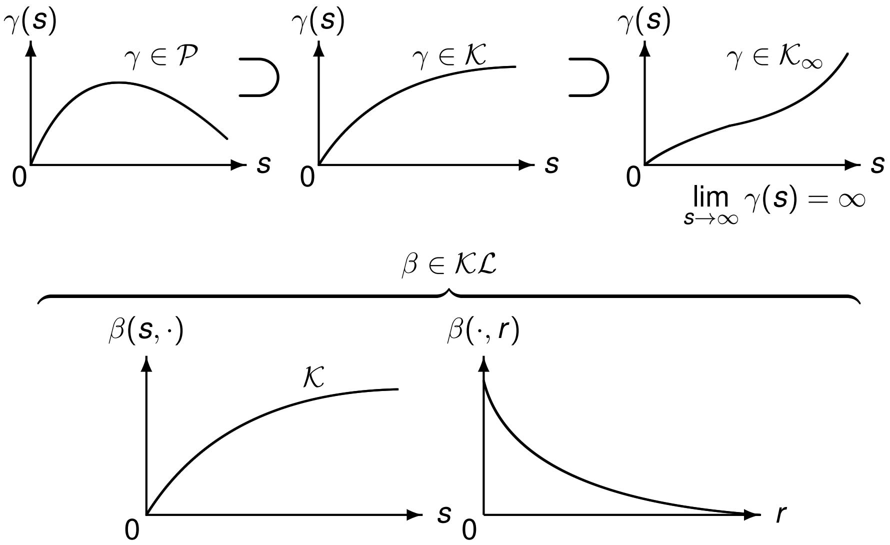
$$\begin{cases} \dot{x}(t) = Ax(t) + f(x(t), u(t)), & x(t) \in D(A) \subset X, \\ x(0) = x_0. \end{cases}$$

- $\mathcal{U} = PC(\mathbb{R}_+, U)$
- $Ax = \lim_{t \to +0} \frac{1}{t}(T(t)x - x)$.
- T is a C_0-semigroup.
- f is a Lipschitz continuous perturbation.

$x \in C([0, T], X)$ is a mild solution iff

$$x(t) = T(t)x_0 + \int_0^t T(t - s)f(x(s), u(s))ds.$$

Comparison functions

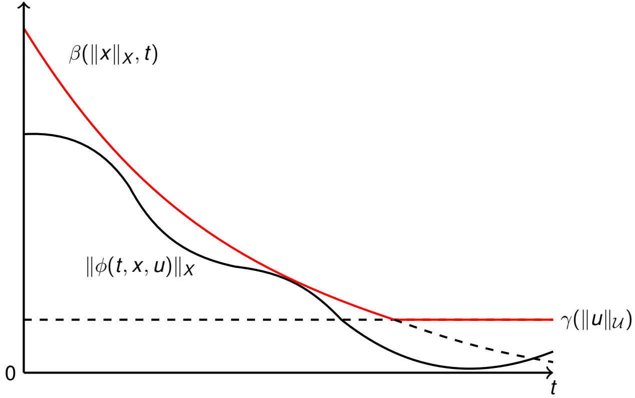


Input-to-state stability

Definition (ISS)

ISS $:\Leftrightarrow \quad \exists \beta \in \mathcal{KL}, \gamma \in \mathcal{K}_\infty: \quad \forall t \geq 0, \forall x \in X, \forall u \in \mathcal{U}$

$$\|\phi(t,x,u)\|_X \leq \max\Big\{\beta(\|x\|_X, t), \underbrace{\gamma}_{Gain}\big(\sup_{s\in[0,t]} \|u(s)\|_U\big)\Big\}.$$

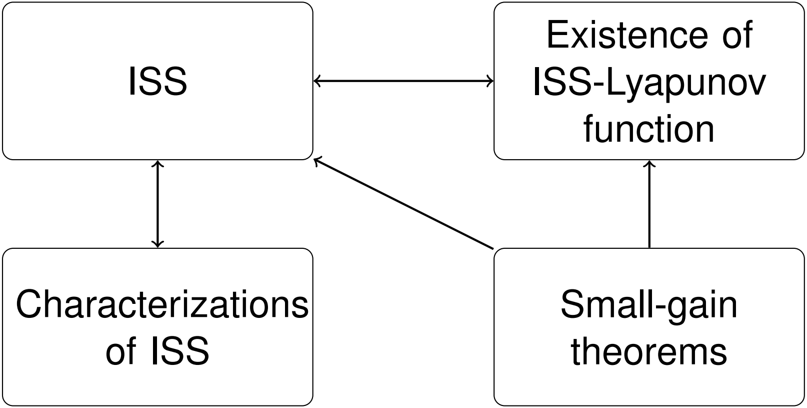


Motivation

Why ISS?

1. Unified theory of internal and external stability
   - E. D. Sontag. *Input to State Stability: Basic Concepts and Results.* In Nonlinear and Optimal Control Theory, chapter 3, 2008.
2. Robust stabilization of nonlinear systems
   - M. Krstić, I. Kanellakopoulos, P. Kokotović. Nonlinear and adaptive control design, Wiley, 1995.
3. Design of robust nonlinear observers
   - M. Arcak, P. Kokotović. Nonlinear observers: a circle criterion design and robustness analysis, 2001.
4. Stability of networks of nonlinear control systems
   - Z.-P. Jiang, I. Mareels, Y. Wang. A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems, Automatica, 1996.
   - S. Dashkovskiy, B. Rüffer, F. Wirth. Small Gain Theorems for Large Scale Systems and Construction of ISS Lyapunov Functions, SICON, 2010.
5. . . .

Fundamentals of ISS-Theory for ODEs



Topics in infinite-dimensional ISS theory

- Linear systems $\dot{x} = Ax + Bu$
  B. Jacob, I. Karafyllis, M. Krstic, J.R. Partington, C. Prieur, E. Witrant, ...
- Constructions of Lyapunov functions. Small-gain theorems. Applications.
  A. Chaillet, S. Dashkovskiy, H. Ito, I. Karafyllis, F. Mazenc, AM, Y. Orlov, A. Papachristodoulou, A. Pisano, C. Prieur, A. Tanwani, S. Tarbouriech, ...
- Lur'e systems
  B. Jayawardhana, H. Logemann, E. P. Ryan
- Impulsive systems
  S. Dashkovskiy, AM
- Characterizations of ISS
  AM, F. Wirth

See the final slides for an up-to-date list of papers in ∞-dim ISS theory.

Highlights of this talk

1. Lyapunov characterization of LISS and ISS
2. Non-Lyapunov characterizations
   - ISS and its characterizations via novel uniform limit property
   - Strong ISS and its characterization
3. Differences between ISS theory for ODEs and for ∞-dim systems.
4. Open problems

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Characterizations of ISS were proved for

1. ODEs
2. PDEs
3. Evolution equations in Banach spaces
4. Time-delay systems
5. Switched systems
6. ∞ ODE-Ensembles

LISS and 0-UGAS

Definition (GAS uniform w.r.t. state (0-UGAS))

0-UGAS $:\Leftrightarrow$ $\exists \beta \in \mathcal{KL}$: $\forall x \in X, \forall t \geq 0$

$$\|\phi(t,x,0)\|_X \leq \beta(\|x\|_X, t).$$

Definition (LISS)

LISS $:\Leftrightarrow$ $\exists \beta \in \mathcal{KL}, \gamma \in \mathcal{K}_\infty, r > 0$:

$t \geq 0,\ \|x\|_X \leq r,\ \|u\|_{\mathcal{U}} \leq r \quad \Rightarrow$

$$\|\phi(t,x,u)\|_X \leq \max\Big\{\beta(\|x\|_X, t), \underbrace{\gamma}_{Gain}\Big(\sup_{s\in[0,t]} \|u(s)\|_U\Big)\Big\}.$$

Outline

ISS Lyapunov functions

$$\dot{x}(t) = Ax(t) + f(x(t), u(t)),$$

Definition

$V : X \to \mathbb{R}_+$ is a non-coercive ISS-Lyapunov function iff $\exists \psi_2, \sigma, \alpha \in \mathcal{K}_\infty$:

(i) $0 < V(x) \le \psi_2(\|x\|_X) \quad \forall x \neq 0$

(ii) $\dot{V}_u(x) \le -\alpha(\|x\|_X) + \sigma(\|u(0)\|_U) \quad \forall x \in X, \forall u \in \mathcal{U}$,

$$\dot{V}_u(x) = \overline{\lim_{t \to +0}} \frac{1}{t}\big(V(\phi(t,x,u)) - V(x)\big).$$

V is called a coercive ISS-Lyapunov function if

$$\exists \psi_1, \psi_2 \in \mathcal{K}_\infty : \quad \psi_1(\|x\|_X) \le V(x) \le \psi_2(\|x\|_X), \quad \forall x \neq 0.$$

Theorem (Classical Direct Lyapunov theorem)

$\exists$ a coercive (L)ISS Lyapunov function $\quad \Rightarrow \quad$ (L)ISS.

LISS = 0-UAS = $\exists$ coercive LISS-LF

$$\dot{x}(t) = Ax(t) + f(x(t), u(t))$$

Theorem (AM, Sys. & Cont. Lett., 2016)

(i) $\forall C > 0 \; \exists K(C) > 0$:

$$\|x\|_X \leq C, \; \|y\|_X \leq C \quad \Rightarrow \quad \|f(y,v) - f(x,v)\|_X \leq L_f(C)\|y - x\|_X.$$

(ii) $f(x,\cdot)$ *be continuous for all* $x \in X$.

(iii) $\exists \; \sigma \in \mathcal{K}$ *and* $\rho > 0$:

$$\|v\|_U \leq \rho, \; \|x\|_X \leq \rho \quad \Rightarrow \quad \|f(x,v) - f(x,0)\|_X \leq \sigma(\|v\|_U).$$

$$\Downarrow$$

0-UAS $\Leftrightarrow$ $\exists$ *0-UAS LF* $\Leftrightarrow$ $\exists$ *LISS LF* $\Leftrightarrow$ *LISS*

Counterexample: 0-UGAS but not LISS

$$\dot{x}_k(t) = -\frac{1}{1 + k|u(t)|} x_k(t)$$

- $X = l_1 := \{(x_k)_{k=1}^{\infty} : \sum_{k=1}^{\infty} |x_k| < \infty\}$
- $\mathcal{U} := PC(\mathbb{R}_+, \mathbb{R})$

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- Properties (i) and (ii) hold

But property (iii) does not hold!

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- $\forall u, \ \forall x \quad \Rightarrow \quad \|\phi(t,x,u)\|_X \to 0, t \to \infty$
- Properties (i) and (ii) hold

But property (iii) does not hold!

It is not LISS!

Final remarks concerning LISS

$\mathbb{R}^n$-world

1. In $\mathbb{R}^n$ (ii) $\wedge$ (i) $\Rightarrow$ (iii).

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4. Our result is more general and uses other technique.

Open question

1. If (iii) is dropped, will converse LISS Lyapunov theorem still hold, even though 0-UAS $\neq$ LISS in general?

Converse Lyapunov theorems for global ISS

Converse Lyapunov theorems for the global ISS property are more complicated!

Our plan

1. UGAS = $\exists$ UGAS Lyapunov function
2. ISS = UGAS under suitable feedback

ISS and Robust Stability

$$\dot{x}(t) = Ax(t) + f(x(t), u(t)), \quad x(t) \in D(A) \subset X \qquad (\Sigma)$$

- Let $\quad u(t) := d(t)\varphi(x(t))$.
- with $\quad d \in \mathcal{D} = \{d : \mathbb{R}_+ \to D\}, \quad D = \{d \in U : \|d\|_U \leq 1\}$.

Then

$$\dot{x}(t) = Ax(t) + \underbrace{f(x(t), d(t)\varphi(x(t)))}_{g(x(t),d(t))}. \qquad (\tilde{\Sigma})$$

Definition

Σ is weakly uniformly robustly asymptotically stable (WURS), if $\exists$ Lipschitz continuous $\varphi : X \to \mathbb{R}_+$ and $\psi \in \mathcal{K}_\infty$:

- $\varphi(x) \geq \psi(\|x\|_X)$
- $\tilde{\Sigma}$ is UGAS over $\mathcal{D} = \{d : \mathbb{R}_+ \to D\}$.

Converse Lyapunov Theorem for disturbed systems

Lemma

f is bi-Lipschitz on bounded balls $\Rightarrow$ g is Lipschitz continuous on bounded balls, uniformly w.r.t. the second argument.

Theorem (Karafyllis, Jiang, 2011)

$$\dot{x}(t) = Ax(t) + g(x(t), d(t)). \qquad (\tilde{\Sigma})$$

- *g is Lipschitz continuous on bounded balls, uniformly w.r.t. the second argument*
- *$\tilde{\Sigma}$ is UGAS*

$\Rightarrow \exists$ *Lipschitz continuous Lyapunov function for $\tilde{\Sigma}$.*

Proposition

V is a Lipschitz continuous Lyapunov function for $\tilde{\Sigma}$
$\Rightarrow$ V is a Lipschitz continuous ISS Lyapunov function for Σ.

ISS and Robust Stability

$$\dot{x}(t) = Ax(t) + f(x(t), u(t)), \quad x(t) \in D(A) \subset X \qquad (\Sigma)$$

$$\dot{x}(t) = Ax(t) + \underbrace{f(x(t), d(t)\varphi(x(t)))}_{g(x(t),d(t))}. \qquad (\tilde{\Sigma})$$

Theorem

Σ is ISS $\Rightarrow$ Σ is WURS.

$$\Sigma \text{ is ISS} \Longleftarrow \exists \text{ ISS-LF for } \Sigma$$

$$\Sigma \text{ is ISS} \Downarrow \qquad \Uparrow \exists \text{ ISS-LF for } \Sigma$$

$$\tilde{\Sigma} \text{ is UGAS} \Longrightarrow \exists \text{ UGAS-LF for } \tilde{\Sigma}$$

Outline

1. Basic definitions
2. Lyapunov characterizations
   - Lyapunov characterization of LISS
   - Lyapunov characterization of ISS
3. Characterization of ISS
4. Directions for future work

Motivation II

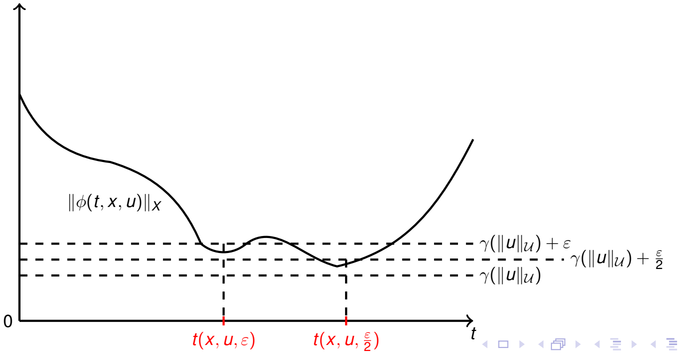
Definition (Stability and Attractivity for zero inputs)

0-ULS $:\Leftrightarrow \quad \exists \sigma \in \mathcal{K}_\infty,\ r > 0$

$$\|x\|_X \le r,\ \ t \ge 0 \quad \Rightarrow \quad \|\phi(t,x,0)\|_X \le \sigma(\|x\|_X)$$

LIM $:\Leftrightarrow \quad \exists \gamma \in \mathcal{K}_\infty \colon \forall x \in X, \forall u \in \mathcal{U}, \forall \varepsilon > 0\ \exists T = T(\varepsilon, x, u) :$

$$\|\phi(t,x,u)\|_X \le \varepsilon + \gamma(\|u\|_{\mathcal{U}}).$$



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Theorem (Characterizations of ISS (Sontag, Wang))

$$\dot{x} = f(x,u). \qquad \text{(ODE)}$$

For a forward complete system (ODE) it holds that

$$\textit{ISS} \quad \Leftrightarrow \quad \textit{LIM} \wedge \textit{0-ULS}$$

- E. D. Sontag and Y. Wang. On characterizations of the input-to-state stability property. Sys. & Cont. Letters, 1995.
- E. D. Sontag and Y. Wang. New characterizations of input-to-state stability. IEEE TAC, 1996.

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For a forward complete system (ODE) it holds that

$$\textit{ISS} \quad \Leftrightarrow \quad \textit{LIM} \wedge \textit{0-ULS}$$

Can we prove a generalization of this theorem to evolution equations?

Thinking about stability theory

Definition

Σ is called forward complete (FC) if for any $x \in X$, any $u \in \mathcal{U}$ the solution $\phi(\cdot, x, u)$ exists, is unique and finite for all times.

Thinking about stability theory

Does forward completeness tell us something beyond mere existence of solutions?

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Definition

FC Σ has bounded reachability sets (BRS) $:=$ $\forall R > 0\ \forall \tau > 0$ $\exists M = M(R,\tau)$:

$$\sup_{\|x\|_X \le R,\ \|u\|_{\mathcal{U}} \le R,\ t \in [0,\tau]} \|\phi(t,x,u)\|_X \le M(R,\tau) < \infty.$$

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Proposition (Lin, Sontag, Wang, 1996)

$$\Sigma_{ODE} : \dot{x} = f(x,u), \qquad x(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^m$$

If Σ_{ODE} is FC $\quad \Rightarrow \quad \Sigma_{ODE}$ is BRS.

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BRS is a "bridge" between solution theory and stability theory

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Infinite-dimensional FC systems are not necessarily BRS!

Linear undisturbed systems

Let A generate C_0-semigroup T, $\phi(t,x) = T(t)x$.

Simple properties

- BRS: $\exists M, \lambda > 0 : \|T(t)x\|_X \leq Me^{\lambda t}\|x\|_X$.
- $\forall x \in X \lim_{t\to\infty} \|T(t)x\|_X = 0 \quad \Rightarrow \quad \exists M > 0$: $\|T(t)\| \leq M$.

Definition (Stability and Attractivity for zero inputs)

0-UGS $:\Leftrightarrow$ $\exists \sigma \in \mathcal{K}_\infty : \forall t \geq 0,\ \forall x \in X$

$$\|\phi(t,x,0)\|_X \leq \sigma(\|x\|_X) \tag{1}$$

0-ULS $:\Leftrightarrow$ (1) holds $\forall x : \|x\|_X \leq r$.

0-GATT $:\Leftrightarrow$ $\forall x \in X \quad \lim_{t\to\infty} \|\phi(t,x,0)\|_X = 0$.

0-GAS $:\Leftrightarrow$ 0-ULS $\wedge$ 0-GATT.

Nonlinear undisturbed systems: examples

Consider systems

$$\dot{x} = Ax + f(x)$$

- FC $\wedge$ 0-GAS $\not\Rightarrow$ BRS

$$X = l_2 := \{(z_i)_{i=1}^{\infty} : \sum_{i=1}^{\infty} |z_i|^2 < \infty\}, \quad z_i = (x_i, y_i) \in \mathbb{R}^2.$$

$$\Sigma : \begin{cases} \Sigma_i : \begin{cases} \dot{x}_i = -x_i + x_i^2 y_i - \frac{1}{i^2} x_i^3, \\ \dot{y}_i = -y_i. \end{cases} \\ i = 1, \ldots, \infty \end{cases}$$

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- FC $\wedge$ BRS $\wedge$ 0-GAS $\not\Rightarrow$ 0-UGS
    - Pick previous system.
    - Change (nonuniformly) time clocks: $\tilde{x}_k(t) := x_k(\frac{t}{k})$
    - Resulting system is 0-GAS, BRS, but not 0-UGS.

Moral

What we know:

- closed unit ball is never compact
- FC $\wedge$ 0-GAS $\not\Rightarrow$ BRS
- FC $\wedge$ BRS $\wedge$ 0-GAS $\not\Rightarrow$ 0-UGS

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Generalizing results from ODE theory is not straightforward!

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- FC $\wedge$ BRS $\wedge$ 0-GAS $\not\Rightarrow$ 0-UGS

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Non-Lyapunov characterizations due to Sontag, Wang do not hold for in ∞-dim!

Moral

What we know:

- closed unit ball is never compact
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- FC $\wedge$ BRS $\wedge$ 0-GAS $\not\Rightarrow$ 0-UGS

Generalizing results from ODE theory is not straightforward!

Non-Lyapunov characterizations due to Sontag, Wang do not hold for in ∞-dim!

Key: Uniform weak attractivity!

Definition

$$\text{UGS} \;:\Leftrightarrow\quad \exists \sigma, \gamma \in \mathcal{K}_\infty:$$

$$t \geq 0, x \in X, u \in \mathcal{U} \;\Rightarrow\; \|\phi(t,x,u)\|_X \leq \sigma(\|x\|_X) + \gamma(\|u\|_{\mathcal{U}}).$$

$$\text{ULS} \;:\Leftrightarrow\quad \exists \sigma, \gamma \in \mathcal{K}_\infty, r > 0:$$

$$t \geq 0, \|x\|_X \leq r, \|u\|_{\mathcal{U}} \leq r \;\Rightarrow\; \|\phi(t,x,u)\|_X \leq \sigma(\|x\|_X) + \gamma(\|u\|_{\mathcal{U}}).$$

$$\text{LIM} \;:\Leftrightarrow\quad \exists \gamma \in \mathcal{K}_\infty\colon \forall x \in X, \forall u \in \mathcal{U}, \forall \varepsilon > 0\; \exists T = T(\varepsilon, x, u):$$

$$\|\phi(t,x,u)\|_X \leq \varepsilon + \gamma(\|u\|_{\mathcal{U}}).$$

$$\text{ULIM} \;:\Leftrightarrow\quad \exists \gamma \in \mathcal{K}_\infty \cup \{0\}\colon \forall \varepsilon, \delta > 0\; \exists T = T(\varepsilon, \delta):$$

$$u \in \mathcal{U},\; \|x\|_X \leq \delta \quad\Rightarrow\quad \exists t \leq T : \|\phi(t,x,u)\|_X \leq \varepsilon + \gamma(\|u\|_{\mathcal{U}}).$$

Proposition (AM, F. Wirth, 2017)

For forward complete ODEs it holds that:

$$\text{LIM} \quad\Leftrightarrow\quad \text{ULIM}$$

Characterizations of ISS

Theorem (AM, F. Wirth, 2017)

Let Σ be BRS. Then the following statements are equivalent:

(i) Σ *is ISS*

(ii) Σ *is ULIM* $\wedge$ *ULS*

(iii) Σ *is ULIM* $\wedge$ *UGS*

If in addition $\exists\ \sigma \in \mathcal{K}$ *and* $\rho > 0$*:*

$$\|v\|_U \le \rho,\ \|x\|_X \le \rho \quad \Rightarrow \quad \|f(x,v) - f(x,0)\|_X \le \sigma(\|v\|_U).$$

then the following notion is also equivalent to ISS:

(iv) Σ *is ULIM* $\wedge$ *0-ULS*

Characterizations of strong ISS

$$\dot{x}(t) = Ax(t) + f(x(t), u(t)), \quad x(t) \in D(A) \subset X \qquad (\Sigma)$$

Definition

Σ is strongly input-to-state stable (sISS), if $\exists \gamma \in \mathcal{K}$, $\sigma \in \mathcal{K}_\infty$ and $\beta : X \times \mathbb{R}_+ \to \mathbb{R}_+$:

1. $\beta(x, \cdot) \in \mathcal{L}$ for all $x \in X$, $x \neq 0$
2. $\beta(x, t) \leq \sigma(\|x\|_X)$ for all $x \in X$ and all $t \geq 0$
3. for all $x \in X$, all $u \in \mathcal{U}$ and all $t \geq 0$ it holds that

$$\|\phi(t, x, u)\|_X \leq \beta(x, t) + \gamma(\|u\|_\mathcal{U}).$$

sLIM $:\Leftrightarrow \quad \exists \gamma \in \mathcal{K}_\infty : \forall x \in X, \forall \varepsilon > 0 \; \exists T = T(\varepsilon, x) :$

$$u \in \mathcal{U} \quad \Rightarrow \quad \exists t \leq T : \|\phi(t, x, u)\|_X \leq \varepsilon + \gamma(\|u\|_\mathcal{U}).$$

Theorem

$$sISS \quad \Leftrightarrow \quad sAG \wedge UGS \quad \Leftrightarrow \quad sLIM \wedge UGS$$

Summary: ISS and sISS characterizations

$$\text{ULIM} \wedge \text{0-ULS} \wedge \text{BRS} \Leftrightarrow \text{ULIM} \wedge \text{ULS} \wedge \text{BRS} \Leftrightarrow \text{ULIM} \wedge \text{UGS} \Longleftrightarrow \text{ISS}$$

$$\text{ULIM} \wedge \text{UGS} \;\Downarrow\; \not\Uparrow \; \text{sLIM} \wedge \text{UGS}$$

$$\text{sLIM} \wedge \text{UGS} \Longleftrightarrow \text{sISS}$$

$$\text{sLIM} \wedge \text{UGS} \Downarrow \text{LIM} \wedge \text{UGS}$$

Corollary

For ODEs all these notions are equivalent, since:

- *FC* $\Leftrightarrow$ *BRS*
- *LIM* $\Leftrightarrow$ *ULIM*

Results of Sontag and Wang are a special case of our results!

Summary: ISS and sISS characterizations

$\text{ULIM} \wedge \text{0-ULS} \wedge \text{BRS} \Leftrightarrow \text{ULIM} \wedge \text{ULS} \wedge \text{BRS} \Leftrightarrow \text{ULIM} \wedge \text{UGS} \Longleftrightarrow \text{ISS}$

$\text{ULIM} \wedge \text{UGS} \Downarrow \not\Uparrow \text{sLIM} \wedge \text{UGS}$

$\text{sLIM} \wedge \text{UGS} \Longleftrightarrow \text{sISS}$

$\text{sLIM} \wedge \text{UGS} \Downarrow \text{LIM} \wedge \text{UGS}$

- A.M. *Local input-to-state stability: Characterizations and counterexamples.* Sys. & Con. Lett., 2016
- A.M., F. Wirth. *Characterizations of input-to-state stability for infinite-dimensional systems.* Submitted to TAC, 2017
- A.M. *Criteria for input-to-state practical stability.* Submitted, 2017.

Summary: ISS and sISS characterizations

$$\text{ULIM} \wedge \text{0-ULS} \wedge \text{BRS} \Leftrightarrow \text{ULIM} \wedge \text{ULS} \wedge \text{BRS} \Leftrightarrow \text{ULIM} \wedge \text{UGS} \Longleftrightarrow \text{ISS}$$

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$$\text{sLIM} \wedge \text{UGS} \Longleftrightarrow \text{sISS}$$

$$\text{sLIM} \wedge \text{UGS} \Downarrow \text{LIM} \wedge \text{UGS}$$

What we discussed

- Uniformity matters!
- Characterizations of ISS and sISS

Summary: ISS and sISS characterizations

$$\text{ULIM} \wedge \text{0-ULS} \wedge \text{BRS} \Leftrightarrow \text{ULIM} \wedge \text{ULS} \wedge \text{BRS} \Leftrightarrow \text{ULIM} \wedge \text{UGS} \Longleftrightarrow \text{ISS}$$

$$\Downarrow \;\not\Uparrow$$

$$\text{sLIM} \wedge \text{UGS} \Longleftrightarrow \text{sISS}$$

$$\Downarrow$$

$$\text{LIM} \wedge \text{UGS}$$

What we could discuss

- Generalizations of these results to wide classes of systems, including:
  - DEs in Banach spaces
  - Time-delay systems
  - Switched systems
- Characterizations of practical ISS
- Relations of ULIM and non-coercive Lyapunov functions.

Summary: ISS and sISS characterizations

$$\text{ULIM} \wedge \text{0-ULS} \wedge \text{BRS} \Leftrightarrow \text{ULIM} \wedge \text{ULS} \wedge \text{BRS} \Leftrightarrow \text{ULIM} \wedge \text{UGS} \Longleftrightarrow \text{ISS}$$

$$\text{ULIM} \wedge \text{UGS} \Downarrow \not\Uparrow \text{sLIM} \wedge \text{UGS}$$

$$\text{sLIM} \wedge \text{UGS} \Longleftrightarrow \text{sISS}$$

$$\text{sLIM} \wedge \text{UGS} \Longdownarrow \text{LIM} \wedge \text{UGS}$$

Open problems

- How far can these results can be tightened for special classes of systems, as e.g. time-delay systems.

2-nd Workshop "Stability and Control of Infinite-Dimensional Systems"

Scope

- Stability and control of partial differential equations
- Stability and control of time-delay systems
- Input-to-state stability of infinite-dimensional systems
- Stabilizability of infinite-dimensional systems
- Semigroup and admissibility theory

- Venue: University of Würzburg, Germany
- Date: 10–12 October, 2018.
- Organisers: S. Dashkovskiy, B. Jacob, A. Mironchenko, F. Wirth.

Previous workshop: Passau, Germany, 12–14 October, 2016

- 47 Participants from 11 countries; 22 Invited speakers
- http://www.fim.uni-passau.de/en/dynamical-systems/workshop/

Outline

0.1 strong ISS, strong LIM, strong AG

Motivation

- We have only basic results concerning characterizations of AG.
- Relations between LIM and AG are not clear, even for ODEs
- Relations between sLIM and LIM are again not clear, even for ODEs
- The same about sAG and AG.

Aim

- Obtain deep characterizations of strong ISS property.

Motivation

- For linear systems with admissible operators iISS $\Rightarrow$ ISS
- But it is unclear, whether ISS $\Rightarrow$ iISS, unless further assumptions are made
  (B. Jacob, F. Schwenninger, J. Partington, 2017).
- Relation between iISS and ISS for nonlinear systems.

0.2 Characterizations of integral ISS

Motivation

- For linear systems with admissible operators iISS $\Rightarrow$ ISS
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  (B. Jacob, F. Schwenninger, J. Partington, 2017).
- Relation between iISS and ISS for nonlinear systems.

Aim

- Characterizations of integral ISS property are needed!

0.2 Characterizations of integral ISS

Motivation

- For linear systems with admissible operators iISS $\Rightarrow$ ISS
- But it is unclear, whether ISS $\Rightarrow$ iISS, unless further assumptions are made
  (B. Jacob, F. Schwenninger, J. Partington, 2017).
- Relation between iISS and ISS for nonlinear systems.

Open problems on "Characterizations"

- Characterizations in metric spaces
- Characterizations of sISS, sLIM, sAG notions
- Characterizations of integral ISS

1. Non-coercive Lyapunov functions

$$\dot{x}(t) = Ax(t) + f(x(t)),$$

$V : X \to \mathbb{R}_+$ is a non-coercive Lyapunov function iff $\exists \psi_2, \sigma, \alpha \in \mathcal{K}_\infty$:

(i) $0 < V(x) \leq \psi_2(\|x\|_X) \quad \forall x \neq 0$

(ii) $\dot{V}(x) \leq -\alpha(\|x\|_X) \quad \forall x \in X,$

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Theorem (AM, F. Wirth, 2017, submitted)

Let:

- Σ *be RFC*
- 0 *be a robust equilibrium of* Σ

If V is a noncoercive Lyapunov function for $\Sigma \quad \Rightarrow \quad \Sigma$ *is UGAS*

1. Non-coercive Lyapunov functions

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Theorem (AM, F. Wirth, 2017, submitted)

Let:

- *Σ be RFC*
- *0 be a robust equilibrium of Σ*

If V is a noncoercive Lyapunov function for Σ $\Rightarrow$ Σ is UGAS

Can we derive ISS from non-coercive ISS Lyapunov functions?

2. Linear systems with unbounded input operators

$$\dot{x} = Ax + Bu.$$

Proposition

ISS $\Leftrightarrow$ 0-UGAS $\wedge$ admissibility of B

Questions

- Lyapunov characterization of ISS
- Is it enough to consider coercive ISS Lyapunov functions?
- Characterization of integral ISS
- What is the relationship between ISS and integral ISS?

3. Robust boundary control of PDEs

$$(\Sigma_1)\quad \begin{array}{l} \frac{\partial x}{\partial t}(z,t) = \frac{\partial^2 x}{\partial z^2}(z,t) + ax(z,t) \\ x(0,t) = 0 \quad \forall t \geq 0 \\ x(1,t) = u(t) + d(t) \quad \forall t \geq 0. \end{array} \quad \xrightarrow[\text{Volterra tr.}]{} \quad \begin{array}{l} \frac{\partial w}{\partial t}(z,t) = \frac{\partial^2 w}{\partial z^2}(z,t) \\ w(0,t) = 0 \quad \forall t \geq 0 \\ w(1,t) = d(t) \quad \forall t \geq 0, \end{array} \quad (\Sigma_2)$$

Σ_1 is transformed into Σ_2 by means of

- $w(z,t) = x(z,t) + \int_0^z k(z,y)x(y,t)dy$
- $u(t) = -\int_0^1 k(1,y)x(y,t)dy$

and we naturally come to the question of ISS of Σ_2.

Questions

- ISS of the boundary control systems
- Robust stabilization of PDE systems with boundary controls
- Boundary interconnections of PDE systems

We have:

- Small-gain theorems in Lyapunov form for interconnections of n systems with in-domain inputs

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We would like to have:

- Small-gain theorems in trajectory form

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- Small-gain theorems in Lyapunov form for interconnections of n systems with in-domain inputs

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- Small-gain theorems in trajectory form
  A major difficulty here is that ISS $\neq$ AG $\wedge$ UGS

4. Stability of interconnections

We have:

- Small-gain theorems in Lyapunov form for interconnections of n systems with in-domain inputs

We would like to have:

- Small-gain theorems in trajectory form
  A major difficulty here is that ISS $\neq$ AG $\wedge$ UGS
- SGT for boundary interconnected PDEs
- SGT for infinite interconnections

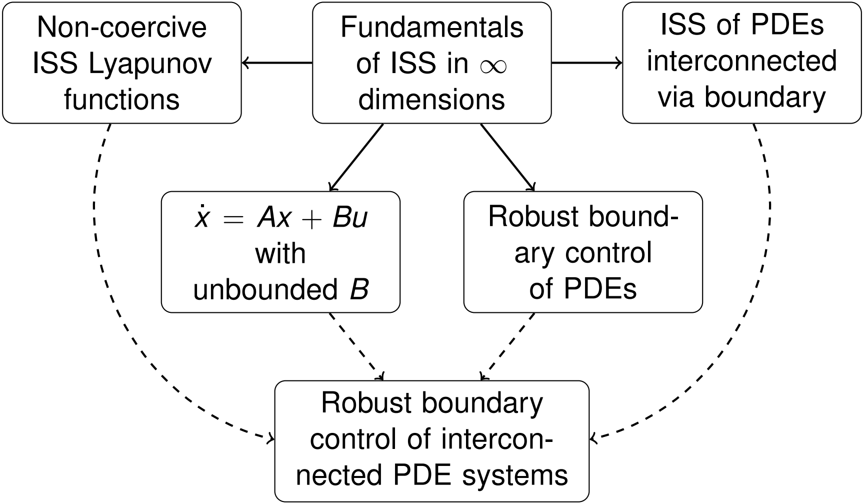
5. Construction of LFs for wide classes of systems

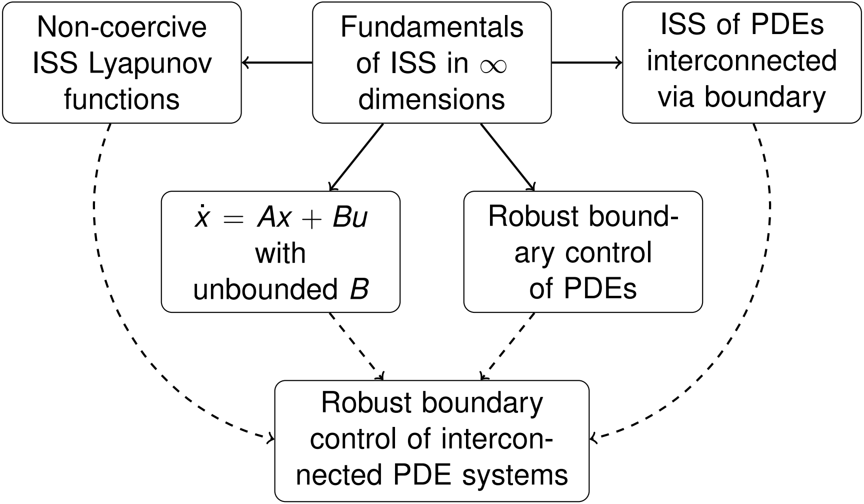
Important for applications

- ISS stabilization
- Construction of nonlinear observers
- etc.

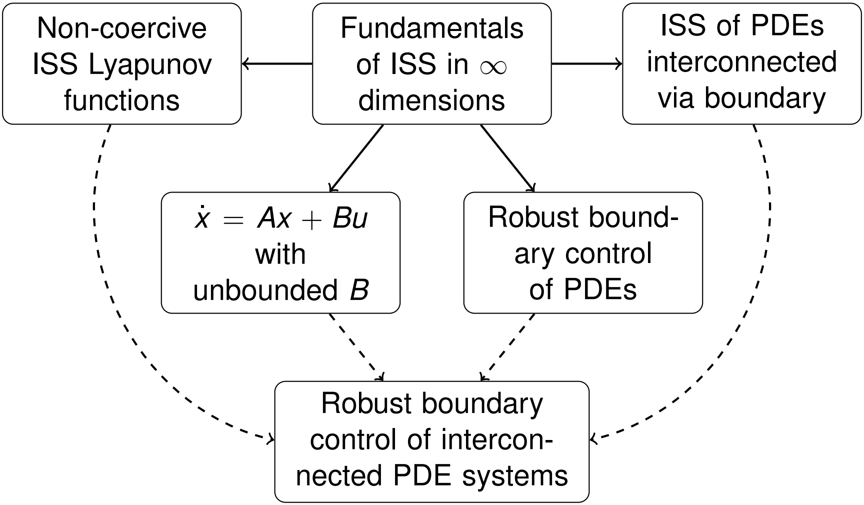
- Till now almost exclusively parabolic PDEs have been studied.
- A few results on systems of conservation laws and higher-order equations.

- No results for wave equations
- No results for nonlinear systems with boundary inputs
- No results ISS of feedbacks of different kinds of systems (ODE-PDE, heat-wave etc.)


Non-coercive ISS Lyapunov functions
Fundamentals of ISS in ∞ dimensions
ISS of PDEs interconnected via boundary
ẋ = Ax + Bu with unbounded B
Robust boundary control of PDEs
Robust boundary control of interconnected PDE systems



Thank you for attention!



Papers and slides can be found at
www.mironchenko.com

Literature overview: linear systems

$$\dot{x} = Ax + Bu$$

- Systems with unbounded B are studied
- PDEs with boundary control fall into this class
- This theory is just in the beginning of the development

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Literature overview: nonlinear systems 3

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